Lepton masses and mixing in modular groups

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The Flavor Problem

- Hierarchy problem

$$m_{f_3} \gg m_{f_2} \gg m_{f_1}$$

f stands for all elementary fermion fields.

- Tiny neutrino mass

$m_\nu <<< m_l, m_q$, l=e,μ,τ, q = u, c, t, d, s, b.

$m_\nu \sim < 0.5$ eV, $m_l \geq 0.511$ MeV, mq ~ > 2 MeV

- The lepton mixing is large while the quark mixing is small.
- The patterns of neutrino mixing of two large and one small angles.

Flavor symmetry

- Discrete symmetry

1. Usual discrete symmetry models
2. Modular discrete symmetry models

   a. Holomorphic modular groups (supersymmetric models)

   b. Non-holomorphic modular groups (non-supersymmetric models) [1]

[1] B.-Y. Qu and G.-J. Ding, "Non-holomorphic modular flavor symmetry," JHEP 08 (2024)

Modular invariance

- In the models based on modular flavor invariance, one can consider a scalar field (the modular τ) that can break the modular symmetry via its vev.
- The Yukawa couplings (modular forms) are charged under modular group.
- The modular invariance can be achieved by modular weight (k), hence there is no need to impose z_i symmetries

Modular group

- The modular group $\bar{\Gamma}$ is defined as linear fractional transformations on the upper half of the complex plan ***H*** and has the form

$$\gamma : \tau \rightarrow \gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in Z, \quad ad - bc = 1,$$

Where τ is a complex number belongs to ***H.***

- The modular group $\bar{\Gamma}$ is isomorphic to the projective special linear group

$$PSL(2, Z) = SL(2, Z)/\{I, -I\},$$

where

$$SL(2, Z) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in Z, ad - bc = 1 \right\}.$$

Modular group

- The group $\bar{\Gamma}$ has two generators S and T satisfying

$$S^2 = (ST)^3 = I\,,$$

where their action on the complex number τ is given by

$$S\colon\tau \to \frac{-1}{\tau}\,, \qquad T\colon\tau \to \tau + 1\,.$$

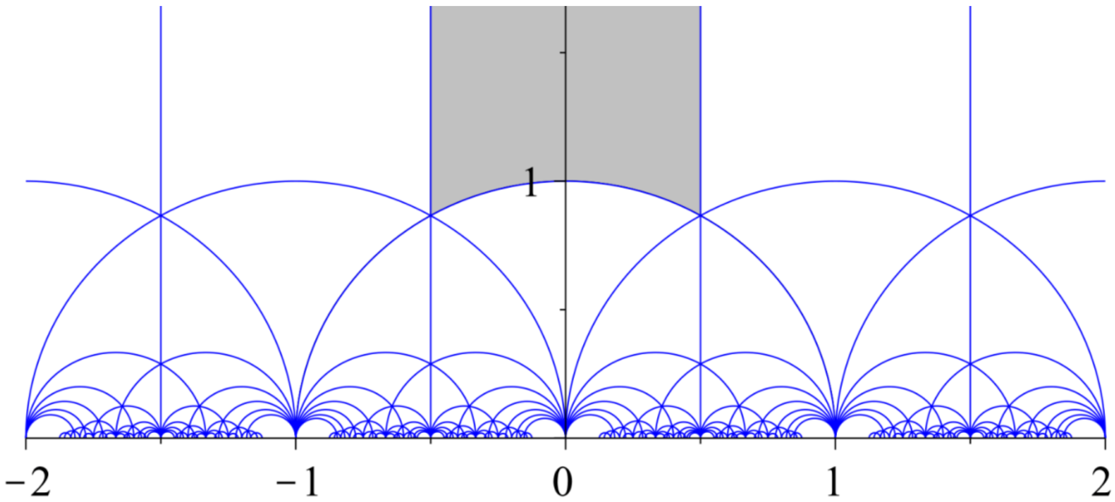
- S and T can be represented as

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\,.$$

The Fundamental Domain

- The fundamental domain is defined as

$$\mathcal{D} = \left\{ \tau \in \mathcal{H} \middle| -\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2},\ |\tau| \geq 1 \right\}.$$



Fundamental domain. (2024, March 20). In *Wikipedia*. https://en.wikipedia.org/wiki/Fundamental_domain

Finite modular groups

- Define the infinite modular groups Γ(N), N = 1, 2, 3, …….. As

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} mod\ N \right\}.$$

For $N = 1$,

$$\Gamma(1) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z), \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} mod\ 1 \right\}.$$

$$\Gamma(1) \equiv SL(2,Z).$$

- For N = 1, 2, we define

$$\bar{\Gamma}(N) = \Gamma(N)/\{I, -I\}$$

- For N>2,

$$\bar{\Gamma}(N) = \Gamma(N)$$

Finite modular groups

- It is straightforward to notice that

$$\bar{\Gamma}(1) = PSL(2, Z) = \bar{\Gamma}.$$

- The group $\bar{\Gamma}$ and its subgroup $\bar{\Gamma}(N)$ are discrete but infinite, while the quotient modular group

$$\Gamma_N = \bar{\Gamma} / \bar{\Gamma}(N)$$

  is finite.

- The group Γ_N is called **<u>the finite modular group</u>** and can be obtained by extending the conditions on the generators with the condition $T^N = 1$.
- For some N<5, the finite modular group Γ_N is isomorphic to a permutation group, for instance,

$$\Gamma_2 \cong S_3, \Gamma_3 \cong A_4, \Gamma_4 \cong S_4 \text{ and, } \Gamma_5 \cong A_5.$$

Modular function and modular forms

- The modular function f(τ) of weight 2k is a meromorphic function of the complex variable τ which satisfies

$$f(\gamma(\tau)) = f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{2k} f(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N),$$

  k is an integer, $k \geq 0$.

- If the modular function is holomorphic everywhere, it is called “modular form” of weight 2k.
- The modular forms of level N and weight 2k form a linear space of finite dimension. In the basis at which the transformation of a set of modular forms $f_i(\tau)$ is described by a unitary representation ρ(γ), one can get

$$f_i(\gamma(\tau)) = (c\tau + d)^{2k} \rho_{ij}(\gamma) f_j(\tau), \qquad \gamma \in \Gamma(N).$$

Supermultiplet transformation

- The supermultiplets ϕ^I transform under Γ_N in the representation $\rho(\gamma)$ as

$$\phi^{(I)}(\tau) \to \phi^{(I)}(\gamma(\tau)) = (c\tau + d)^{-2k_I} \rho^{(I)}(\gamma)\phi^{(I)}(\tau).$$

  where:
  k is the modular weight,
  $\rho(\gamma)$ is a unitary representation of Γ_N ,
  I refers to different sectors in the theory.

- Consider the superpotential W(τ, ϕ)

$$W(\tau, \phi) = \sum_I \sum_n Y_{I_1 I_2 \dots I_n}(\tau)\phi^{I_1} \dots \phi^{I_n}.$$

- The invariance of the superpotential W(τ, ϕ) under the modular transformation requires $Y_{I_1 I_1 \dots I_n}(\tau)$ to be a modular form transforming in the representation

$$Y_{I_1\ I_2\ \dots I_n}(\gamma\tau) = (c\tau + d)^{2k_Y} \rho(\gamma) Y_{I_1\ I_2\ \dots I_n}(\tau).$$

Modular invariance condition

- The modular invariance forces the condition

$$k_Y = k_{I_1} + k_{I_2} + ... + k_{I_n}.$$

Modular forms of level 3

- The group A_4 has one triplet representation ***3*** and three singlets 1, 1', and 1" and is generated by two elements S and T satisfying the conditions

$$S^2 = T^3 = (ST)^3 = \mathbf{1}.$$

- The modular form of level 3 has the form

$$f_i(\gamma(\tau)) = (c\tau + d)^{2k} \rho_{ij}(\gamma) f_j(\tau), \qquad \gamma \in \Gamma(3).$$

- The modular form of Γ_3 has the dimension = 2k+1, so for the lowest value of k=1, the modular form of Γ_3 is transformed as a triplet of weight 2.

Modular forms of level 3

- The modular form of weight 2 and level 3 transforms as a triplet and is given by $Y_3^{(2)} = (y_1, y_2, y_3)$ where

$$
\begin{aligned}
y_1(\tau) &= \frac{i}{2\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right], \\
y_2(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right], \\
y_3(\tau) &= \frac{-i}{\pi}\left[\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right].
\end{aligned}
$$

- where $\omega = e^{2\pi i/3}$ and the Dedekind eta-function $\eta(z)$ is defined as

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q = e^{2\pi i\tau}.$$

Higher weight modular forms

- Modular forms of weight 4 are constructed via multiplication of two triplets of weight 2. Using A4 multiplication rules of two triplets, one can get one triplet and three singlets all of weight 4 as

$$Y_3^{(4)} = \begin{pmatrix} y_1^2 - y_2\ y_3 \\ y_3^2 - y_2\ y_1 \\ y_2^2 - y_1\ y_3 \end{pmatrix}, \ Y_1^{(4)} = y_1^2 + 2y_2\ y_3,\ Y_2^{(4)} = y_3^2 + 2y_2\ y_1,\ Y_3^{(4)} = y_2^2 + 2y_1\ y_3.$$

- The representations of the above singlets are

$$Y_1^{(4)} \sim 1, \qquad Y_2^{(4)} \sim 1', \qquad Y_3^{(4)} \sim 1''.$$

- At all values of , the condition $Y_3^{(4)} = 0$ is satisfied.

Residual Symmetries of A4

- There are three independent fixed points,

$$\tau_1 = e^{\frac{2\pi i}{3}} = -\frac{1}{2} + \frac{\sqrt{2}}{3}$$
$$\tau_2 = i$$
$$\tau_3 = i\infty$$

- τ_1 is invariant under ST transformation

$$\tau_1 \xrightarrow{\text{ST}} \tau_1$$

$$A_4 \xrightarrow{\text{At } \tau_1} Z_3 = \{I, ST, (ST)^2\}$$

Residual Symmetries of A4 (cont.)

- τ_2 is invariant under S transformation

$$\tau_2 \xrightarrow{\text{S}} \tau_2$$

$$A_4 \xrightarrow{\text{At } \tau_2} Z_2 = \{I, S\}$$

- τ_3 is invariant under T transformation

$$\tau_3 \xrightarrow{\text{T}} \tau_3$$

$$A_4 \xrightarrow{\text{At } \tau_3} Z_3 = \{I, T, T^2\}$$

A_4 Modular invariance model

- The lepton content in the model is extended by adding a triplet of chiral supermutiplets N^c as a right-handed neutrino and three SM singlets S_i to get the neutrino masses via the inverse seesaw mechanism.
- A gauge singlet scalar χ transforming trivially under A_4 is added to get the masses of the singlet fermions N^c and S.
- The modular weights are chosen such that the following relations are satisfied:

$$
\begin{aligned}
k_L + k_{H_d} + k_E &= 2, \\
k_L + k_{H_u} + k_N &= 2, \\
2k_S + 4k_\chi &= 0, \\
k_S + k_N + k_\chi &= 0.
\end{aligned}
$$

fields	L	E_1^c	E_2^c	E_3^c	N^c	S	H_d	H_u	χ
A_4	3	1	$1''$	$1'$	3	3	1	1	1
k_I	3	-1	-1	-1	-1	2	0	0	-1

Assignment of flavors under A_4 and the modular weight k_I

The lepton modular A_4 invariant superpotential can be written as

$$\begin{aligned} w_l = & \lambda_1 E_1^c H_d (L \otimes Y_3^{(2)})_1 + \lambda_2 E_2^c H_d (L \otimes Y_3^{(2)})'_1 \\ & + \lambda_3 E_3^c H_d (L \otimes Y_3^{(2)})''_1 + g_1((N^c H_u L)_{3S} Y_3^{(2)})_1 \\ & + g_2((N^c H_u L)_{3A} Y_3^{(2)})_1 + h(N^c \otimes S)_1 \chi \\ & + \frac{f}{\Lambda^3} (S \otimes S)_1 \chi^4, \end{aligned}$$

A_4 Modular invariance model (cont.)

- where
  Λ: is the nonrenormalizable scale,
  g_1: is the coupling constant of the term of the symmetric triplet arising from the product of the two triplets L and Y,
  g_2: is the coupling of the antisymmetric triplet term.
- The fields H_u, H_d and χ acquire vevs namely v_u, v_d and v' respectively, where $v' > v_u, v_d$.
- assume that v' satisfies the relation

$$\frac{v'}{\Lambda} \sim O(\lambda_C)$$

where $\lambda_c = 0.225$ is the Cabibbo angle.

A_4 Modular invariance model (cont.)

- The charged lepton mass matrix is

$$m_e = v_d \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \times \begin{pmatrix} y_1 & y_3 & y_2 \\ y_2 & y_1 & y_3 \\ y_3 & y_2 & y_1 \end{pmatrix}.$$

- It is convenient to use the Hermitian matrix $M_e = m_e^\dagger m_e$ which can be diagonalized as

$$M_e^{\text{diag}} = U_e^\dagger M_e U_e.$$

A_4 Modular invariance model (cont.)

- The neutrino mass matrices are

$$\mu_s = f v' \lambda_c^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad M_R = h v' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$m_D = v_u \begin{pmatrix} 2g_1 y_1 & (-g_1 + g_2) y_3 & (-g_1 - g_2) y_2 \\ (-g_1 - g_2) y_3 & 2g_1 y_2 & (-g_1 + g_2) y_1 \\ (-g_1 + g_2) y_2 & (-g_1 - g_2) y_1 & 2g_1 y_3 \end{pmatrix}.$$

- The neutrino mass matrix in the basis (ν_L, N^c, S) is given by

$$M = \begin{pmatrix} 0 & m_D & 0 \\ m_D^T & 0 & M_R \\ 0 & M_R^T & \mu_s \end{pmatrix}.$$

A_4 Modular invariance model (cont.)

- The masses of the light neutrino state is

$$m_\nu = m_D M_R{}^{-1} \mu_s M_R^{T-1} m_D^T.$$

- It is convenient to diagonalize the Hermitian matrix $M_\nu = m_\nu^\dagger m_\nu$,

$$M_\nu^{\text{diag}} = U_\nu^\dagger M_\nu U_\nu.$$

The lepton mixing U_{PMNS} matrix is given by

$$U_{PMNS} = U_e^\dagger U_\nu.$$

The mixing angles can be calculated from the relations

$$Sin^2(\theta_{13}) = |(U_{PMNS})_{13}|^2, \quad Sin^2(\theta_{12}) = \frac{|(U_{PMNS})_{12}|^2}{1 - |(U_{PMNS})_{13}|^2}, \quad Sin^2(\theta_{23}) = \frac{|(U_{PMNS})_{23}|^2}{1 - |(U_{PMNS})_{13}|^2}$$

A_4 Modular invariance model (cont.)

- The parameter g_2/g_1 is complex in general, so we can write it as
$$\frac{g_2}{g_1} = g e^{i\phi},$$

where ϕ is the relative phase of g_1 and g_2.

	$\frac{\Delta m_{12}^2}{(10^{-5}\ \mathrm{eV}^2)}$	$\frac{\lvert\Delta m_{23}^2\rvert}{(10^{-3}\ \mathrm{eV}^2)}$	$r = \frac{\Delta m_{12}^2}{\lvert\Delta m_{23}^2\rvert}$	$\theta_{12}/^\circ$	$\theta_{23}/^\circ$	$\theta_{13}/^\circ$	δ_{CP}/π
Best fit	7.39	2.51	0.0294	33.82	49.8	8.6	1.57
3σ range	6.79–8.01	2.41–2.611	0.026–0.033	31.61–36.27	40.6–52.5	8.27–9.03	1.088–2

where $$\Delta m_{12}^2 = m_2^2 - m_1^2, \qquad \lvert\Delta m_{23}^2\rvert = \lvert m_3^2 - (m_2^2 + m_1^2)/2\rvert.$$

Results

- The parameters are scanned and one can get the following benchmarks
- For the normal hierarchy,

$$\tau = -0.245 + 0.5236i, \quad g = 2.503, \quad \phi = -0.105\pi, \quad \frac{\lambda_1}{\lambda_3} = 0.00031, \quad \frac{\lambda_2}{\lambda_1} = 0.063, \text{ with}$$

$$\begin{aligned} r &= \frac{\Delta m_{12}^2}{|\Delta m_{23}^2|} = 0.0293, \quad \frac{m_e}{m_\tau} = 0.0003, \quad \frac{m_\mu}{m_\tau} = 0.061, \\ \theta_{12} &= 33.25^\circ, \quad \theta_{23} = 41.678^\circ, \quad \theta_{13} = 8.73^\circ. \end{aligned}$$

Results (cont.)

- For inverted neutrino mass hierarchy,

1. $\tau = -0.494 + 0.55i, \quad g = 2.05, \quad \phi = -\pi/2, \quad \frac{\lambda_1}{\lambda_3} = 0.0009, \quad \frac{\lambda_2}{\lambda_1} = 0.07,$ with

$$r = \frac{\Delta m_{12}^2}{|\Delta m_{23}^2|} = 0.0286, \qquad \frac{m_e}{m_\tau} = 0.0003, \qquad \frac{m_\mu}{m_\tau} = 0.061,$$
$$\theta_{12} = 32.4^\circ, \quad \theta_{23} = 49.26^\circ, \quad \theta_{13} = 8.54^\circ.$$

2. $\tau = 0.0962 + 0.984i, \quad g = 2.05, \phi = -\pi/2, \quad \frac{\lambda_1}{\lambda_3} = 0.0009, \quad \frac{\lambda_2}{\lambda_1} = 0.07,$ with

$$r = \frac{\Delta m_{12}^2}{|\Delta m_{23}^2|} = 0.0296, \qquad \frac{m_e}{m_\tau} = 0.0003, \qquad \frac{m_\mu}{m_\tau} = 0.061,$$
$$\theta_{12} = 32.36^\circ, \quad \theta_{23} = 49.24^\circ, \quad \theta_{13} = 8.73^\circ.$$

Results (cont.)

- The two points $\langle\tau\rangle = 0.494 + 0.55i$ and $\langle\tau\rangle = 0.0962 + 0.984i$ are close to the fixed points $\tau_2' = 0.5 + 0.5i$ and $\tau_2 = i$ respectively.
- The two fixed points are related to each other

$$\tau_2' = \mathrm{ST}\,\tau_2$$

$$\tau_2' \xrightarrow{\mathrm{ST^2ST}} \tau_2'$$

$$A_4 \xrightarrow{\text{At } \tau_2'} Z_2 = \{I, ST^2ST\}$$

Lepton masses at fixed points

- At $\tau_2 = \mathrm{i}$,

$$Det(M_e) = 0$$

So τ_2 can not be used to lead to the correct lepton masses and mixing.

- At $\tau_2' = 0.5 + 0.5\,\mathrm{i}$,

1. One of the eigenvalues of M_e is zero
2. One of the eigenvalues of M_ν is zero and $\mathrm{Det}(M_\nu) = 0$. The mixing matrix in this case has two vanishing mixing angles and a nearly maximal angle.

- The observed lepton masses and mixing are consequences of breaking modular residual symmetry by deviation from τ_2 and τ_2'

Quark sector

fields	Q_1	Q_2	Q_3	u_1^c	u_2^c	u_3^c	d_1^c	d_2^c	d_3^c
A_4	1	$1'$	$1''$	$1''$	1	$1'$	1	$1''$	$1'$
k_I	3	2	0	4	4	1	0	-1	4

- For <τ>= 0.494 + 0.55i, the A_4 invariant superpotential for down quarks can be written as

$$w_d = \frac{h_{11}^d}{\Lambda^3} d_1^c H_d Q_1 \chi^3 + \frac{h_{22}^d}{\Lambda} d_2^c H_d Q_2 \chi + \frac{h_{23}^d}{\Lambda^2} Y_2^{(4)} d_3^c H_d Q_2 \chi^2 + h_{33}^d Y_1^{(4)} d_3^c H_d Q_3 .$$

- If $h_{11}^d / h_{33}^d \sim h_{22}^d / h_{33}^d \sim 1/2$, and $h_{23}^d / h_{33}^d \sim 1$

- The down quark mass matrix is $m_d = h_{33}^d \ < H_d > \begin{pmatrix} \lambda^3/2 & 0 & 0 \\ 0 & \lambda/2 & 0 \\ 0 & Y_2^{(4)} \lambda^2 & Y_1^{(4)} \end{pmatrix}.$

Quark sector (cont.)

- The Hermitian matrix $M_d = m_d^\dagger m_d$ can be diagonalized by

$$V_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.490 - 0.869i & 0.0257 + 0.045i \\ 0 & 0.0524 & 0.9985 \end{pmatrix},$$

with eigenvalues $M_d = diag(\lambda^4/2, \lambda^2/2, 1)\ h_{33}^d\ Y_1^{(4)}\ < H_d >$.

Which is in a good agreement with the experimental data.

- The invariant superpotential for up quarks is

$$\begin{aligned} w_u &= \frac{h_{11}^u}{\Lambda^3} Y_2^{(4)} u_1^c H_u Q_1 \chi^3 + \frac{h_{12}^u}{\Lambda^2} Y_1^{(4)} u_1^c H_u Q_2 \chi^2 + \frac{h_{21}^u}{\Lambda^3} Y_1^{(4)} u_2^c H_u Q_1 \chi^3 \\ &+ h_{23}^u Y_2^{(4)} u_2^c H_u Q_3 + \frac{h_{33}^u}{\Lambda} u_3^c H_u Q_3 \chi. \end{aligned}$$

Quark sector (cont.)

- The up quark mass matrix is

$$m_u = < H_u > \begin{pmatrix} h_{11}^u Y_2^{(4)} \lambda^3 & h_{12}^u Y_1^{(4)} \lambda^2 & 0 \\ h_{21}^u Y_1^{(4)} \lambda^3 & 0 & h_{23}^u Y_2^{(4)} \\ 0 & 0 & h_{33}^u \lambda \end{pmatrix}.$$

The Hermitian matrix $M_u = m_u^\dagger m_u$ can be diagonalized by

$$V_u = \begin{pmatrix} -0.478 + 0.848i & -0.11 + 0.198i & 0.0017 - 0.0035i \\ -0.118 - 0.195i & 0.504 + 0.83i & 1.5 \times 10^{-7} \\ 0.00349 & 0.0008245 & 0.999942 \end{pmatrix}$$

with the corresponding eigenvalues $M_u^{diag} = h_{33}^u < H_u > Y_1^{(4)} diag(\lambda^7, \lambda^3, 1),$

The quark mixing matrix, V_{CKM} takes the form

$$|V_{CKM}| = |V_u^\dagger V_d| = \begin{pmatrix} 0.9737 & 0.227 & 0.006 \\ 0.227 & 0.9723 & 0.05 \\ 0.005 & 0.05 & 0.9986 \end{pmatrix}$$

CONCLUSION

1. The model is free from large number of flavons or extra symmetries like Z_N symmetries.
2. The predicted lepton mixing and mass ratios are compatible with the recent data at values of τ near fixed points for inverted hierarchy scenario.
3. The model is valid also in quark sector at one value of τ near a fixed point.

Thank you