Spinography

Everything you wanted to know (and more) about Clifford algebras and Pin

A D Kennedy

University of Edinburgh

Apology

Je n'ai fait celle-ci plus longue que parce que je n'ai pas eu le loisir de la faire plus courte

Blaise Pascal (1657)

Verb. Sap.

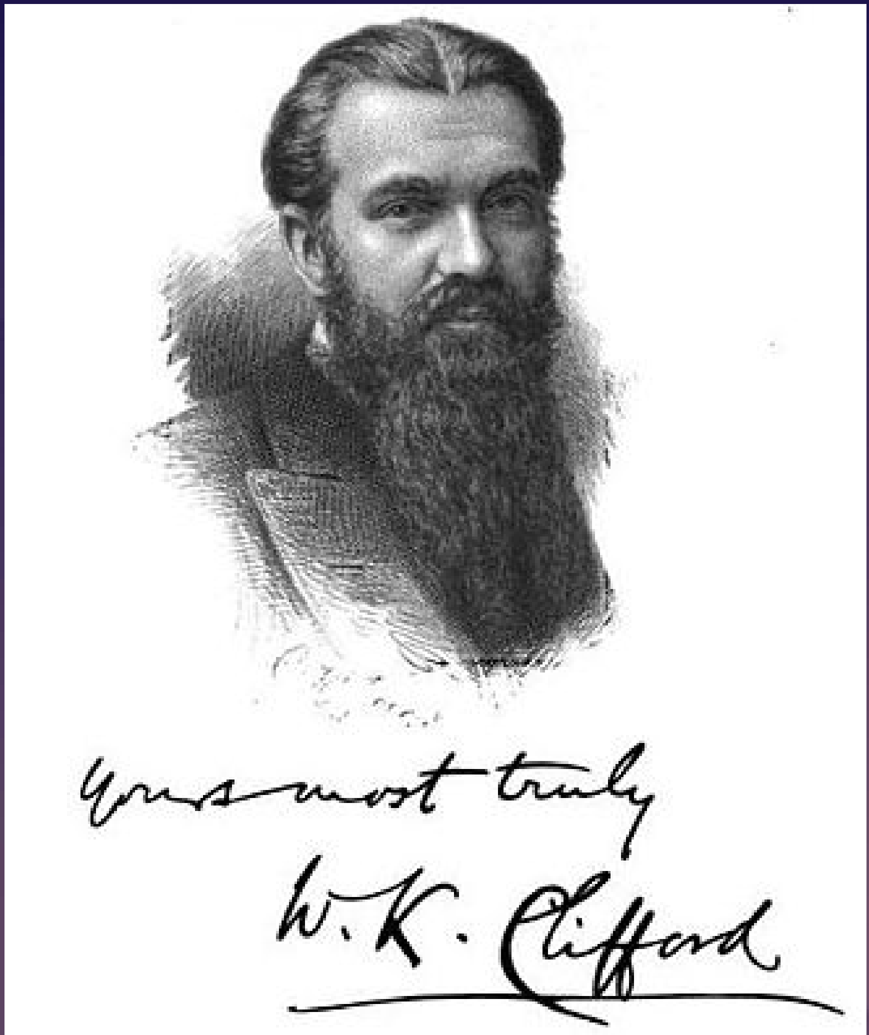
- $\mathfrak{so}(4;\mathbb{C})$ is not simple
  - $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{su}(2;\mathbb{C}) \oplus \mathfrak{su}(2;\mathbb{C})$
  - Look at the Dynkin (Coxeter) diagram
- The real forms are
  - $\mathfrak{so}(4,0;\mathbb{R}) \cong \mathfrak{su}(2,\mathbb{R}) \oplus \mathfrak{su}(2,\mathbb{R})$
  - $\mathfrak{so}(3,1;\mathbb{R}) \cong \mathfrak{sl}(2;\mathbb{C})$
- This does not work for general n
  - Needed for dimensional regularization
  - $\mathfrak{so}(6;\mathbb{C}) \cong \mathfrak{su}(4;\mathbb{C})$

Periodicity, Spin, and Pin

References

- Yvonne Choquet-Bruhat and Cécile DeWitt-Morette, *Analysis, Manifolds and Physics*, Part II, Elsevier (2000)
- Claude Chevalley, *The Algebraic Theory of Spinors and Clifford Algebras*, Springer (19960
- Michael Atiyah, Raoul Bott, and Arnold Shapiro†, *Clifford Modules,* Topology, **3** (Suppl. 1) 3–38 (1964)
- Marcus Berg, Cécile DeWitt-Morette, Shangjr Gwo, and Eric Kramer, *The Pin Groups in Physics: C, P, and T*, arXiv:math-ph/0012006 (2000)

Introduction



- Let V be a d dimensional real vector space with a metric $g_{\mu\nu}$ of signature (m, n)
- The abstract Clifford algebra $\mathcal{C}(m, n)$ associated with this space is generated by vectors $\{e_1, \dots, e_d\}$ that satisfy the anticommutation relations $\{e_\mu, e_\nu\} = 2g_{\mu\nu}$

Introduction

- As a vector space $\mathcal{C}(m,n)$ is spanned by $e_{\mu_1} \dots e_{\mu_k}$ with $k = 0, \dots, d$ and $\mu_1 < \mu_2 < \cdots < \mu_k$, and thus has 2^d dimensions
- By the Skolem—Noether theorem the complexified Clifford algebra $\mathcal{C}(m,n;\mathbb{C}) = \mathcal{C}(m,n;\mathbb{R}) \otimes \mathbb{C}$ for $d = 2p$ is isomorphic to the algebra $M_{2^p}(\mathbb{C})$ of $2^p \times 2^p$ matrices
  - We shall write this as the representation $\rho: \mathcal{C}(m,n;\mathbb{C}) \to M_{2^p}(\mathbb{C})$, $\rho(e_\mu) = \gamma_\mu$
  - Remember that the γ_μ are matrices, so we can do things like taking their trace, whereas the e_μ are elements of an abstract algebra

Representations of Clifford Algebras

Pauli Matrices

- Let's consider some low-dimensional examples:
  - For d =2 consider the familiar Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Which satisfy $\sigma_j \sigma_k = \varepsilon_{jk\ell} i \sigma_\ell$, $\sigma_\ell^2 = \mathbb{I}$, and $\sigma_1 \sigma_2 \sigma_3 = i\mathbb{I}$

- These clearly provide a representation of the complex Clifford algebra $\mathcal{C}(2; \mathbb{C})$
- It is illuminating to consider the representation they provide for the real algebras $\mathcal{C}(2,0; \mathbb{R})$, $\mathcal{C}(1,1; \mathbb{R})$, and $\mathcal{C}(0,2; \mathbb{R})$

Pauli Matrices

- $\mathcal{C}(2,0;\mathbb{R})$
  - We need two generators whose square is 1, so we may choose σ_1 and σ_3. Since $\sigma_3\sigma_1 = i\sigma_2$ we find that $\mathcal{C}(2,0;\mathbb{R}) \cong \mathrm{M}_2(\mathbb{R})$ consists of all real 2×2 matrices
- $\mathcal{C}(1,1;\mathbb{R})$
  - We need one generator whose square is 1 and one whose square is -1, so we may choose σ_1 and $i\sigma_2$. Since $\sigma_1 i\sigma_2 = -\sigma_3$ we again find that $\mathcal{C}(1,1;\mathbb{R}) \cong \mathrm{M}_2(\mathbb{R})$ consists of all real 2×2 matrices
- $\mathcal{C}(0,2;\mathbb{R})$
  - We need two generators whose square is -1, so we may choose $i\sigma_1$ and $i\sigma_2$. Since $i\sigma_1 i\sigma_2 = -i\sigma_3$ we find that $\mathcal{C}(0,2;\mathbb{R}) \cong \mathbb{H}$ consists of quaternions (with real coefficients)

The Matrix γ^*

- The generalization of γ_5 is $\gamma^* = \gamma_1\gamma_2\cdots\gamma_d$
- This satisfies $\{\gamma^*, \gamma_\nu\} = 0$ if $d = 2p$ is even
- Its square is ${\gamma^*}^2 = (-)^{p+n}\mathbb{I}$
  - Note that $d = 2p \Rightarrow \frac{d(d-1)}{2} = p(2p-1) \equiv p \pmod 2$, hence

$$\begin{aligned} {\gamma^*}^2 &= (\gamma_1\gamma_2\cdots\gamma_d)(\gamma_1\gamma_2\cdots\gamma_d) \\ &= (-)^{\frac{d(d-1)}{2}}\gamma_1^2\gamma_2^2\cdots\gamma_d^2 = (-)^{p+n}\mathbb{I} \end{aligned}$$

Higher Dimensions

- We may build representations of <u>even dimensional</u> Clifford algebras recursively by taking tensor products
  - Start with Pauli matrices
- Suppose we have constructed γ matrices for $\mathcal{C}(n, m; \mathbb{R})$ and γ' matrices for $\mathcal{C}(n', m'; \mathbb{R})$
- Consider the $d + d'$ generators

$$\gamma_\mu \otimes \mathbb{I}', \gamma^* \otimes \gamma'_\nu$$

where $\mu = 1, \ldots, d$ and $\nu = 1, \ldots, d'$

Higher Dimensions

- These clearly anticommute, and moreover

$$(\gamma_\mu \otimes \mathbb{I}')^2 = \gamma_\mu^2 \otimes \mathbb{I}'^2 = g_{\mu\mu}(\mathbb{I} \otimes \mathbb{I}')$$

$$(\gamma^* \otimes \gamma'_\nu)^2 = {\gamma^*}^2 \otimes {\gamma'_\nu}^2 = (-)^{p+n} g_{\nu\nu}(\mathbb{I} \otimes \mathbb{I}')$$

- The algebra generated by these is therefore $\mathcal{C}(m + m', n + n' \,; \mathbb{R})$ or $\mathcal{C}(m + n', n + m' \,; \mathbb{R})$ depending on $(-)^{p+n}$

Periodicity Modulo 8

- $\mathcal{C}(m+8,0;\mathbb{R}) \cong \mathcal{C}(m,0;\mathbb{R}) \otimes \mathrm{M}_{16}(\mathbb{R})$
  - Using the previous result and noting that $p+n=1+2=3$ is odd for $\mathcal{C}(0,2;\mathbb{R})$ we have

$$\begin{aligned}\mathcal{C}(m+8,0;\mathbb{R}) &\cong \mathcal{C}(m+6,0;\mathbb{R}) \otimes \mathcal{C}(0,2;\mathbb{R}) \\ &\cong \mathcal{C}(m+4,0;\mathbb{R}) \otimes \mathcal{C}(0,2;\mathbb{R})^{\otimes 2} \\ &\ \ \vdots \\ &\cong \mathcal{C}(m,0;\mathbb{R}) \otimes \mathcal{C}(0,2;\mathbb{R})^{\otimes 4} \\ &\cong \mathcal{C}(m,0;\mathbb{R}) \otimes \mathbb{H}^{\otimes 4}\end{aligned}$$

  - The result follows since $\mathbb{H} \otimes \mathbb{H} \cong \mathrm{M}_4(\mathbb{R})$

Periodicity Modulo 8

- For $m > n$ we have
$$\mathcal{C}(m,n;\mathbb{R}) \cong \mathcal{C}(1,1;\mathbb{R})^{\otimes n} \otimes \mathcal{C}(m-n,0;\mathbb{R})$$
$$\cong M_{2^n}(\mathbb{R}) \otimes \mathcal{C}(m-n,0;\mathbb{R})$$
  - Noting that $p+n=1+1=2$ is even for $\mathcal{C}(1,1;\mathbb{R})$ we have
$$\mathcal{C}(m,n;\mathbb{R}) \cong \mathcal{C}(1,1;\mathbb{R})^{\otimes n} \otimes \mathcal{C}(m-n,0;\mathbb{R})$$
$$\cong M_2(\mathbb{R})^{\otimes n} \otimes \mathcal{C}(m-n,0;\mathbb{R})$$
$$\cong M_{2^n}(\mathbb{R}) \otimes \mathcal{C}(m-n,0;\mathbb{R})$$

Real Clifford Algebras

- We list the structure of the real Clifford vector spaces for all dimensions
  - $\mathcal{C}(p,p;\mathbb{R}) \cong M_{2^p}(\mathbb{R}) \otimes \mathcal{C}(0,0;\mathbb{R}) \cong M_{2^p}(\mathbb{R})$
  - $\mathcal{C}(p,p-1;\mathbb{R}) \cong M_{2^{p-\frac{1}{2}}}(\mathbb{R}) \otimes \mathcal{C}(1,0;\mathbb{R}) \cong M_{2^{p-\frac{1}{2}}}(\mathbb{R}) \oplus M_{2^{p-\frac{1}{2}}}(\mathbb{R})$
  - $\mathcal{C}(p+1,p-1;\mathbb{R}) \cong M_{2^{p-1}}(\mathbb{R}) \otimes \mathcal{C}(2,0;\mathbb{R}) \cong M_{2^p}(\mathbb{R})$
  - $\mathcal{C}(p+1,p-2;\mathbb{R}) \cong M_{2^{p-\frac{3}{2}}}(\mathbb{R}) \otimes \mathcal{C}(3,0;\mathbb{R}) \cong M_{2^{p-\frac{1}{2}}}(\mathbb{C})$
  - $\mathcal{C}(p+2,p-2;\mathbb{R}) \cong M_{2^{p-2}}(\mathbb{R}) \otimes \mathcal{C}(4,0;\mathbb{R}) \cong M_{2^{p-1}}(\mathbb{H})$
  - $\mathcal{C}(p+2,p-3;\mathbb{R}) \cong M_{2^{p-\frac{5}{2}}}(\mathbb{R}) \otimes \mathcal{C}(5,0;\mathbb{R}) \cong M_{2^{p-\frac{3}{2}}}(\mathbb{H}) \oplus M_{2^{p-\frac{3}{2}}}(\mathbb{H})$
  - $\mathcal{C}(p+3,p-3;\mathbb{R}) \cong M_{2^{p-3}}(\mathbb{R}) \otimes \mathcal{C}(6,0;\mathbb{R}) \cong M_{2^{p-1}}(\mathbb{H})$
  - $\mathcal{C}(p+3,p-4;\mathbb{R}) \cong M_{2^{p-\frac{7}{2}}}(\mathbb{R}) \otimes \mathcal{C}(7,0;\mathbb{R}) \cong M_{2^{p-\frac{1}{2}}}(\mathbb{C})$
- This structure is closely related to the Bott periodicity of the stable homotopy groups of the orthogonal groups

Groups

Groups Acting on $\mathcal{C}(m, n; \mathbb{R})$

- The Clifford group $\Gamma(n, m)$ consists of all elements $\Lambda \in \mathcal{C}(m, n; \mathbb{R})$ which are invertible and satisfy $\Lambda v \Lambda^{-1} = v'$ for $v, v' \in V$, i.e., for $v, v' \in \mathrm{span}(\gamma_\mu)$
  - For odd dimensions there is also the twisted Clifford group for which $\alpha(\Lambda) v \Lambda^{-1} = v$ where $\alpha(\Lambda) = \pm\Lambda$ depending on the grading of Λ

Pinor Group $\mathrm{Pin}(m,n)$

- The pinor group $\mathrm{Pin}(m,n)$ is the subgroup of the Clifford group where $|\det \Lambda| = 1$
- $\mathrm{Pin}(m,n)$ is the double cover of $O(m,n)$
  - The name is fortuitous, except for Francophones
    - Remember that it is Un Pin to avoid vulgar French slang
    - According to Atiyah and Bott "This joke is due to J-P. Serre"
    - According to Berg *et al* "The joke has been attributed to J.P. Serre but upon being asked, he did not confirm this"
    - I suspect the joke is due to Nicolas Bourbaki

Pinor Group $\mathrm{Pin}(m, n)$

- It includes discrete operations as well as orthogonal transformations
  - In even dimensions $\gamma^* \gamma_\nu$ gives the reflection $(\gamma^* \gamma_\nu)^{-1} \gamma_\mu (\gamma^* \gamma_\nu) = (1 - 2\delta_{\mu\nu}) \gamma_\mu$, which has odd parity
  - In even dimensions $\Lambda = \gamma^*$ gives the inversion $\gamma^{*-1} \gamma_\mu \gamma^* = -\gamma_\mu$
    - This is a rotation, not a reflection
  - In odd dimensions $\gamma^* \propto \mathbb{I}$
    - There is no parity operator in the irrep

Spinor Group $\mathrm{Spin}(m,n)$

- The spinor group $\mathrm{Spin}(m,n)$ is the subgroup of $\mathrm{Pin}(m,n)$ where $\det\Lambda = 1$
- It is the double cover of $\mathrm{SO}(m,n)$
- A (s)pinor is an element of the linear space carrying the representation of $\mathcal{C}(m,n)$
  - Under $\Lambda \in \mathrm{Pin}(m,n)$ it transforms as $\psi \to \Lambda\psi$
  - So the bilinear $\bar{\psi}\gamma_\mu\psi \mapsto \bar{\psi}\Lambda^{-1}\gamma_\mu\Lambda\psi = L_{\mu\nu}\bar{\psi}\gamma_\nu\psi$

Chiral Symmetry

- In even dimensions there is also another continuous group, chiral symmetry $\Lambda = e^{i\alpha\gamma^*}$
  - This may or may not be a symmetry of the action
  - There is no chiral symmetry in odd dimensions

Real Spinors

- We may define a real form of a (s)pinor just as we define the real form of a complex Lie algebra; namely as the eigenvectors of an involutive automorphism
  - That is a morphism $*: \mathfrak{L} \to \mathfrak{L}$ such that $*^2 = \mathbb{I}$
  - For the operation of complex conjugation (also called charge conjugation) the the real (s)pinors are Majorana s(pinors)
  - For the operation of γ^* conjugation the real (s)pinors are Weyl s(pinors)

Real Forms

- Such real forms do not exist in every dimension

d	4	5	6	7	8	9	10	11
Dirac	$\mathbb{C}^4$	$\mathbb{C}^4$	$\mathbb{C}^8$	$\mathbb{C}^8$	$\mathbb{C}^{16}$	$\mathbb{C}^{16}$	$\mathbb{C}^{32}$	$\mathbb{C}^{32}$
Weyl	$\mathbb{C}^2$	/	$\mathbb{C}^4$	/	$\mathbb{C}^8$	/	$\mathbb{C}^{16}$	/
Majorana	$\mathbb{R}^4$	/	/	/	/	/	$\mathbb{R}^{32}$	$\mathbb{R}^{32}$
Both	/	/	/	/	/	/	$\mathbb{R}^{16}$	/

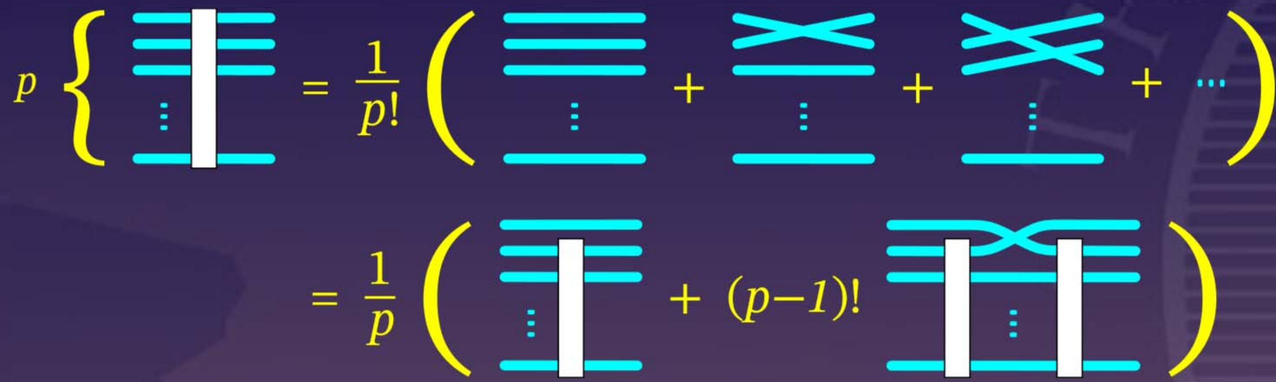
Birdtracks for Spinors

References

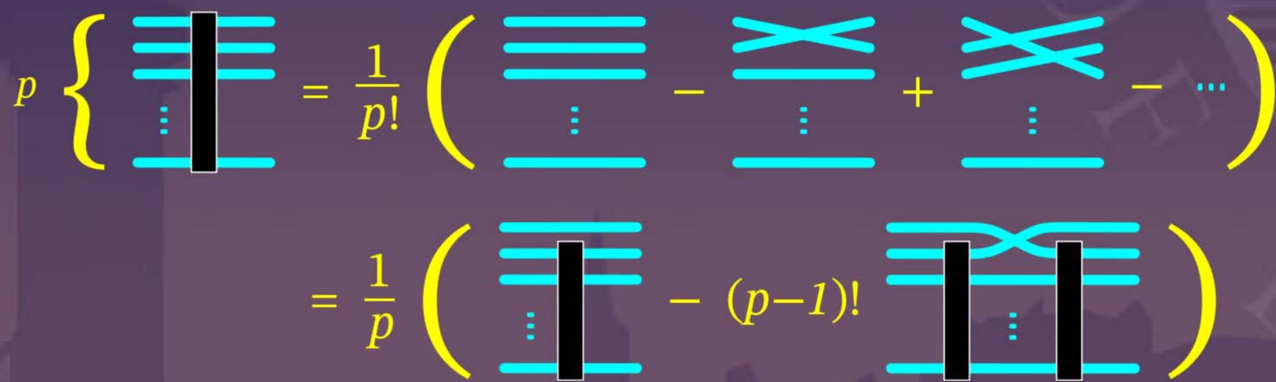
- A. D. Kennedy, *Clifford algebras in 2ω dimensions*, J. Math. Phys, **22**, 1330—1337 (1981)
- Predrag Cvitanović and A. D. Kennedy, *Spinors in Negative Dimensions*, Physica Scripta, **26**, 5—14 (1982)
- Predrag Cvitanović, *Group Theory – Birdtracks, Lie's, and Exceptional Groups*, Princeton University Press (2008)
  - Excellent value: available for the price of two coffees and one croissant at the Frankfurt Airport Hotel or two pints of beer in Svalbard

(Anti)symmetrizers

- Symmetrizer



- Antisymmetrizer



Definitions

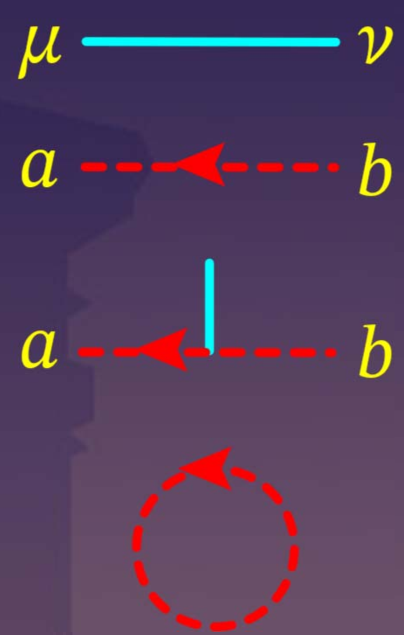
- Definitions

$g^{\mu\nu} =$

$\mathbb{1}_{ab} =$

$(\gamma_\mu)_{ab} =$

$\mathrm{tr}\,\mathbb{1} =$

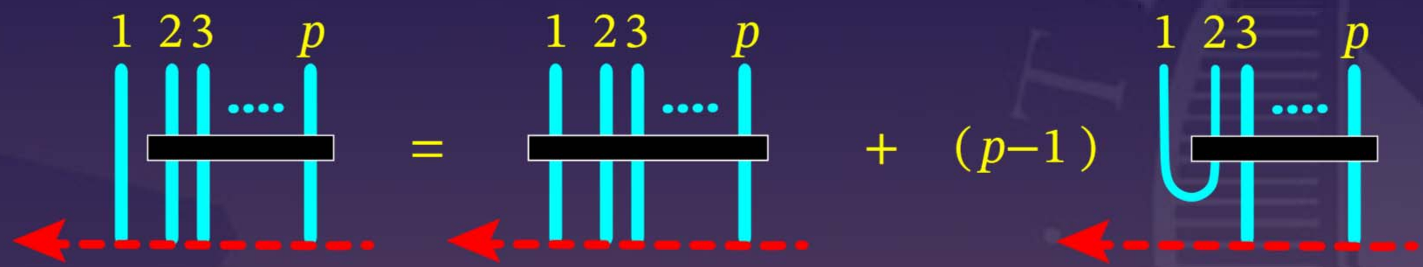


- Clifford Algebra

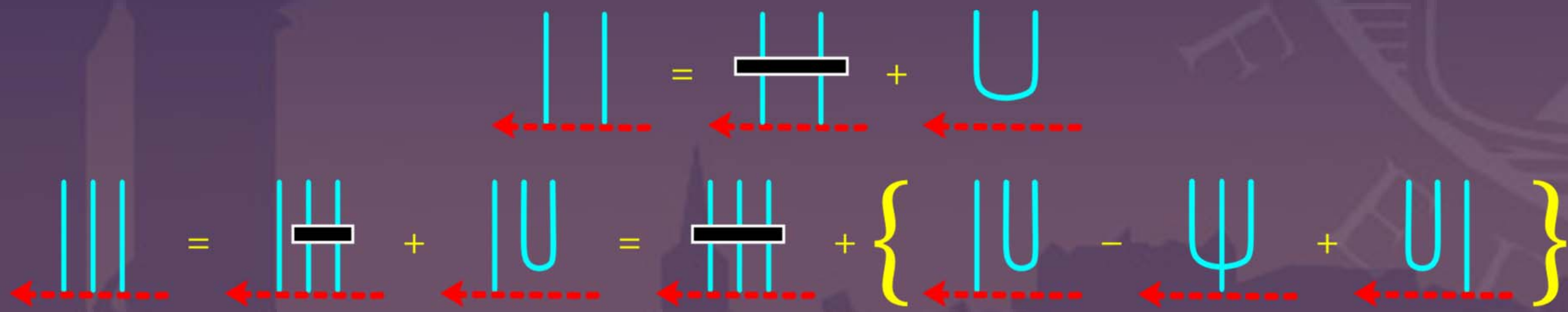
$$\tfrac{1}{2}\{\gamma_\mu,\gamma_\nu\} = g_{\mu\nu}\mathbb{1}$$

Products of γ matrices

- From Clifford algebra



- Examples



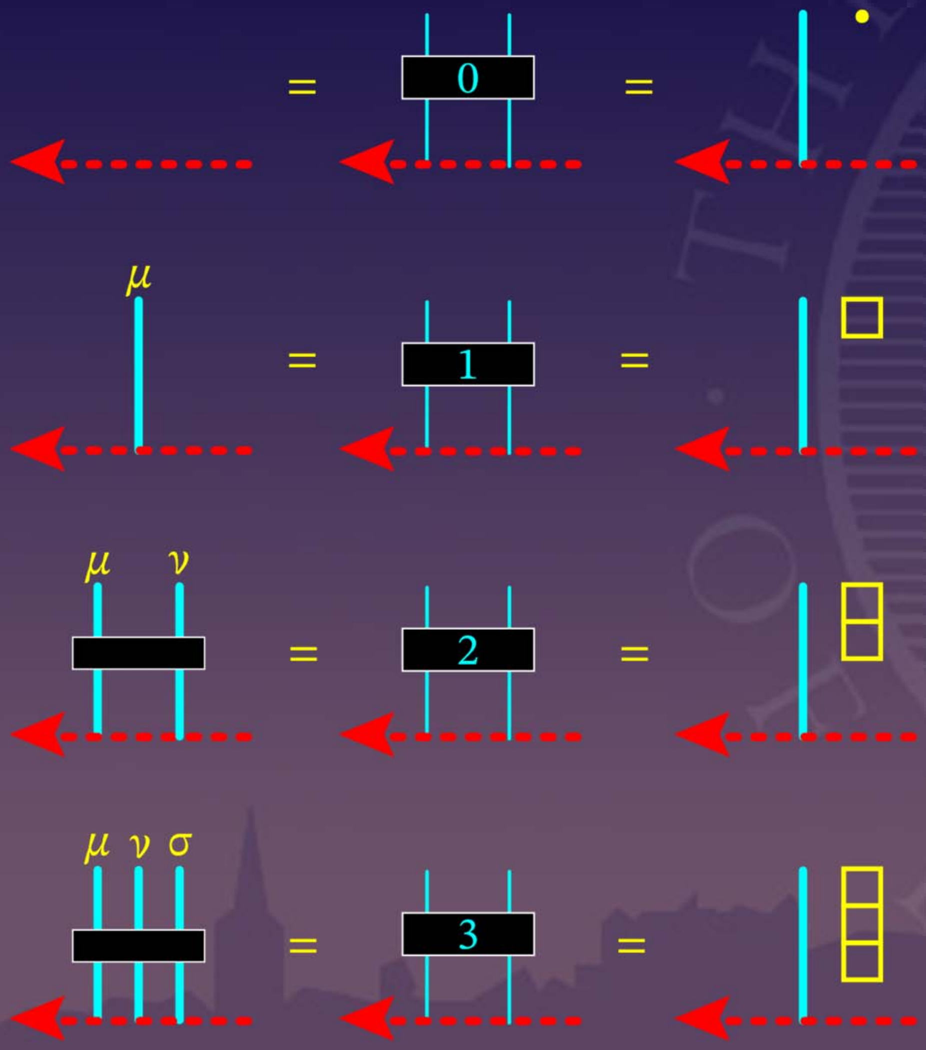
Antisymmetric Basis

$$\Gamma^{(0)} = \mathbb{1} =$$

$$\Gamma^{(1)}_{\mu} = \gamma_{\mu} =$$

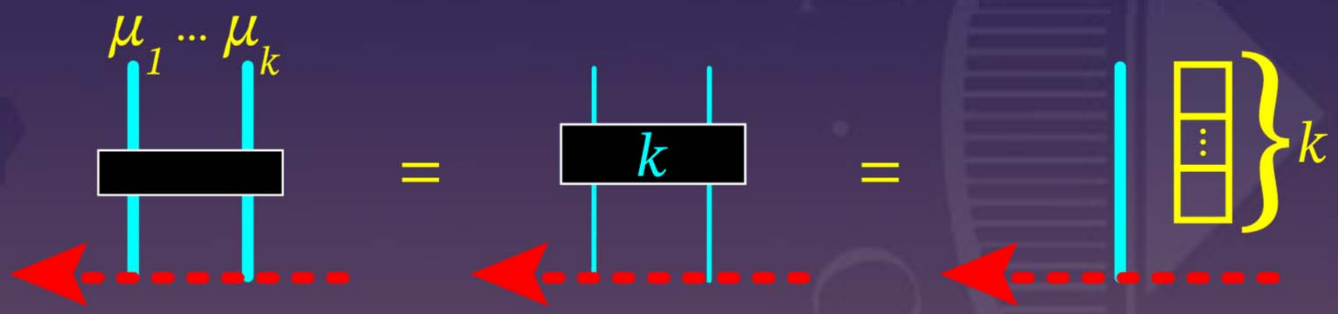
$$\Gamma^{(2)}_{\mu\nu} = \gamma_{[\mu}\gamma_{\nu]} =$$

$$\Gamma^{(3)}_{\mu\nu\sigma} = \gamma_{[\mu}\gamma_{\nu}\gamma_{\sigma]} =$$



Basis Element k

$$\Gamma^{(k)}_{\mu_1\ldots\mu_k} = \gamma_{[\mu_1}\cdots\gamma_{\mu_k]} =$$

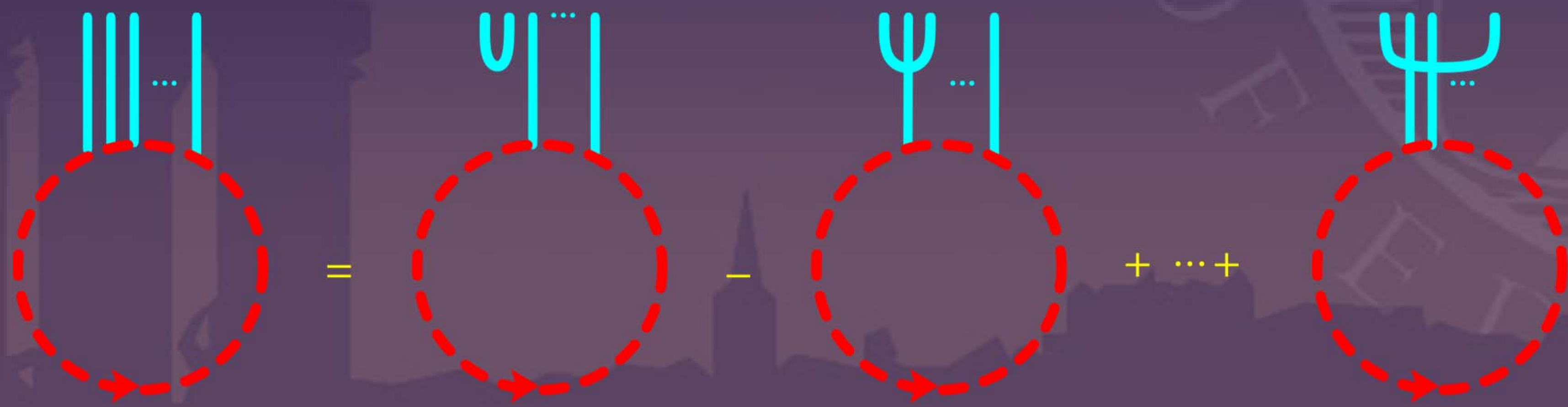


Traces

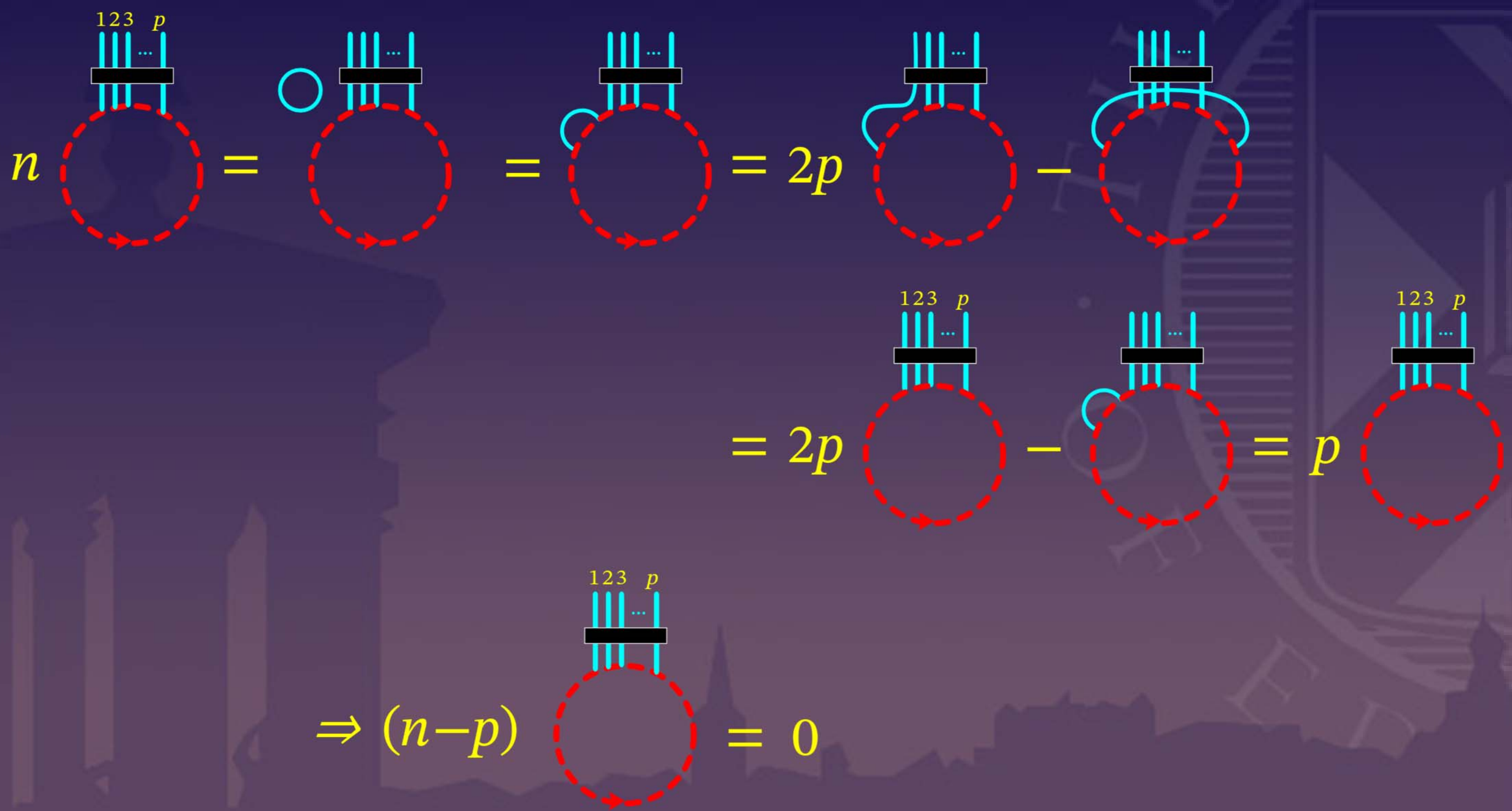
Trace Identity for p even

- Anticommuting the leftmost leg to the right

- Thus, for p even

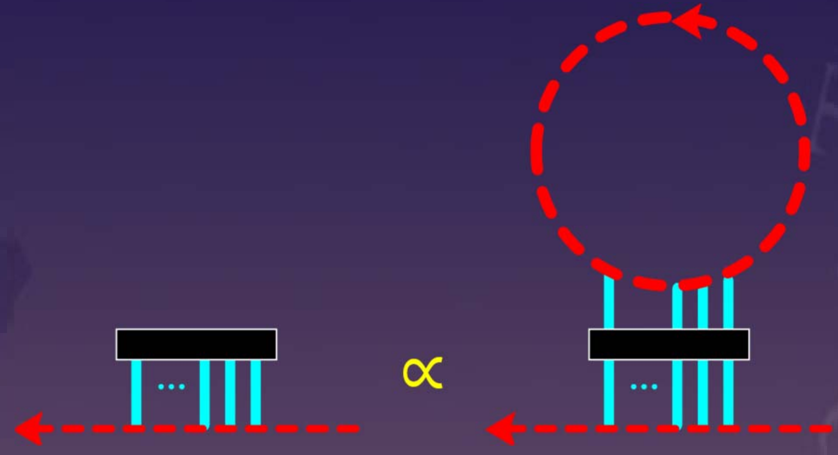


Trace Identity for p odd



Trace identity for p odd

- For n odd and $p = n$ we have

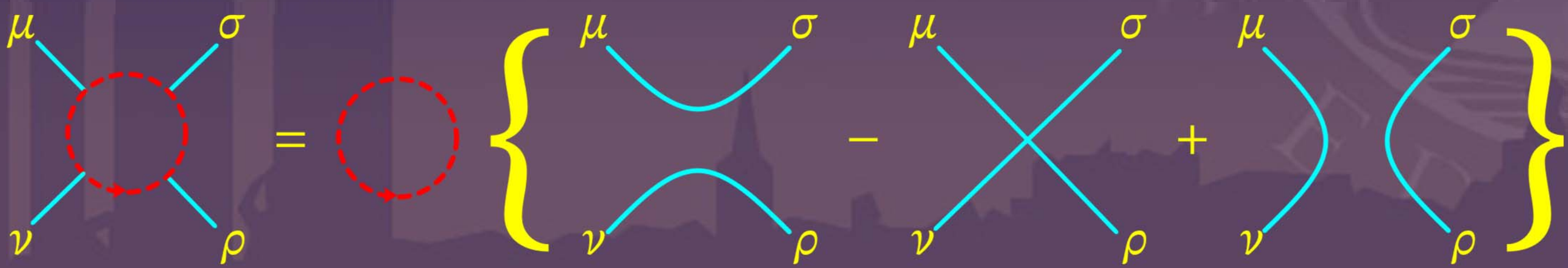


- $\gamma^* = \varepsilon^{\mu_1 \cdots \mu_n} \gamma_{\mu_1} \ldots \gamma_{\mu_n}$ is the generalization of γ_5
- In odd dimensions γ^* commutes with γ_μ
  - by Schur's lemma $\gamma^* \propto \mathbb{I}$ and is not traceless

Trace Reduction

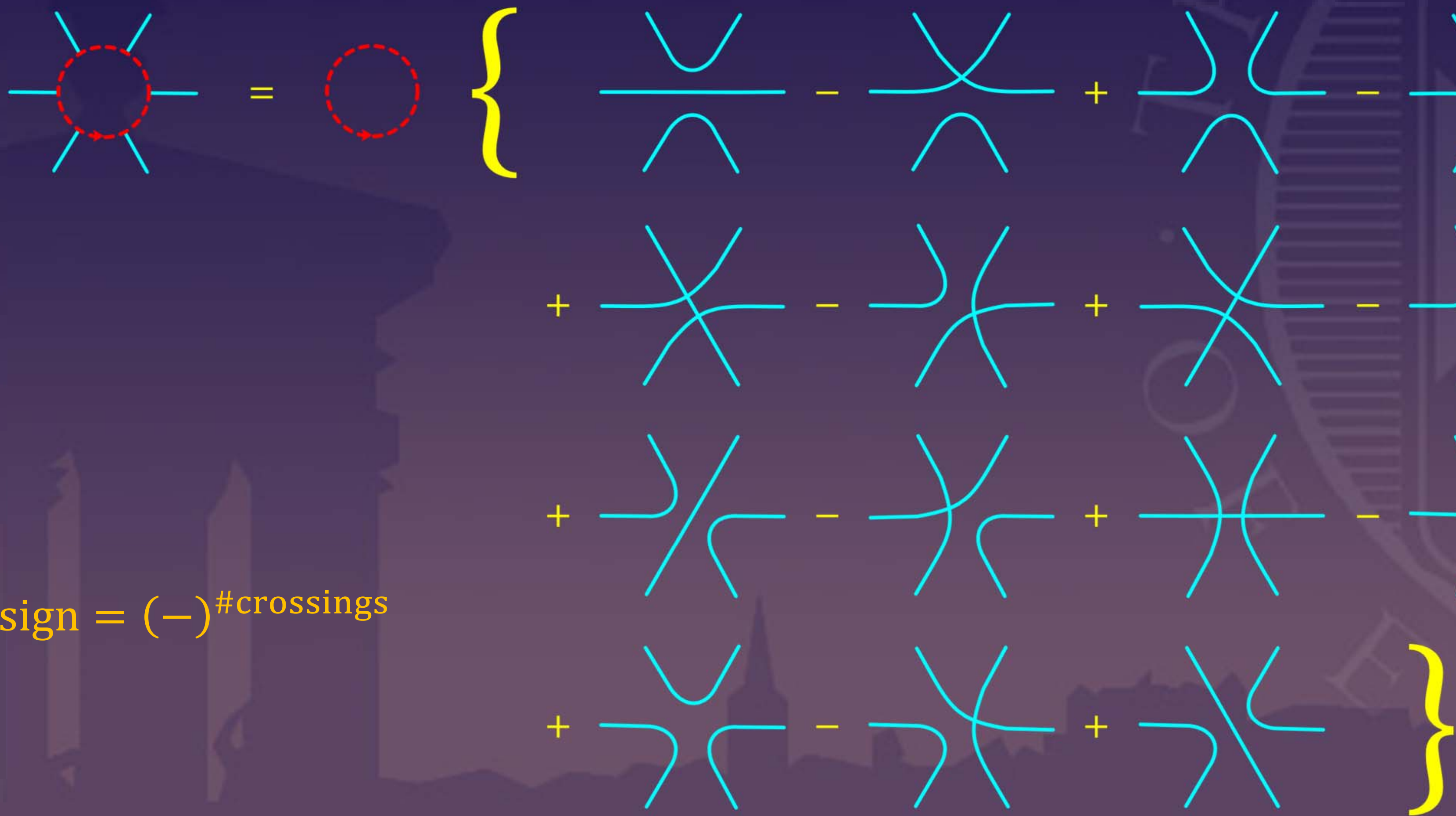
- From the trace identity we see that the trace of an even number of γ matrices is a sum of all $(p-1)!!$ ways of pairing the p legs
  - $p = 2$

  - $p = 4$



Trace Reduction

– $p = 6$



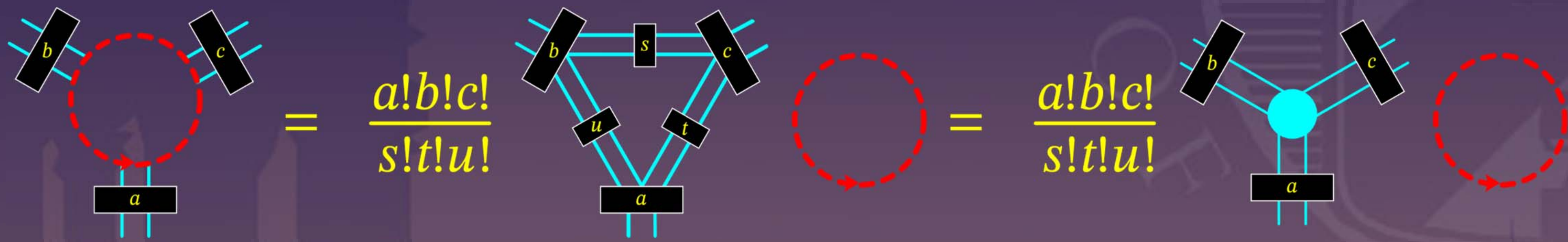
sign $= (-)^{\#\text{crossings}}$

Wignerism

Basis Trace Reduction

- Trace of two basis elements

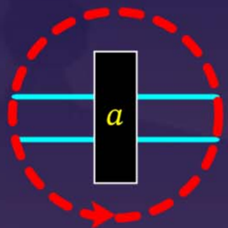
- Trace of three basis elements



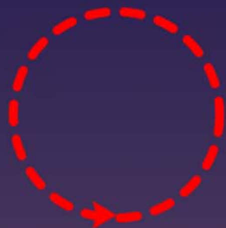
  – Unique connection: $s = \frac{b+c-a}{2}, t = \frac{c+a-b}{2}, u = \frac{a+b-c}{2}$
  – Proportional to $\mathrm{SO}(n)$ vertex for antisymmetric irreps

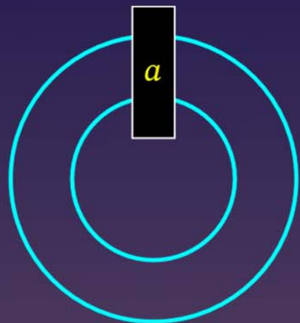
$3j$ Coefficients

- The spinor $3j$ coefficients are thus



$= a!$

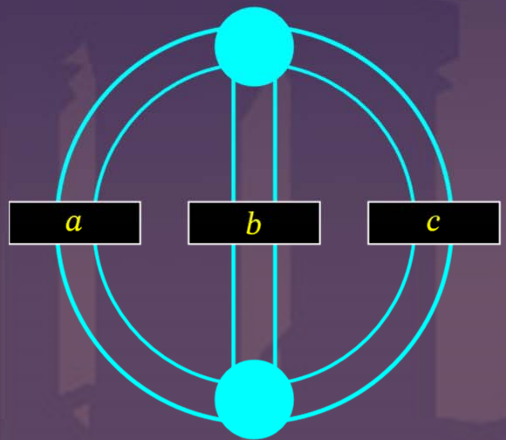




$= a!$

$\binom{n}{a}$

- and the $SO(n)$ antisymmetric $3j$ coefficient are



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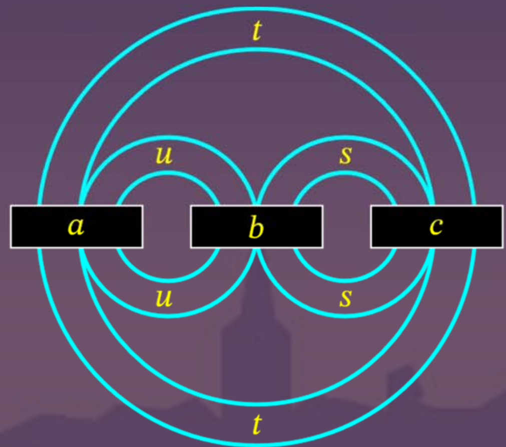
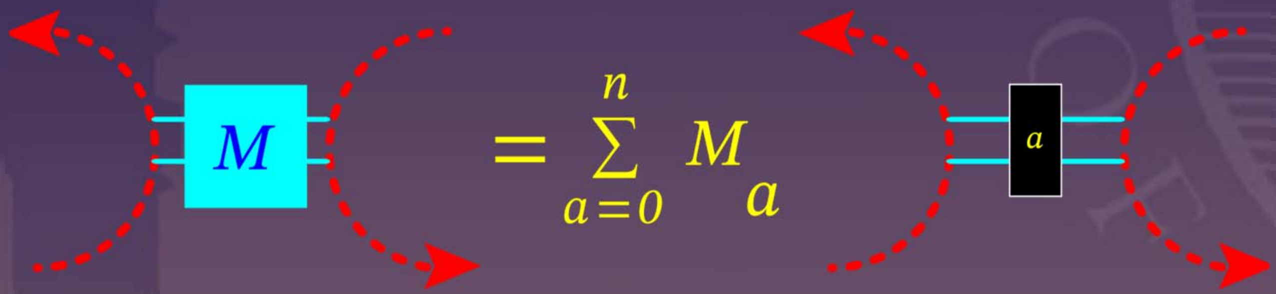


$$= \frac{n!}{s!t!u!(n-s-t-u)!}$$

Completeness Relation

- The basis elements $\Gamma^{(c)}_{\mu_1,\ldots,\mu_c}$ span the Clifford algebra $\mathcal{C}(n)$ over $\mathbb{C}$
  - They have elements have $\binom{n}{c}$ components, so $\dim \mathcal{C}(n) = \sum_{c=0}^{n} \binom{n}{c} = 2^n$
- We therefore have the completeness relation



  - $M \in \mathcal{C}(n) \otimes \mathcal{C}(n)$, but this is complete if $\mathcal{C}(n)$ is

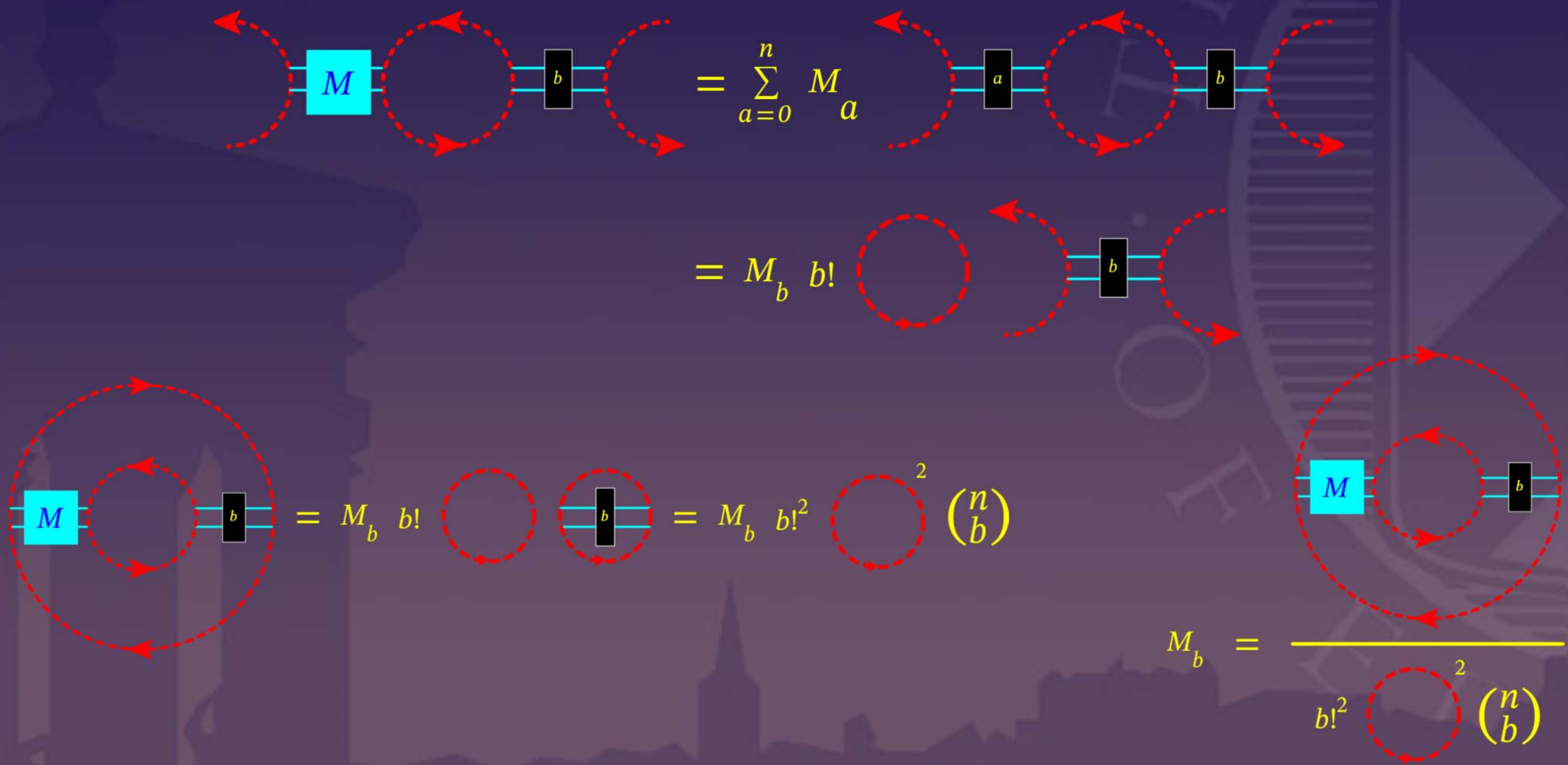
Richard Brauer and Hermann Weyl, *Spinors in n dimensions,* Am. J. Math., **57**, 425–449 (1935)

Completeness Relation

- For even n the Clifford algebra $\mathcal{C}(n)$ over $\mathbb{C}$ is represented faithfully and irreducibly by $2^{n/2} \times 2^{n/2}$ matrices
- For odd n recall that we showed that γ^* lies in the centre $Z = \mathrm{span}(\mathbb{I}, \gamma^*)$
  - So this representation may be reduced onto the two eigenspaces of γ^*
    - It is represented faithfully but not irreducible by $2^{n/2} \times 2^{n/2}$ matrices
    - It is represented irreducibly but not faithfully by $2^{(n-1)/2} \times 2^{(n-1)/2}$ matrices

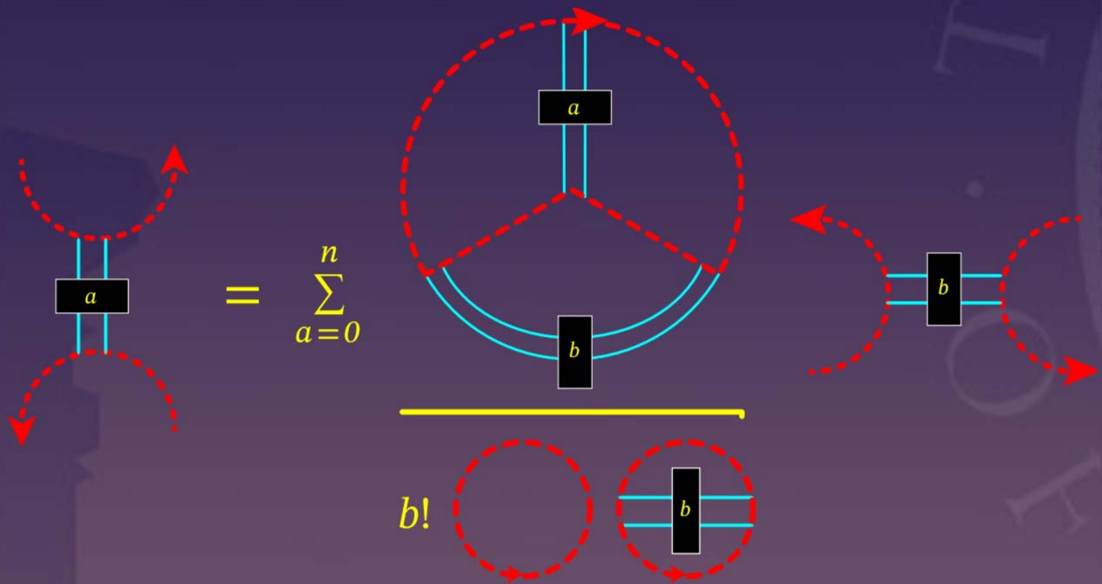
Wigner—Eckart

- We can take traces to compute the coefficients



Spinor $6j$ Coefficients

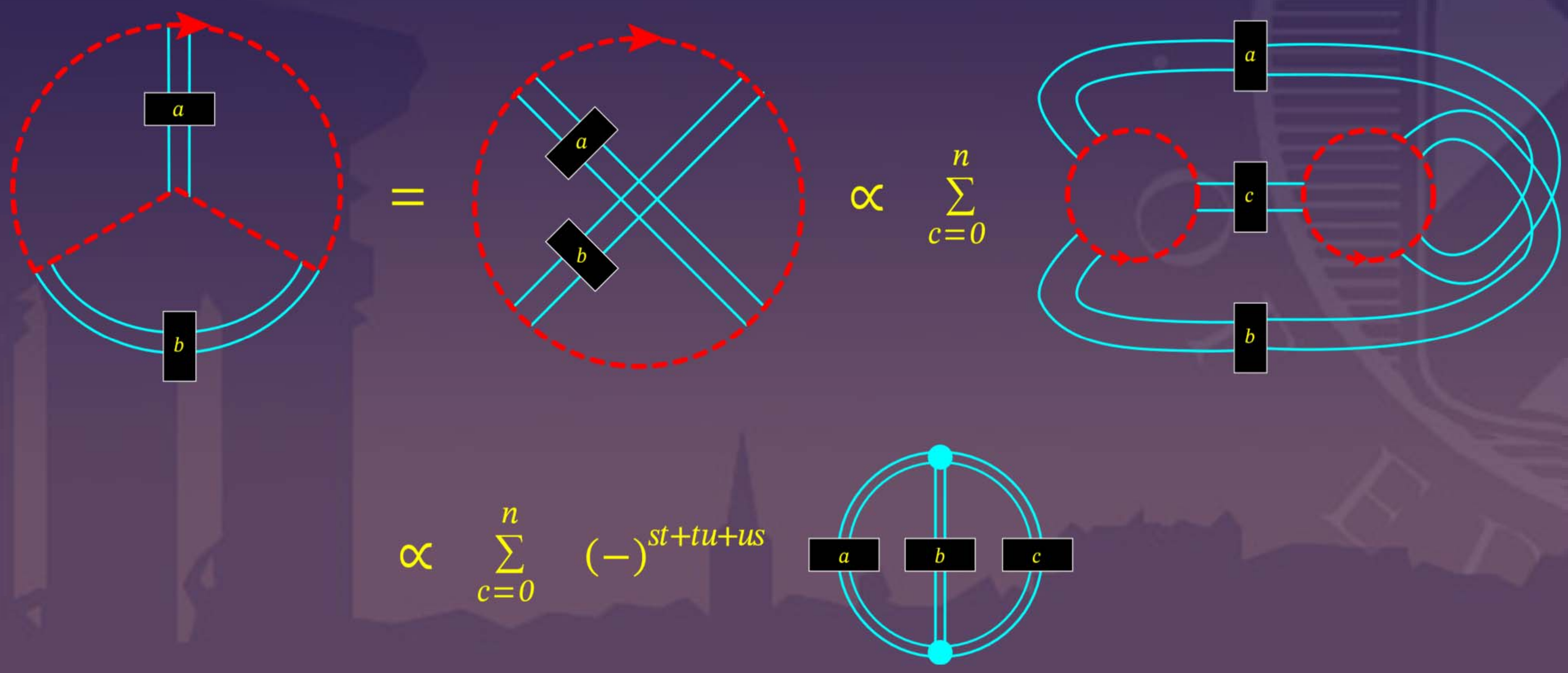
- As usual, we can use the completeness relation to derive a recoupling relation



  - This is the Fierz transformation
  - The $6j$ coefficients are known as Fierz coefficients

Fierz Coefficients Evaluation

- Fierz coefficients may be expressed in terms of $SO(n)$ $3j$ coefficients
  - Using completeness in the second step



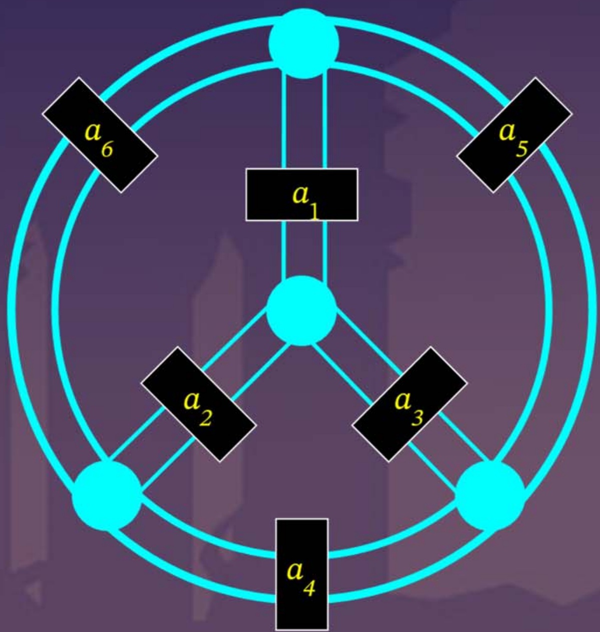
γ Simplificiation

γ Matrix Simplification

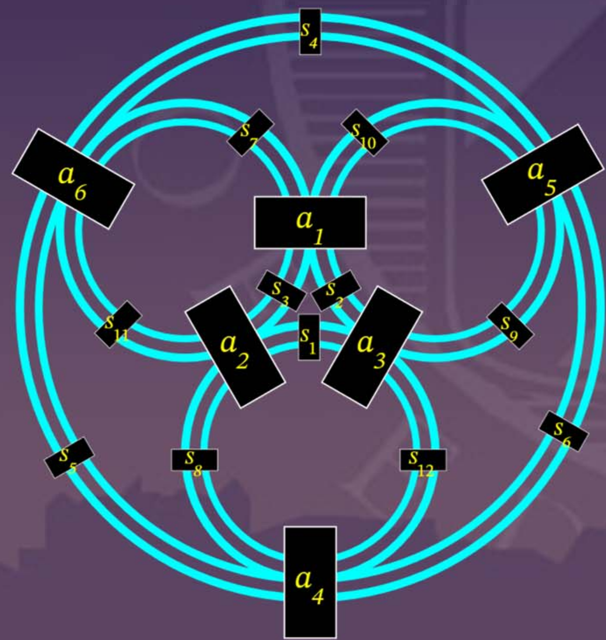
- We may use the recoupling relation to simplify γ matrix expressions
  - The results can be expressed as sums of products of $0j$, $3j$, and $6j$ coefficients
  - As we have seen, the spinorial $6j$ (Fierz) coefficients can be expressed in terms of known $SO(n)$ $3j$ coefficients
  - This is much more efficient than using brute-force trace reduction
- The calculations can be done in $n = 2(\omega + \varepsilon)$ dimensions
  - This is necessary in dimensional regularization as the $\mathcal{O}(\varepsilon)$ terms lead to finite contributions when multiplied by pole terms

$SO(n)$ $6j$ Coefficients

- For sufficiently complicated graphs more complicated $SO(n)$ $6j$ coefficients arise
  - All purely antisymmetric irrep $6j$ are known in closed form

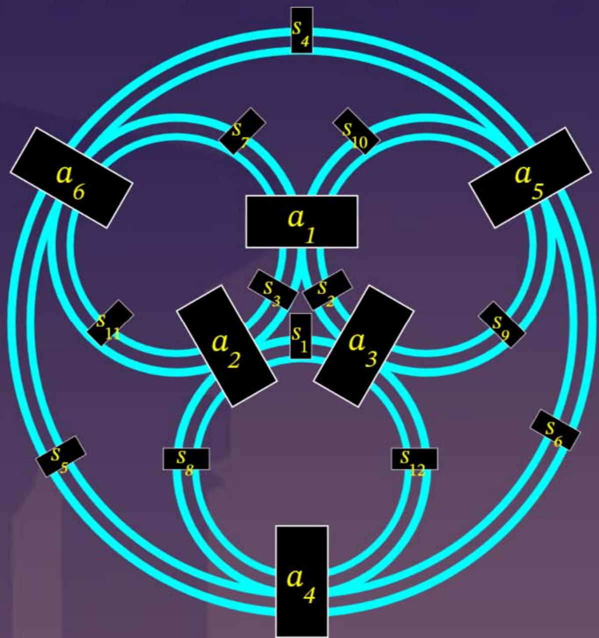


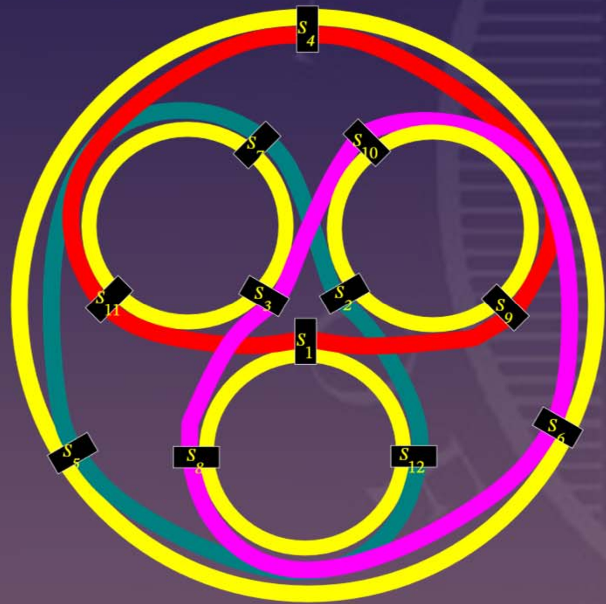
$$= \frac{\prod_{i=1}^{6} a_i!}{\prod_{j=1}^{12} s_j!}$$



$SO(n)$ $6j$ Coefficients

- This can be evaluated because it has four "mini tours" and three "grand tours"





$$\binom{n}{t}\frac{t!}{\prod_{i=1}^{7} t_i!}\frac{\prod_{i=1}^{12} s_i!}{\prod_{j=1}^{6} a_j!} \quad \text{with} \quad t = \sum_{i=1}^{7} t_i$$

Conclusions, Outstanding Problems, and Future Work

Unsolved Problems

- Even more complicated graphs can require $SO(n)$ $6j$ coefficients involving irreps labelled by Young diagrams with more than one column
  - While these may be evaluated by "brute force" it would be nice to have a more efficient algorithm
  - Perhaps making use of representations of Brauer algebras?

Unsolved Problems

- In general, all irreps other than spinor ones can be projected from tensor powers $V^{\otimes k}$ of the n-dimensional defining matrix irrep V
- These are still uniquely labelled by Young diagrams
  - Some Young diagrams do not correspond to irreps
  - E.g., those whose first two columns are longer than n
- Traceless Young projectors are necessary