Quark-hadron duality and lattice QCD

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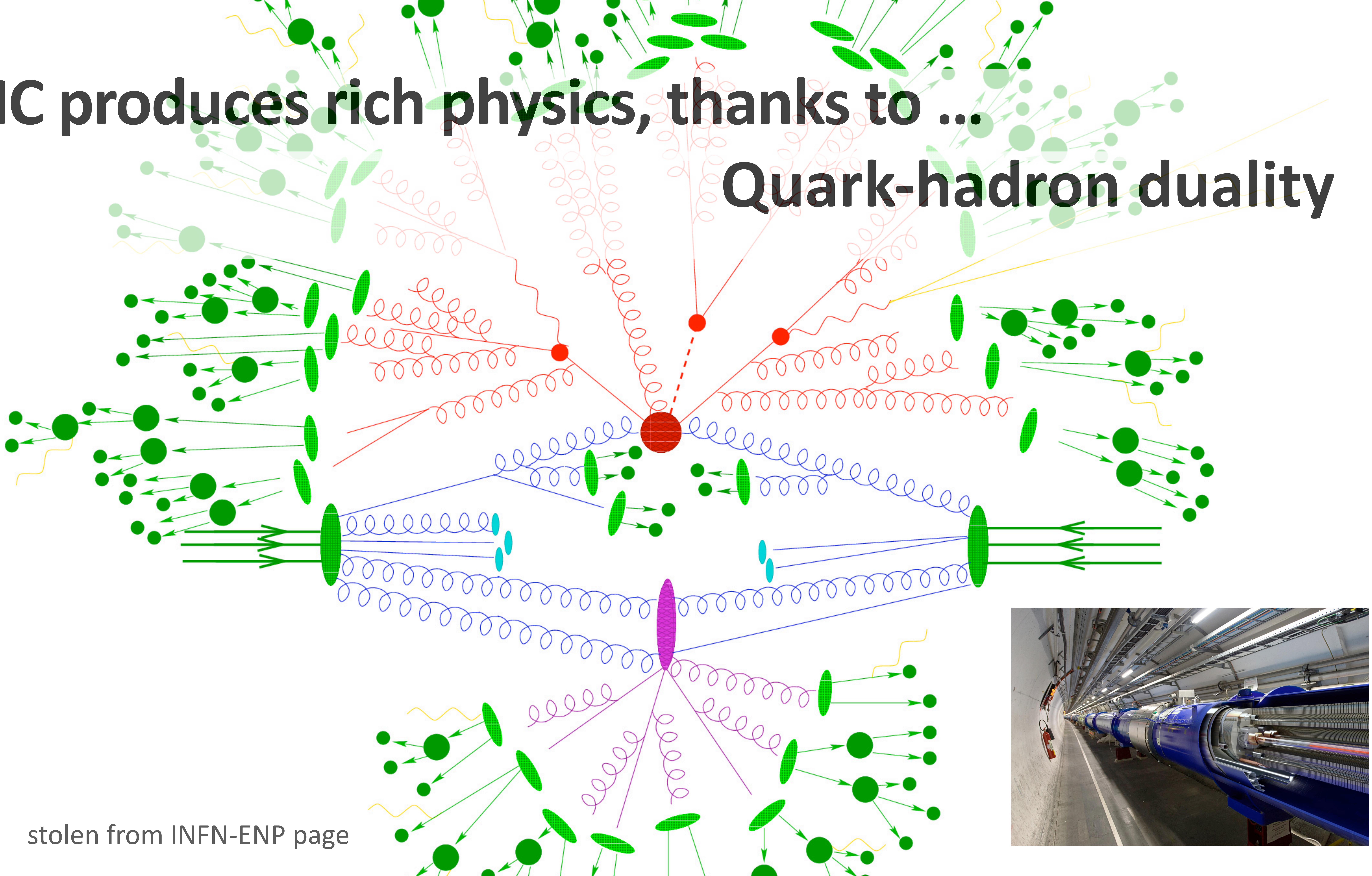
Mar 6, 2024

LHC produces rich physics, thanks to …

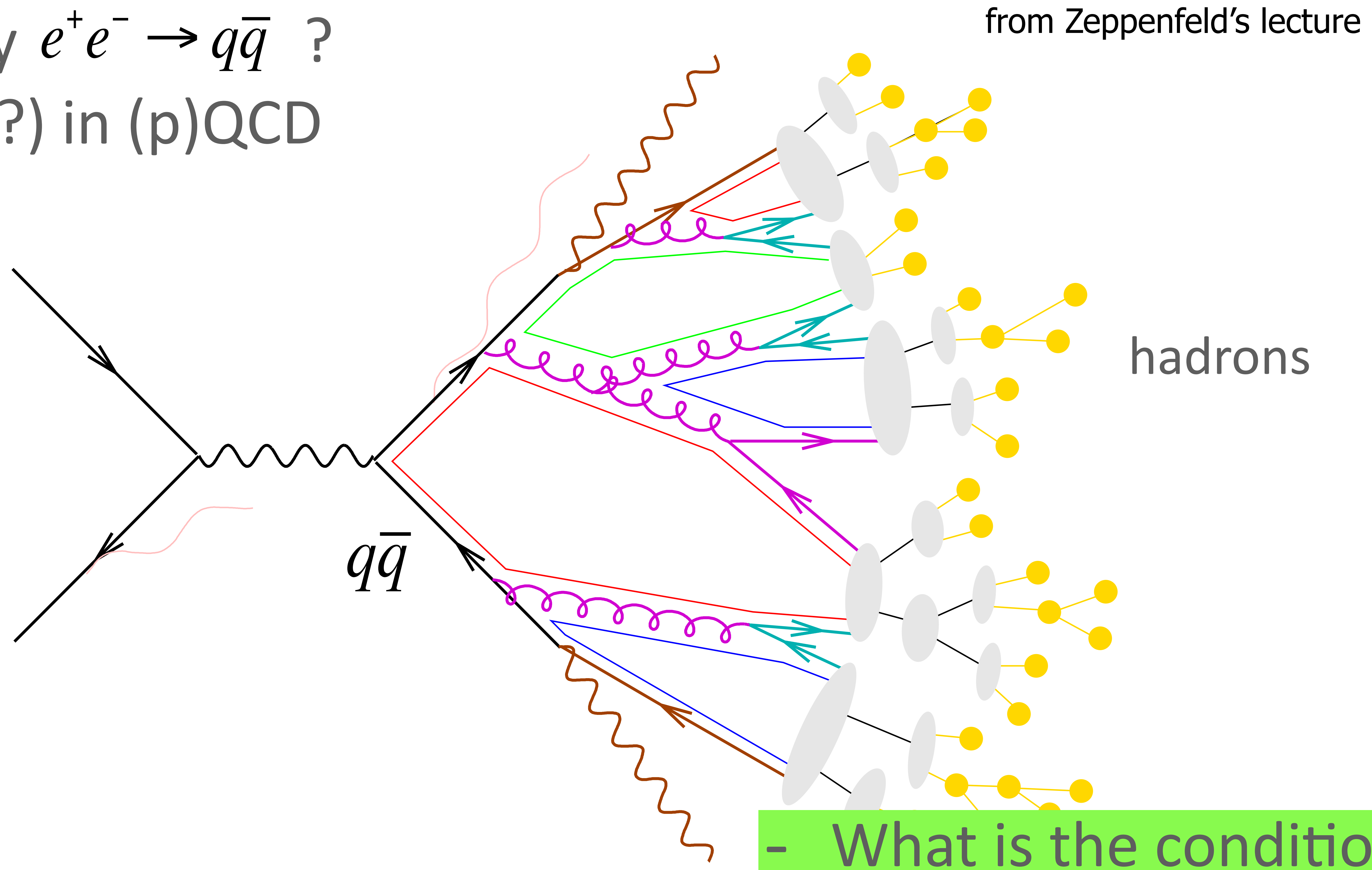
Quark-hadron duality



stolen from INFN-ENP page

Quark-hadron duality ?

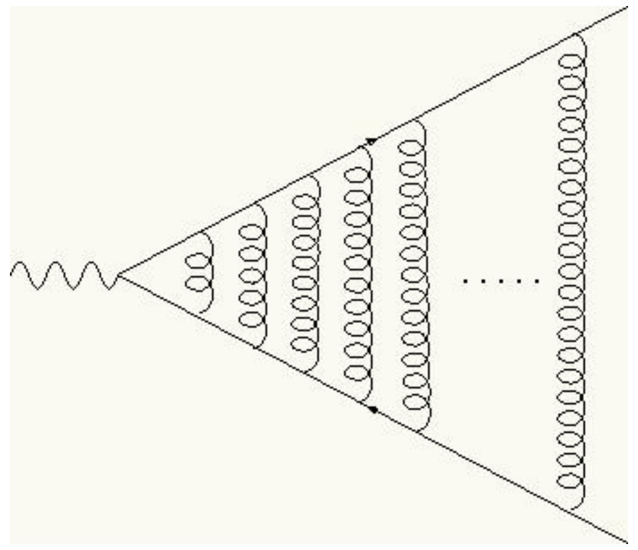
Well approximated by $e^+e^- \to q\bar{q}$?
= Basic **assumption** (?) in (p)QCD



- What is the condition?
- How do you estimate the error?
- What can be done if not satisfied?

Duality badly violated…

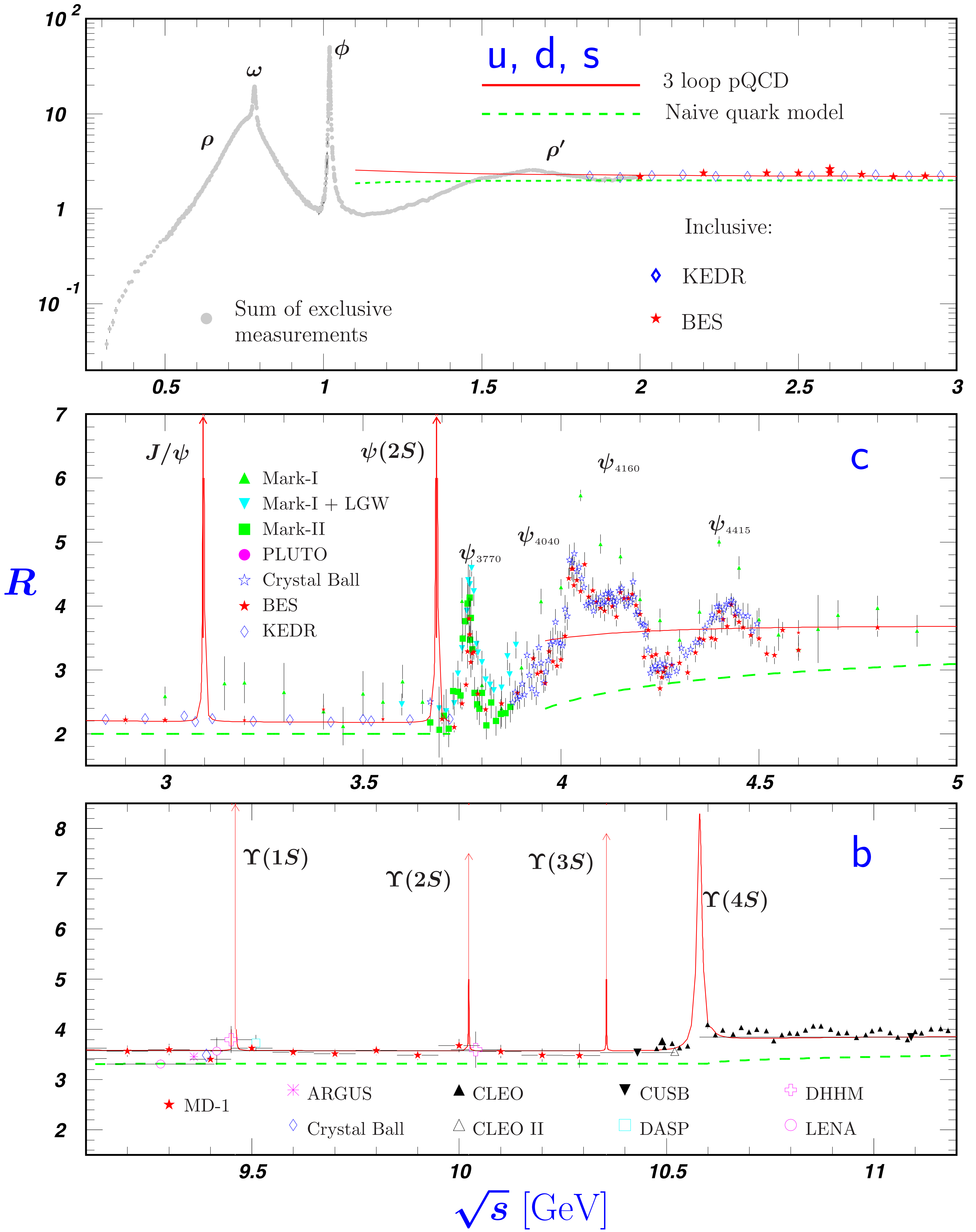
- A lot of resonances!
  - Highly non-perturbative even for quarkonium.



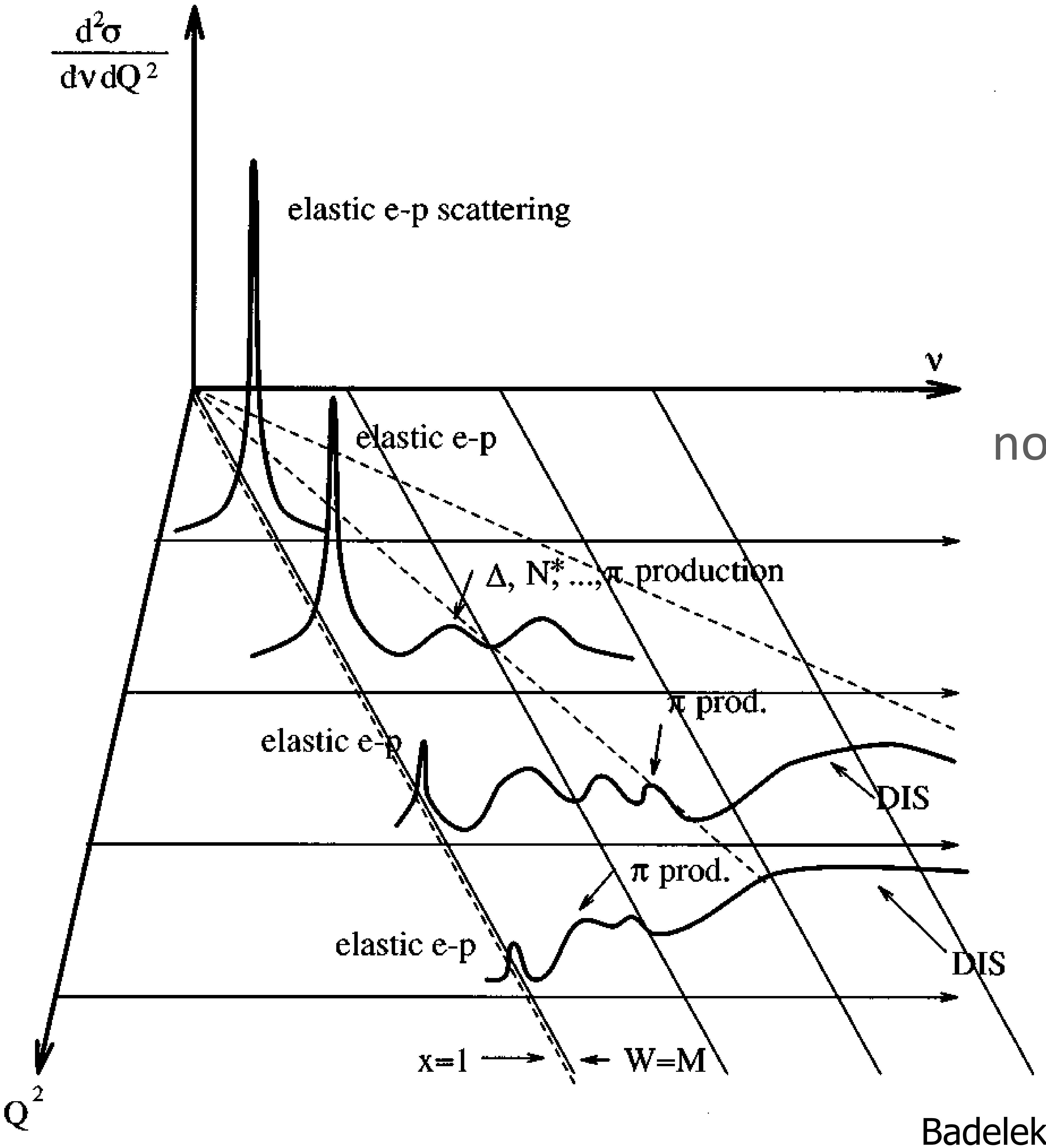
$$\sim \left(\frac{\alpha_s}{v}\right)^n$$

Need to resum, yet incomplete

  - More complicated for the light sector



e-p scattering:



non-perturbative

??

perturbative

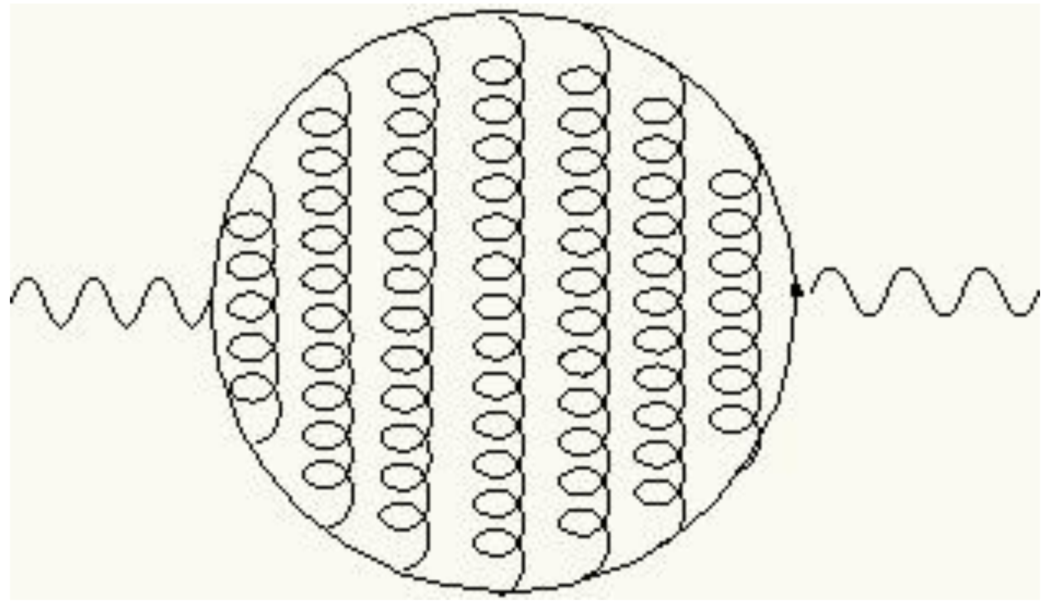
Badelek, Kwiecinski, RMP 68, 445 (1996)

Duality works when…

Poggio, Quinn, Weinberg, PRD13, 1958 (1976)

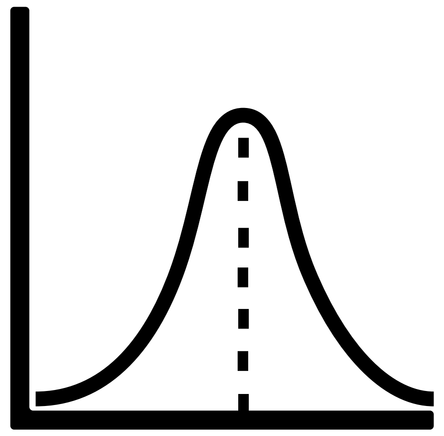
- Sufficiently smeared:

$$\mathrm{Im}\Pi(s) \propto R(s) = \frac{\sigma(e^+e^- \to q\bar{q})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



  - Consider a quantity **smeared** over some range.

$$\bar{R}(s,\Delta) \equiv \frac{\Delta}{\pi}\int_0^\infty ds' \frac{R(s')}{(s-s')^2+\Delta^2}$$
$$= \frac{1}{2\pi i}\int_0^\infty ds' R(s')\left(\frac{1}{s-s'+i\Delta} - \frac{1}{s-s'-i\Delta}\right)$$
$$= \frac{1}{2i}\left[\Pi(s+i\Delta) - \Pi(s-i\Delta)\right]$$



  - One can avoid the threshold singularity.

  - Δ must be larger than Λ_{QCD}^2 to avoid non-perturbative physics.

QCD sum rule

Shifman, Vainshtein, Zakharov, NPB147 385, 448 (1979)

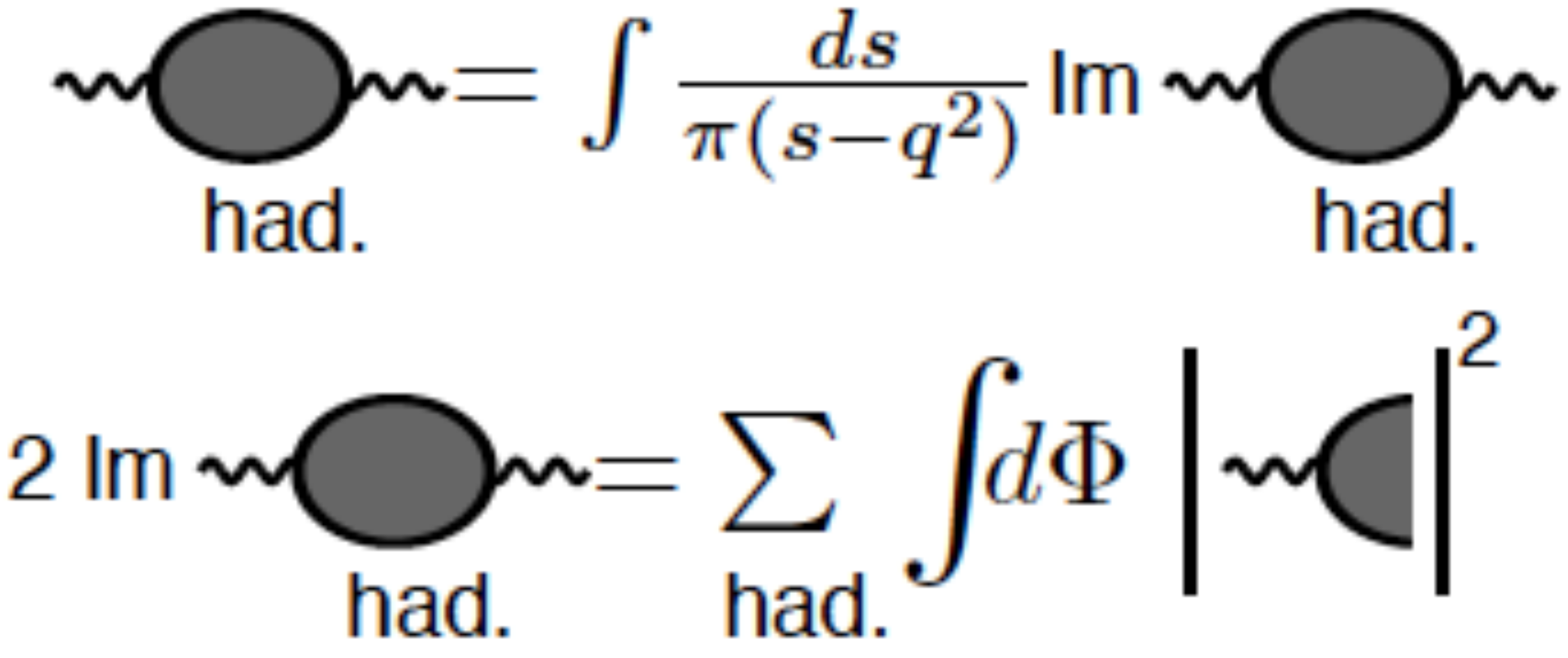
$\Pi(Q^2)$: calculable by pQCD and OPE (+ Borel trans)

plays the role of smearing over energy

$$\Pi(Q^2)$$

space-like region: $Q^2 = -q^2 > 0$

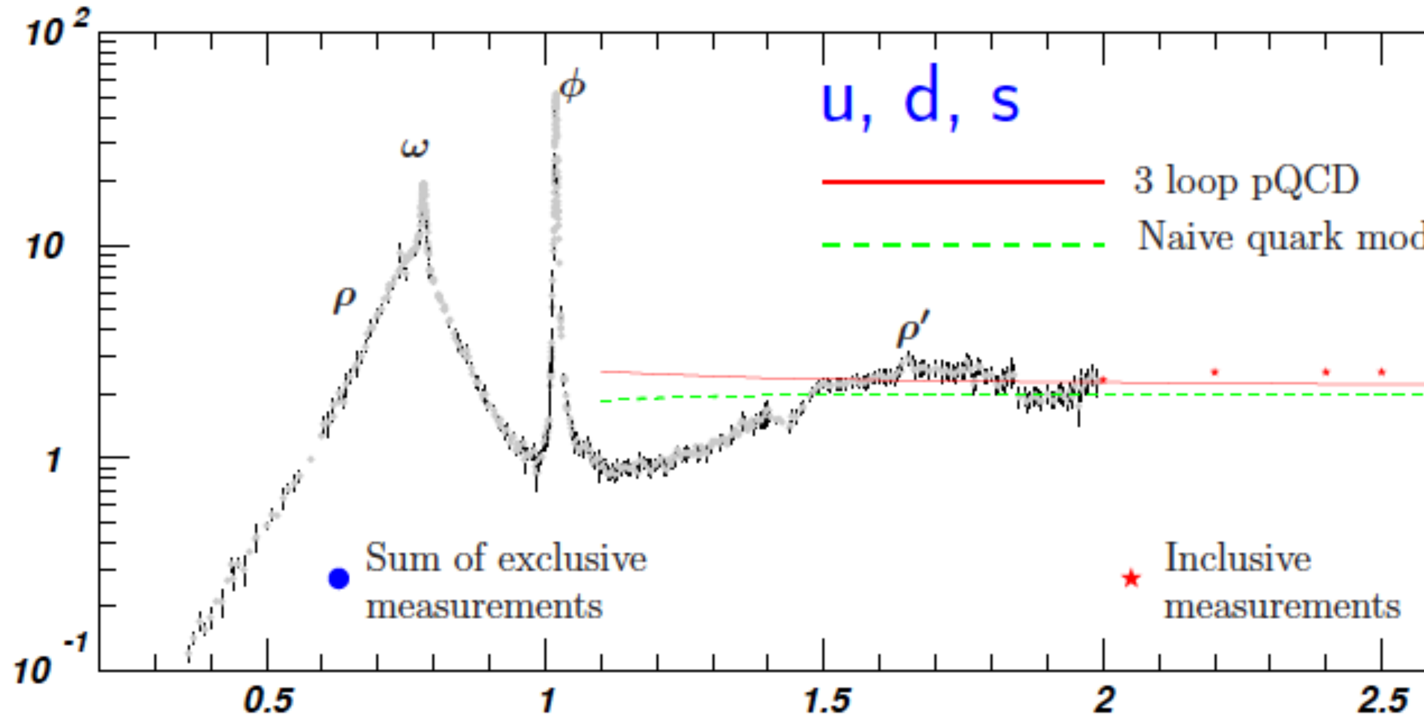
pQCD should work



time-like region: $s = q^2 > 4m_\pi^2$

exp available

Exact analytic relations



QCD sum rule: OPE on the left

Shifman, Vainshtein, Zakharov, NPB147 385, 448 (1979)

$$\Pi(Q^2)$$

Non-perturbative dynamics encoded

$$i\int d^4x\, e^{iqx} T[J(x)J(0)] = C_0(q)\, I + \frac{C_m(q)}{(q^2)^2}\, m\bar{q}q + \frac{C_G(q)}{(q^2)^2}\, \mathrm{Tr}[G_{\mu\nu}^2] + \cdots$$

Perturbative expansion

Suppressed for large $Q^2=-q^2$

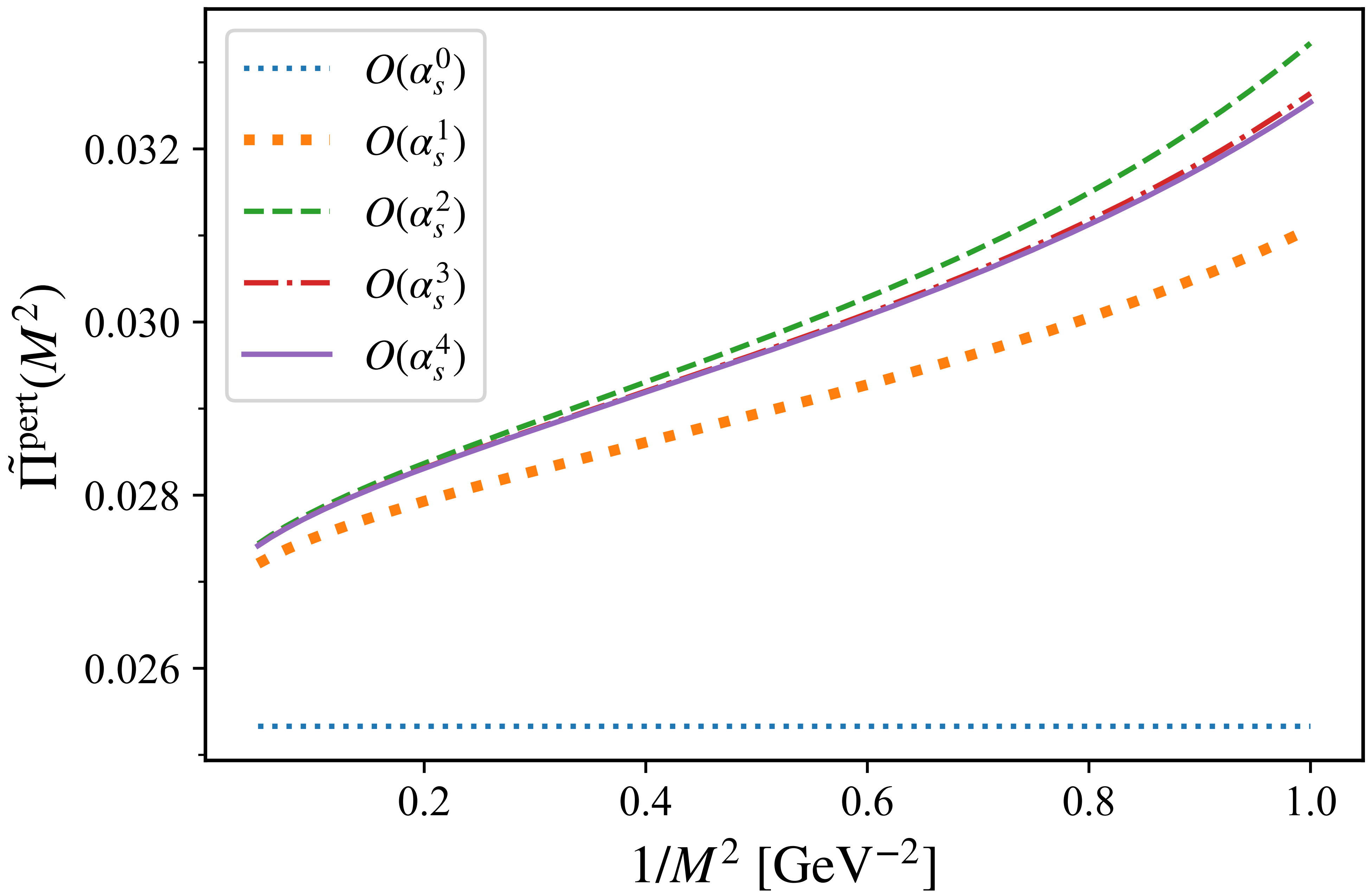
How well does the $1/Q^2$ expansion converge?

Borel transform $\int ds\, e^{-s/M^2} \mathrm{Im}\Pi(s)$

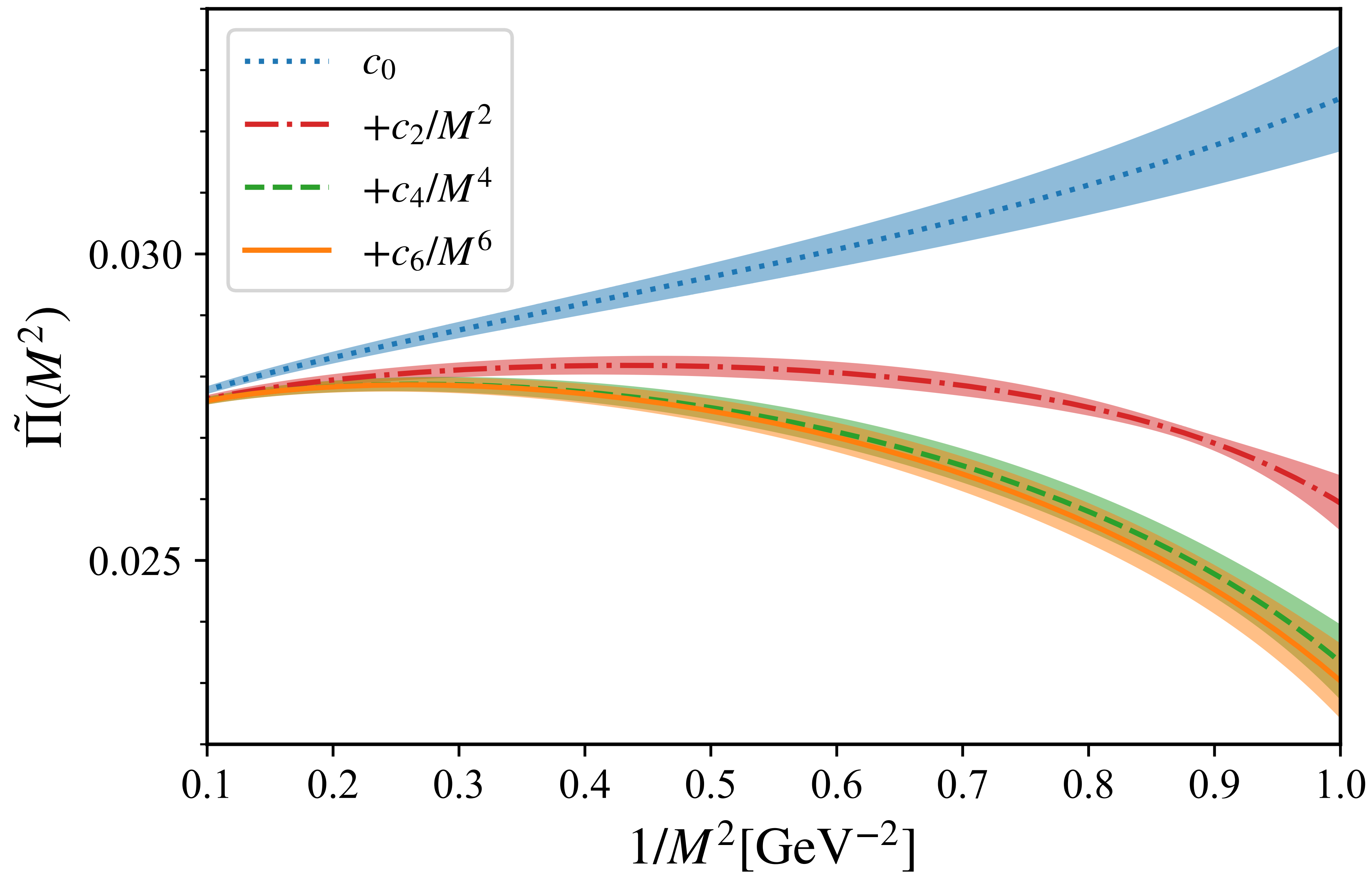
Plots from Ishikawa, SH, arXiv:2103.06539

(another choice of smearing)

perturbative expansion



OPE

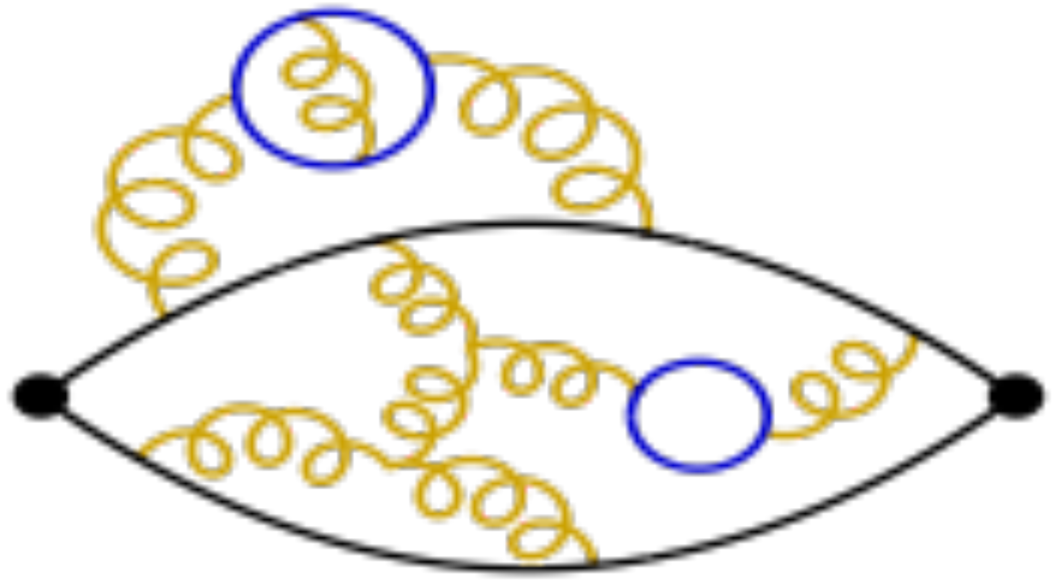


Convergence seems good.
(due to the smearing by the Borel transform).

$\Pi(Q^2)$: why not lattice?

Well, it's surely possible!

$$\Pi_{\mu\nu}(x) = \langle 0|T\{J_\mu(x)J_\nu(0)\}|0\rangle$$

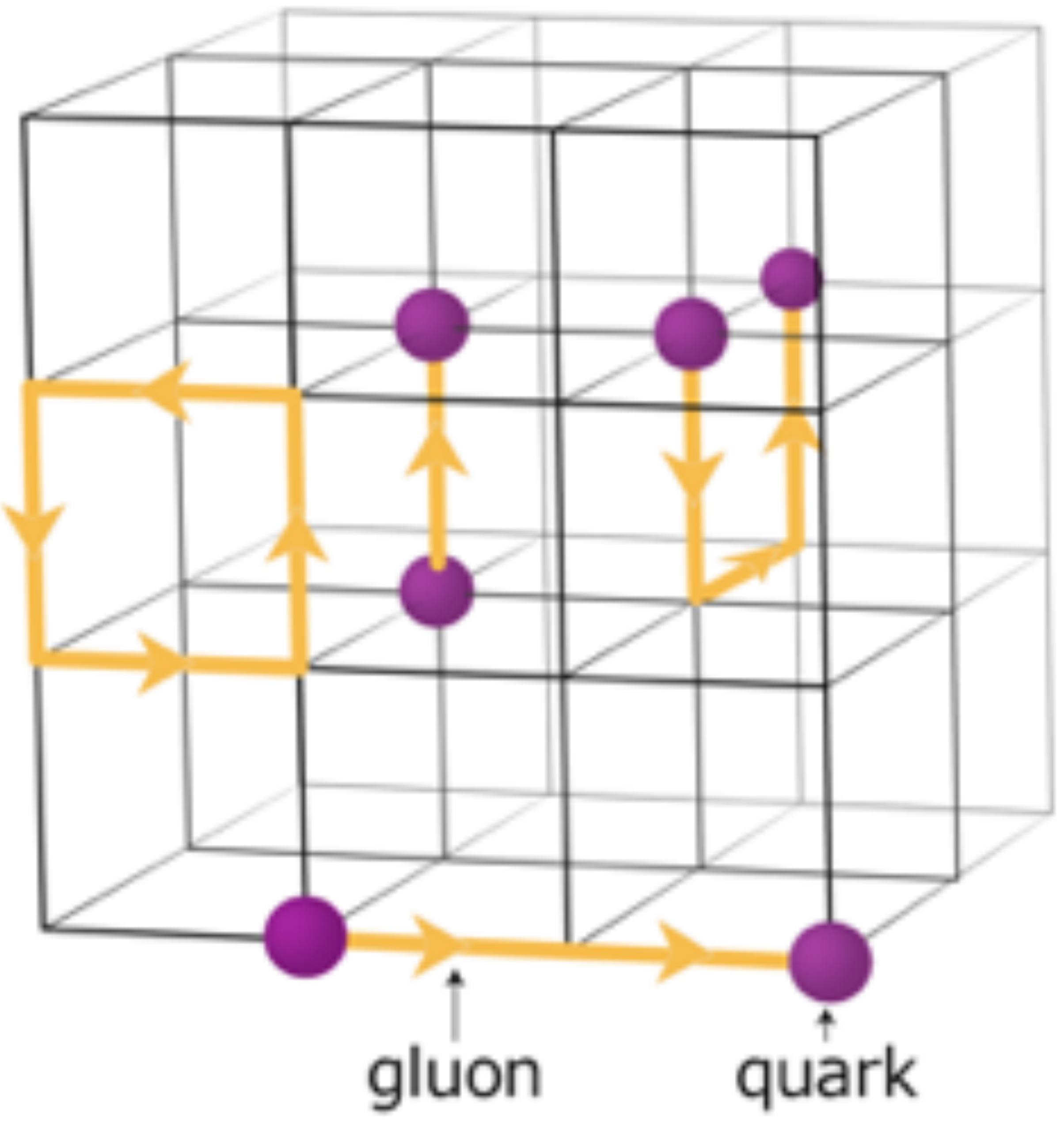


- Fully non-perturbative; no assumption involved.
  - A bread-and-butter calculation, though need large resources to be realistic.
  - An input for hadronic-vacuum-polarization (HVP) contribution of muon g-2.

- Euclidean lattice → only space-like $\Pi(Q^2)$

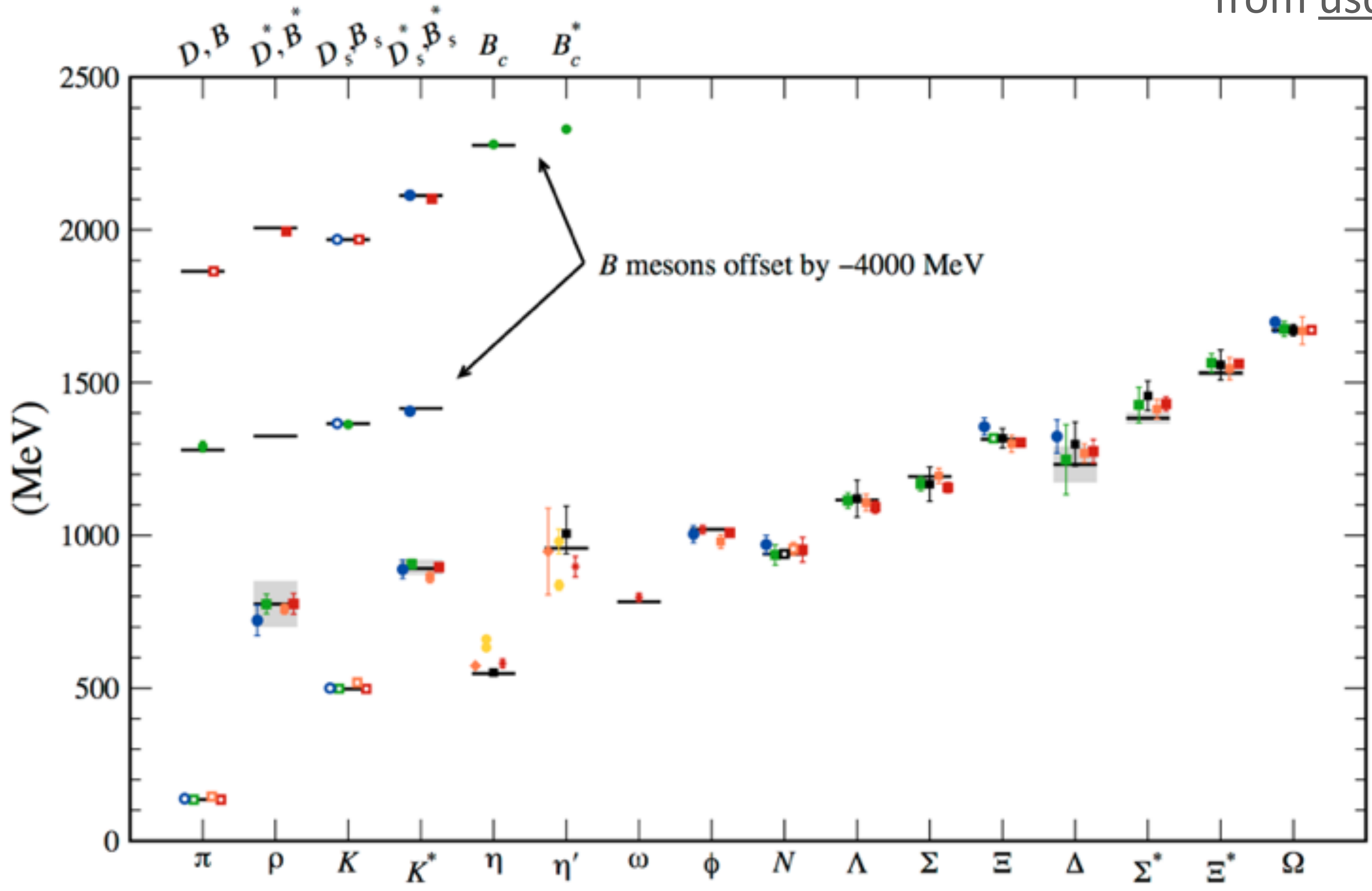
<u>Euclidean</u> lattice QCD

LQCD = ab initio calculation of QCD, on the Euclidean space



- Define the quark and gluon fields on the **Euclidean** lattice.
- Perform the path integral numerically (Monte Carlo).

from usqcd.org



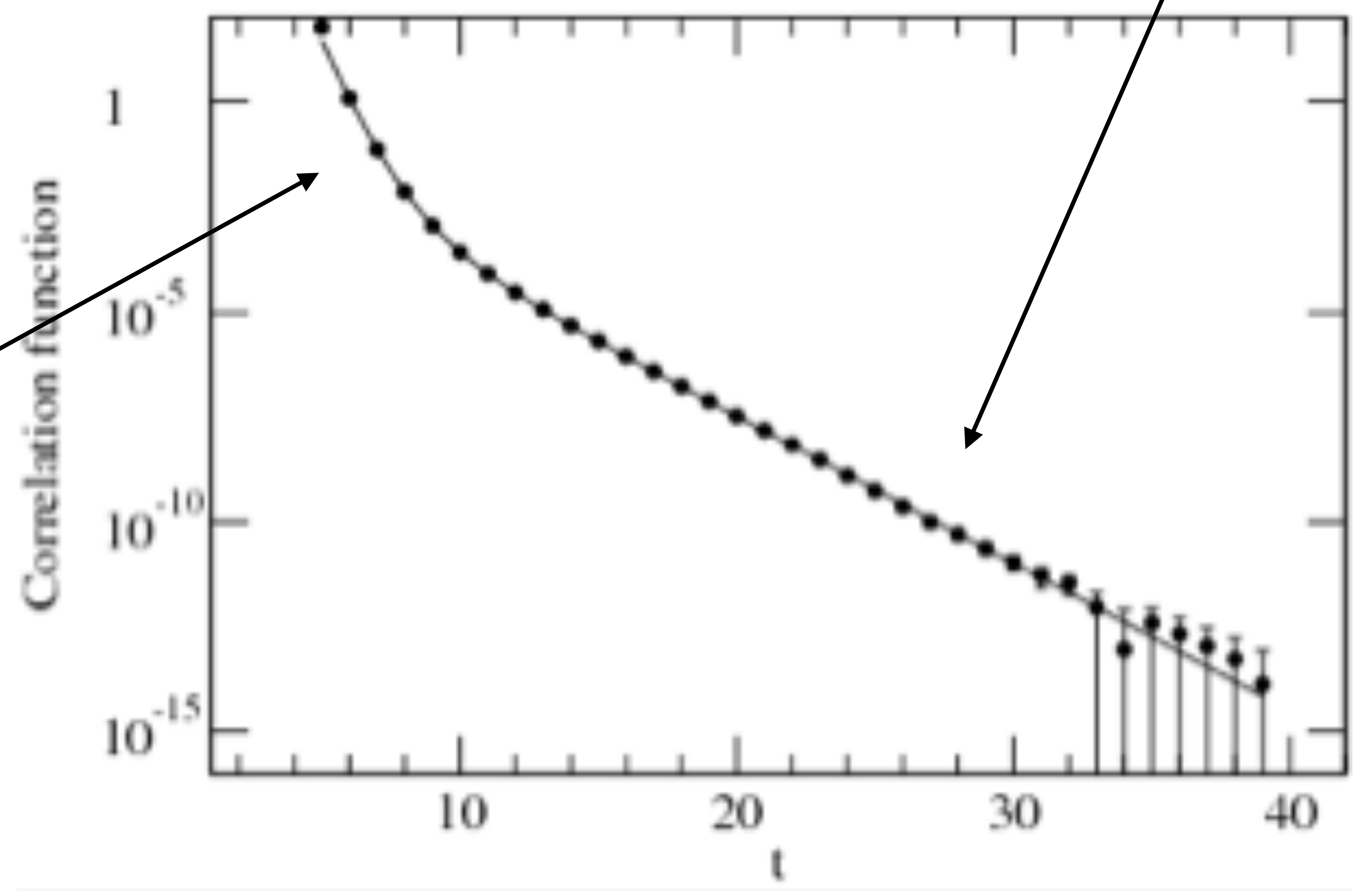
More on vacuum polarization

Euclidean correlator

- e^{-Et} instead of e^{-iEt}

read off the exponential slope at long distances
→ hadron energy (or mass)

$$\int d^3\mathbf{x}\langle\mathcal{O}(\mathbf{x},t)\mathcal{O}^\dagger(0)\rangle$$

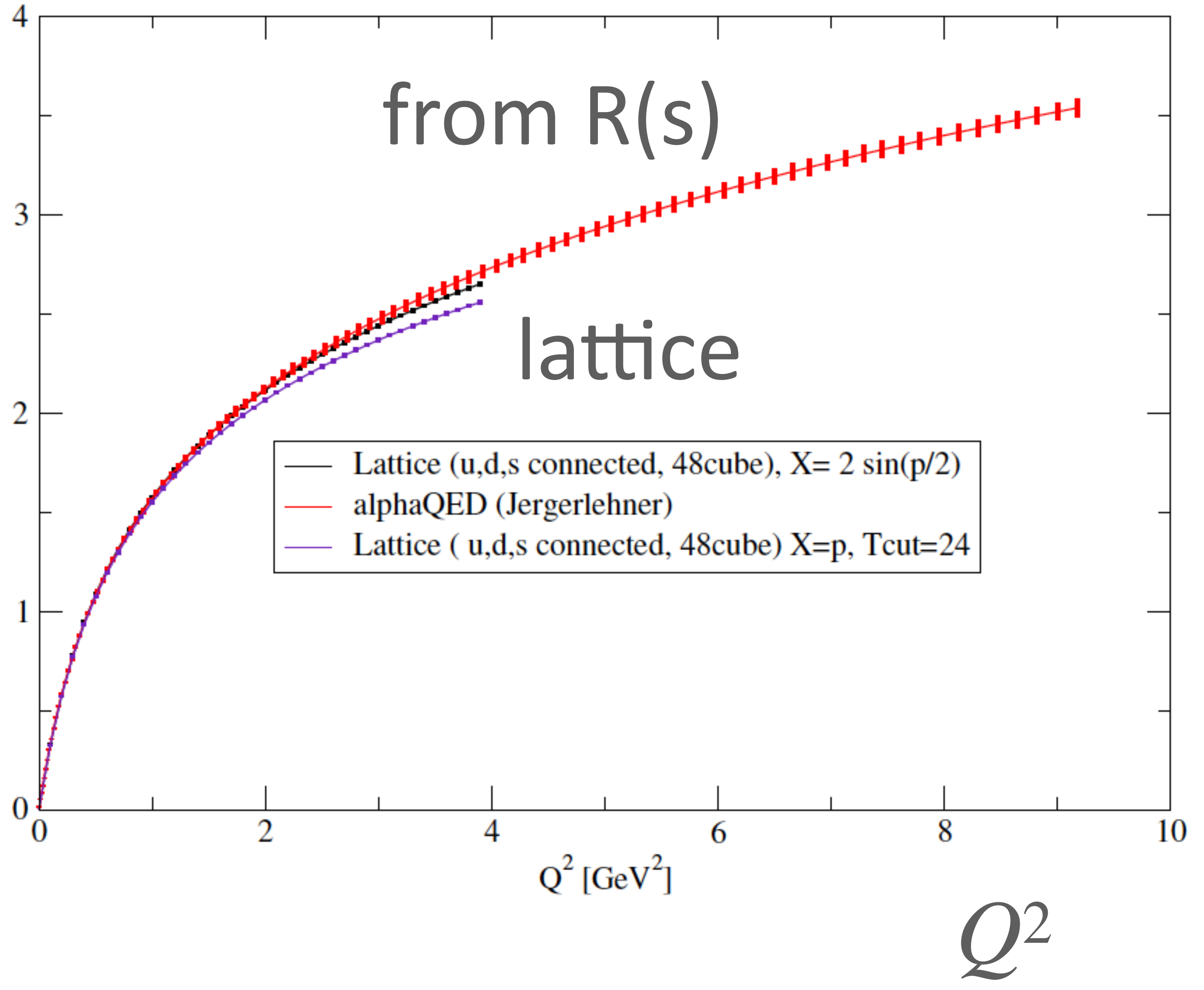


More physics info contained in the short-distance region

Go space-like

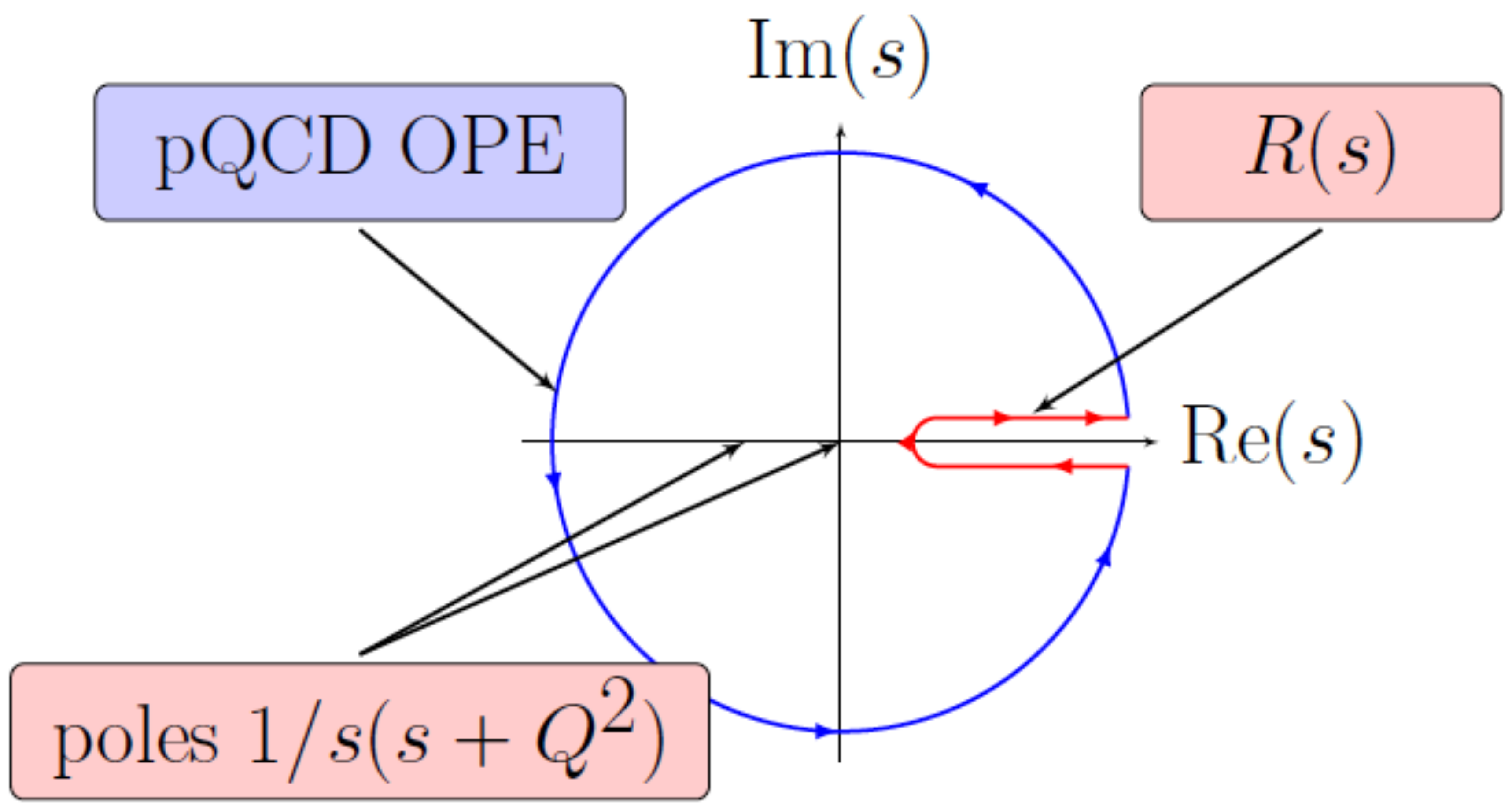
Fourier transform of lattice data
to produce the space-like $\Pi(Q^2)$

RBC/UKQCD:
Izubuchi@g-2 WS (2017)



smearing provided by

$$\hat{\Pi}(Q^2) \;=\; Q^2 \int_0^\infty ds \frac{R(s)}{s(s+Q^2)}$$



Variety of smearings

Some (weighted) integrals:

- Space-like correlator: $$\Pi(-Q^2) = \frac{1}{\pi}\int_0^\infty ds\,\frac{\mathrm{Im}\Pi(s+i\epsilon)}{s+Q^2} = \int_0^\infty ds\,\frac{\rho(s)}{s+Q^2}$$
  - weighted integral over s (or ω)
  - can be written by a Fourier transform of the Euclidean lattice correlator
- HVP contribution to Muon g-2: $$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s}\,\frac{1}{\pi}\mathrm{Im}\Pi(s)\,K(s)$$
  - weighted integral over s (or ω)
  - can also be written as an integral (or a sum) of lattice correlator

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\,C(t)\tilde{f}(t)$$

and more, with some kernel $K(s)$

Bernecker-Meyer (2011)

Connection to the lattice correlator

all possible states contribute ↓

correlator: $$C(t) = \int_0^\infty d\omega\, \rho(\omega) e^{-\omega t} \qquad \sim \langle 0| J\, e^{-\hat{H}t}\, J|0\rangle$$

sum over states:
(or smearing)

$$\Gamma = \int_0^\infty d\omega\, K(\omega)\rho(\omega) \qquad \sim \langle 0| J K(\hat{H}) J|0\rangle$$

Approximation of the form

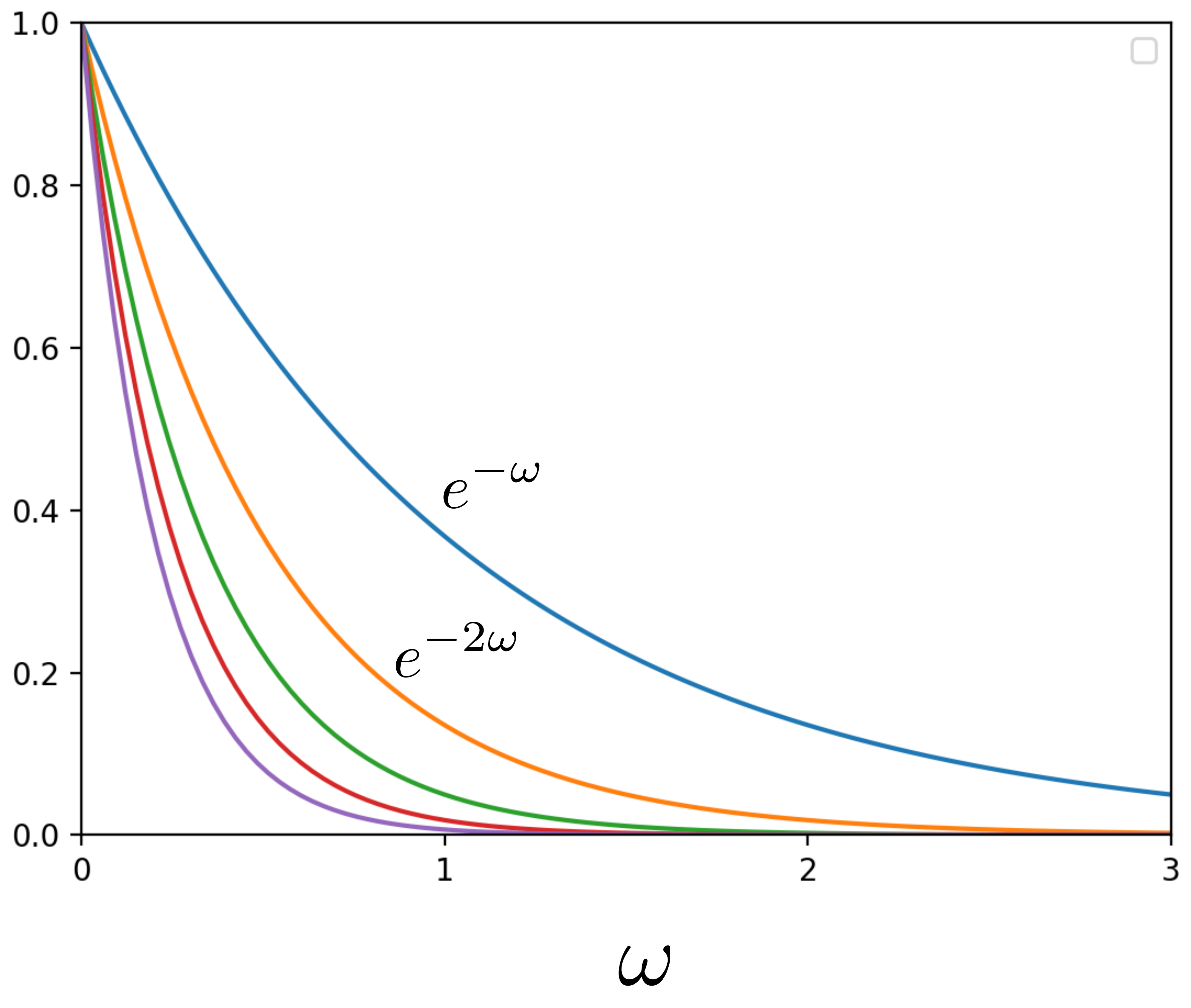
$$K(\hat{H}) = c_0 + c_1 e^{-\hat{H}} + c_2 e^{-2\hat{H}} + c_3 e^{-3\hat{H}} + \cdots$$

can relate Γ to the correlator.

c.f. spectral func: $$\rho(\omega) \propto \sum_X \delta(\omega - E_X) |\langle X|J|0\rangle|^2 \qquad \sim \langle 0| J\, \delta(\omega - \hat{H})\, J|0\rangle$$

Approximation?

$$K(\hat{H}) \simeq k_0 + k_1 e^{-\hat{H}} + k_2 e^{-2\hat{H}} + \cdots + k_N e^{-N\hat{H}}$$



- Not always possible; when the function varies rapidly, in particular.
- Some methods developed recently.

  - Modified Backus-Gilbert

    Hansen, Lupo, Tantalo, arXiv:1903.06476
  - Or, Chebyshev polynomial

    Bailas, Ishikawa, SH, arXiv:2001.11779

Chebyshev polynomials

Bailas, SH, Ishikawa (2000)

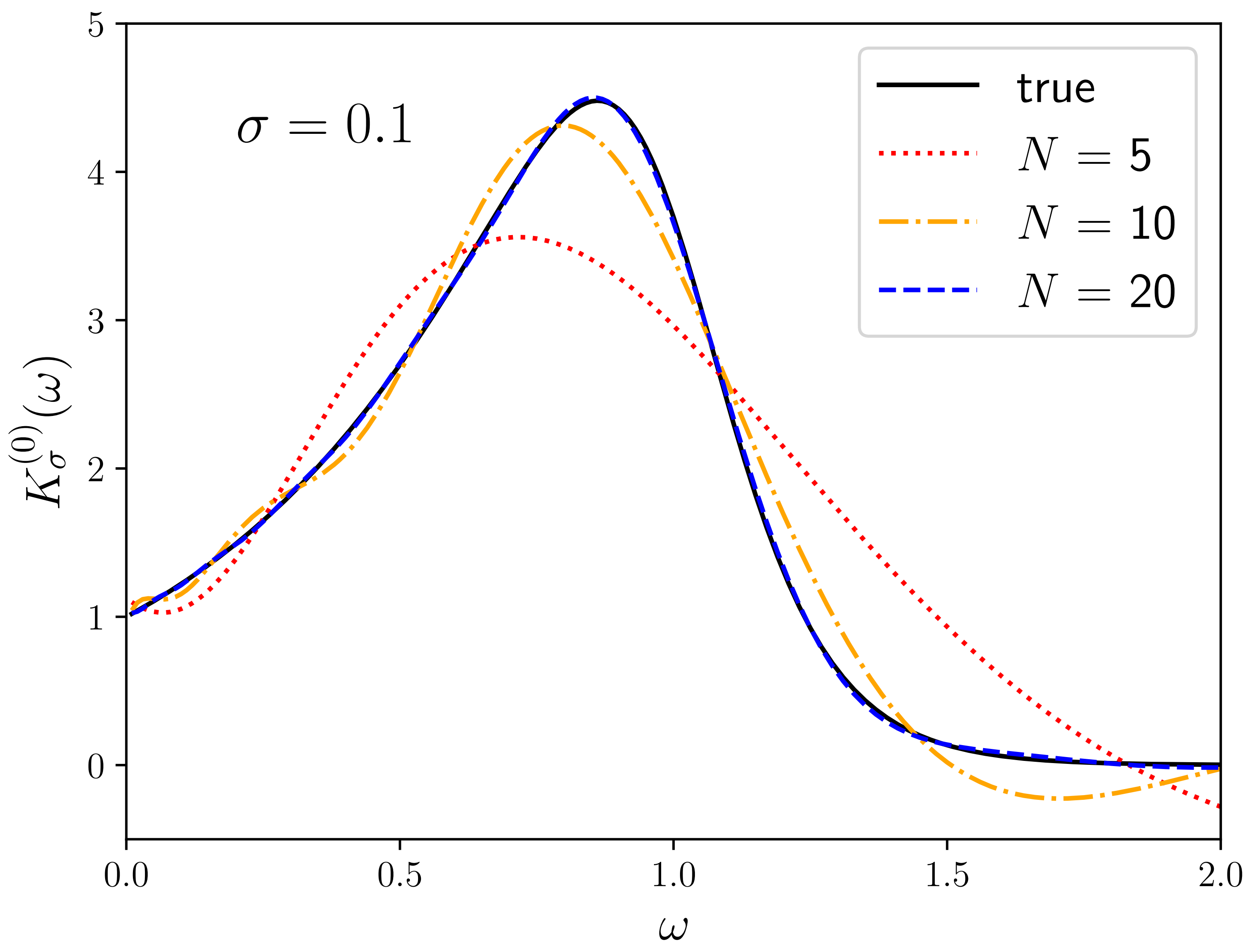
$$K(\hat{H}) \simeq \sum_{j=0}^{N} c_j T_j(e^{-\hat{H}})$$

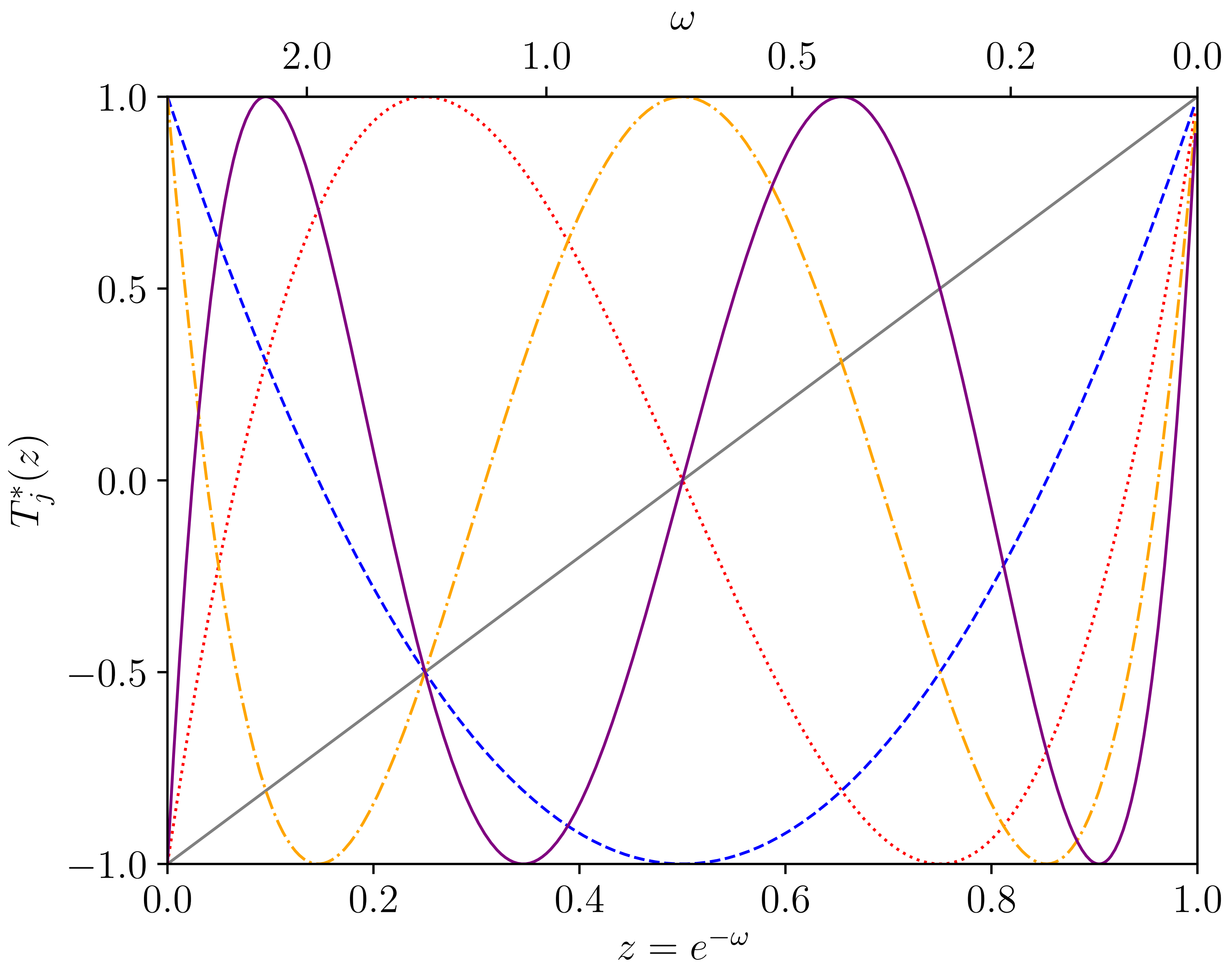
(shifted) Chebyshev polynomials

$$T_0^*(x) = 1$$
$$T_1^*(x) = 2x - 1$$
$$T_2^*(x) = 8x^2 - 8x + 1$$
$$\vdots$$
$$T_{j+1}^*(x) = 2(2x-1)T_j^*(x) - T_{j-1}^*(x)$$

- Coefficients can be easily calculated.
- The “best” approx (= maximal deviation is minimal)
- Only smooth functions can be approximated.
- (The constraint $|T_j(z)| < 1$ helps stabilize.)

example of the Chebyshev approx:



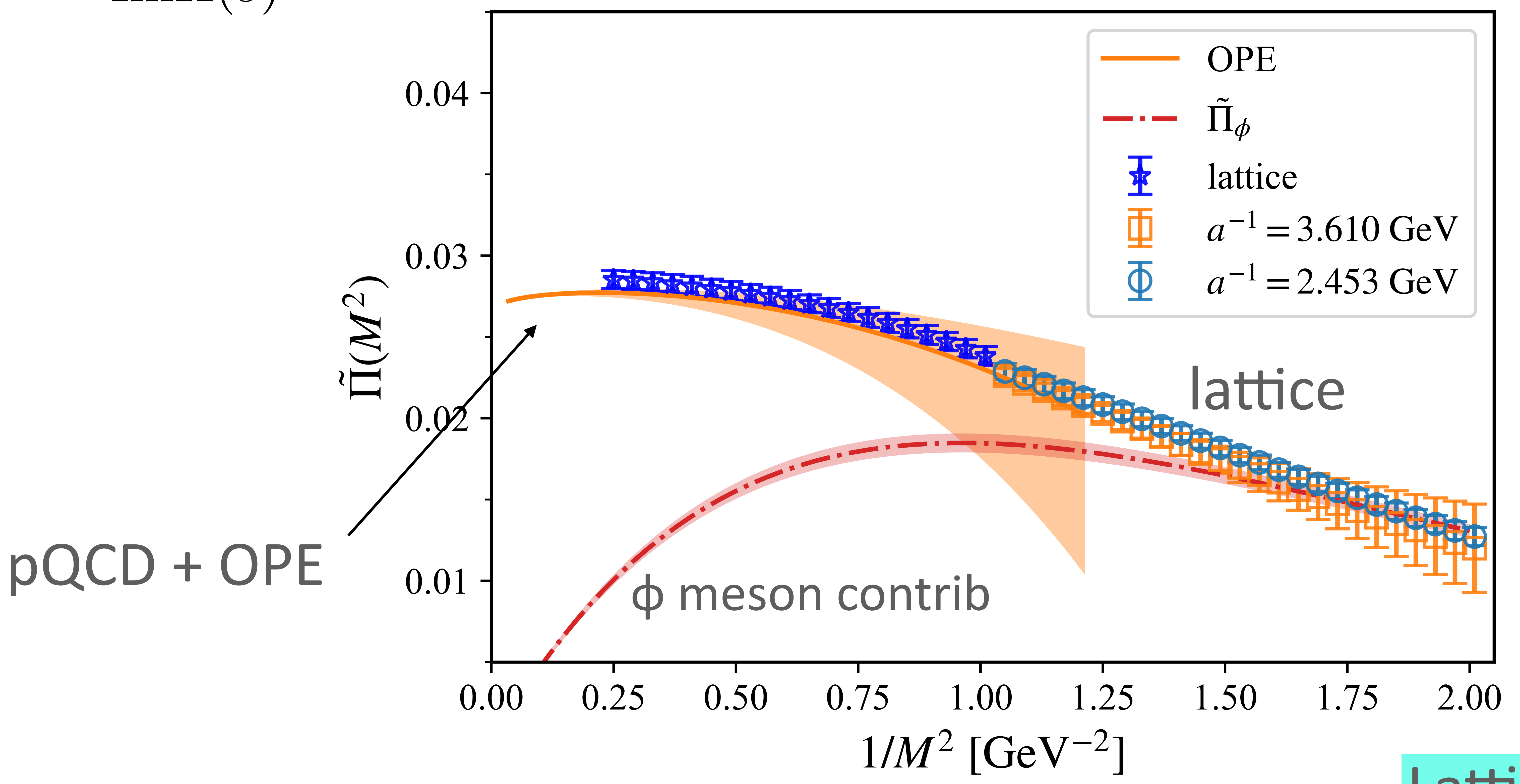


Borel sum (as in QCD sum rule)

Ishikawa, SH, Phys. Rev. D104, 074521 (2021)

$$\int ds\, e^{-s/M^2} \mathrm{Im}\Pi(s)$$

$s\bar{s}$ channel



Lattice can provide the entire energy range with precision.

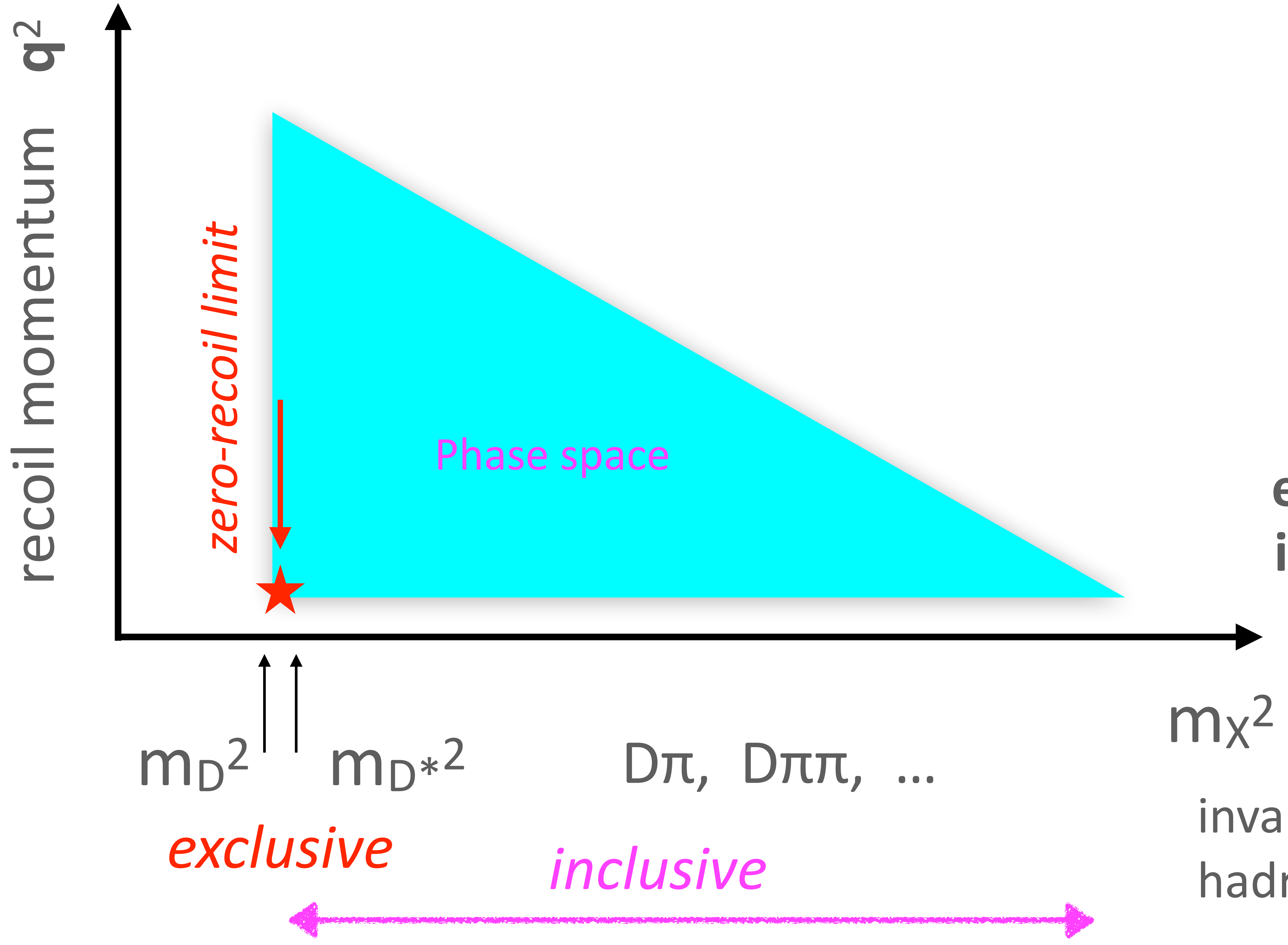
B meson semileptonic decays: total inclusive rate

Based on the collaborations of

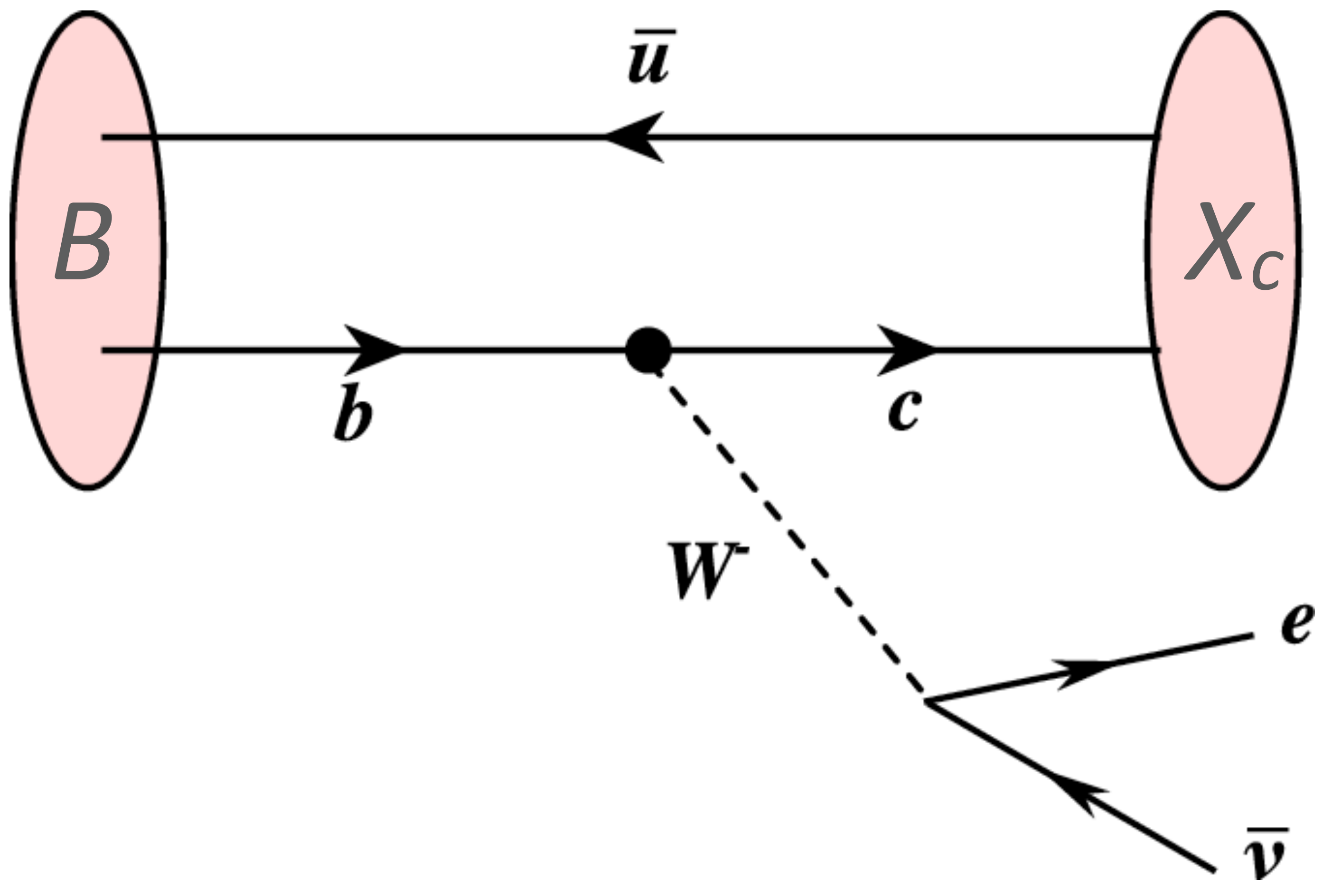
- Gambino, SH, Phys. Rev. Lett. 125 (2020) 032001; arXiv:2005.13730
- Gambino, SH, Machler. Panero, Sanfilippo, Simula, Smecca, Tantalo, JHEP 07 (2022) 083; arXiv:2203.11762
- Barone, Kellerman, SH, Juttner, Kaneko, JHEP 07 (2023) 145; arXiv:2305.14092

see also, Hansen, Meyer, Robaina, Phys. Rev. D96, 094513 (2017); arXiv:1704.08993

Inclusive and exclusive B semileptonic decays



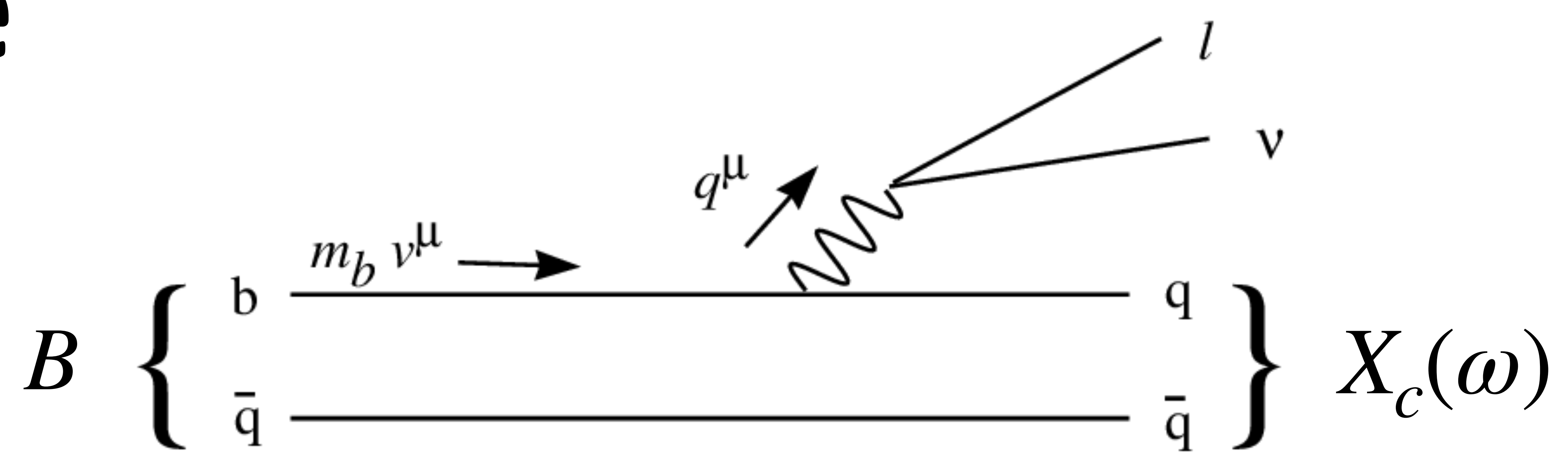
invariant mass of the hadronic system



exclusive particular final states (D, D*, …)

inclusive sum over final states

Inclusive semi-leptonic rate



Differential decay rate:

$$d\Gamma \sim |V_{cb}|^2 l^{\mu\nu} W_{\mu\nu}$$

Structure function (or hadronic tensor):

$$W_{\mu\nu} = \sum_X (2\pi)^2 \delta^4(p_B - q - p_X) \frac{1}{2M_B} \langle B(p_B)|J_\mu^\dagger(0)|X\rangle\langle X|J_\nu(0)|B(p_B)\rangle$$

$$\rightarrow \langle B(\mathbf{0})|\tilde{J}_\mu^\dagger(-\boldsymbol{q};t)\ \delta(\omega - \hat{H})\ \tilde{J}_\nu(\boldsymbol{q};0)|B(\mathbf{0})\rangle$$

Total decay rate:

$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega\, K(\omega;\boldsymbol{q}^2)\langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega - \hat{H})\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle$$

kinematical (phase-space) factor

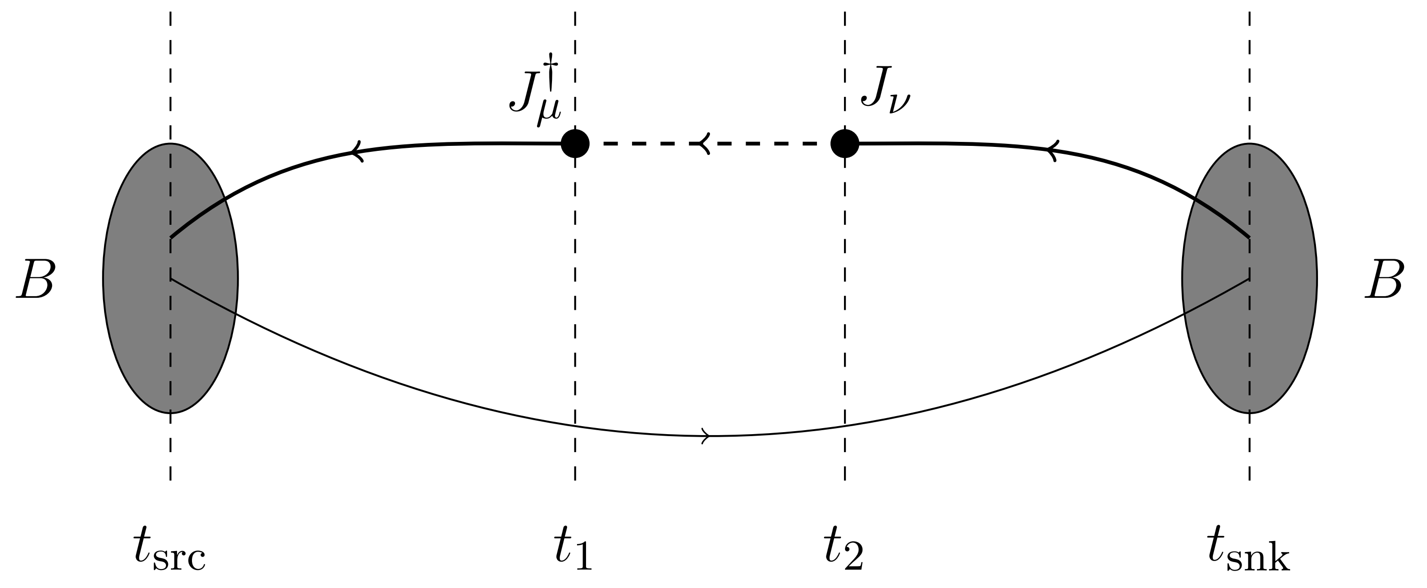
Energy integral to be evaluated:

$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega\, K(\omega;\boldsymbol{q}^2)\langle B(\boldsymbol{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega-\hat{H})\tilde{J}(\boldsymbol{q})|B(\boldsymbol{0})\rangle$$

$$= \langle B(\boldsymbol{0})|\tilde{J}^\dagger(-\boldsymbol{q})K(\hat{H};\boldsymbol{q}^2)\tilde{J}(\boldsymbol{q})|B(\boldsymbol{0})\rangle$$

Compton amplitude obtained on the lattice:

$$\langle B(\boldsymbol{0})|\tilde{J}_\mu^\dagger(-\boldsymbol{q};t)\ \ \tilde{J}_\nu(\boldsymbol{q};0)|B(\boldsymbol{0})\rangle \longrightarrow \langle B(\boldsymbol{0})|\tilde{J}^\dagger(-\boldsymbol{q})e^{-\hat{H}t}\tilde{J}(\boldsymbol{q})|B(\boldsymbol{0})\rangle$$



Using :

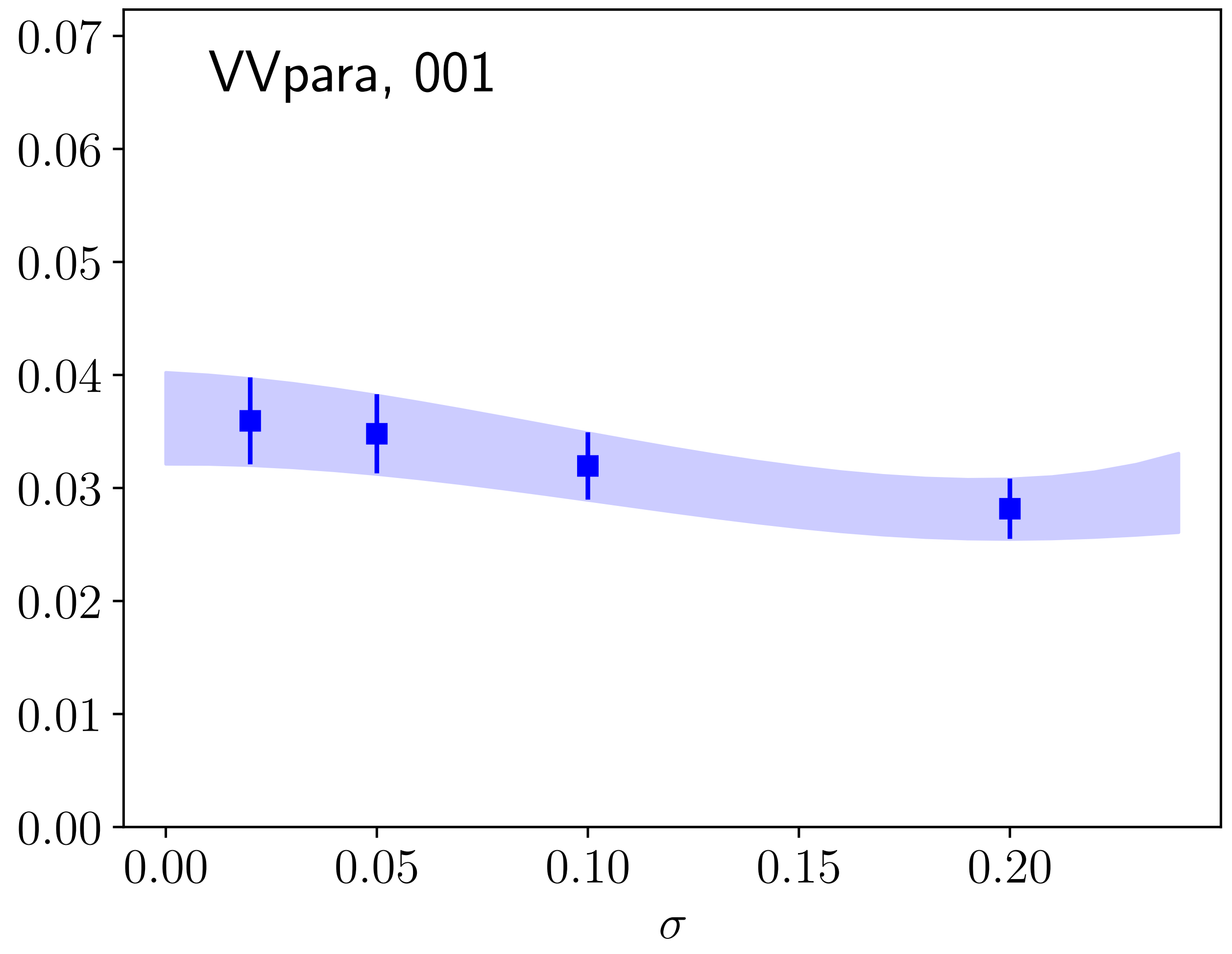
$$K(\hat{H}) = k_0 + k_1 e^{-\hat{H}} + k_2 e^{-2\hat{H}} + \cdots + k_N e^{-k_N\hat{H}}$$

Phase-space factor as a kernel

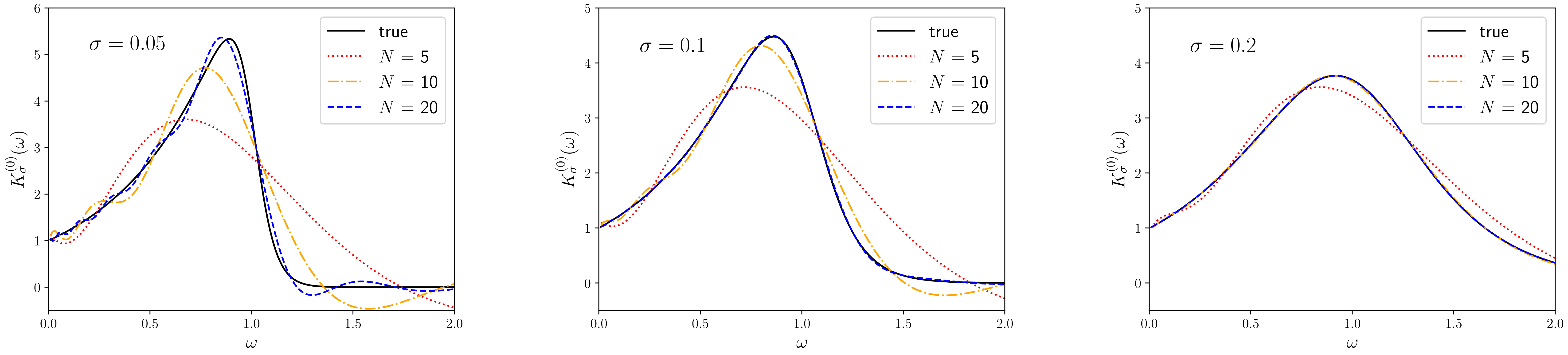
$$K(\omega) \sim e^{2\omega t_0} \underbrace{(m_B - \omega)^l}_{\text{kinematical}} \theta(m_B - |\mathbf{q}| - \omega)$$

upper limit

smear by sigmoid with a width σ;
Need to take the $\sigma \to 0$ limit



narrower — smearing — wider

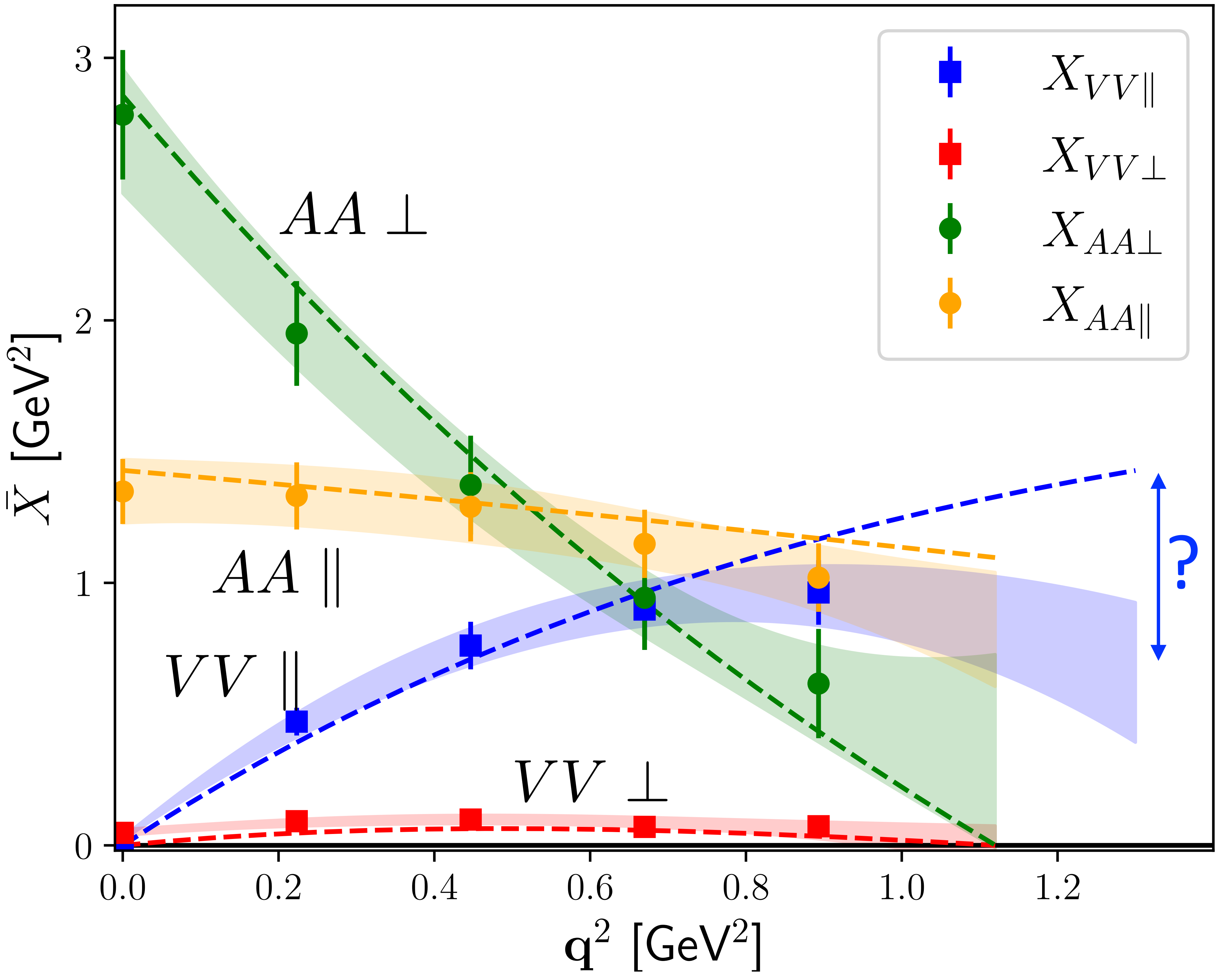


Inclusive decay rate

- Prototype lattice calculation
  - $B_s \to Xc$
  - the b quark is lighter than physical.

- Decay rate in each channel
  - VV and AA
  - parallel or perpendicular to the recoil momentum
  - compared to "exclusive" (dashed lines)
    - $VV_{||}$ is dominated by $B \to D$
    - Others are by $B \to D^*$

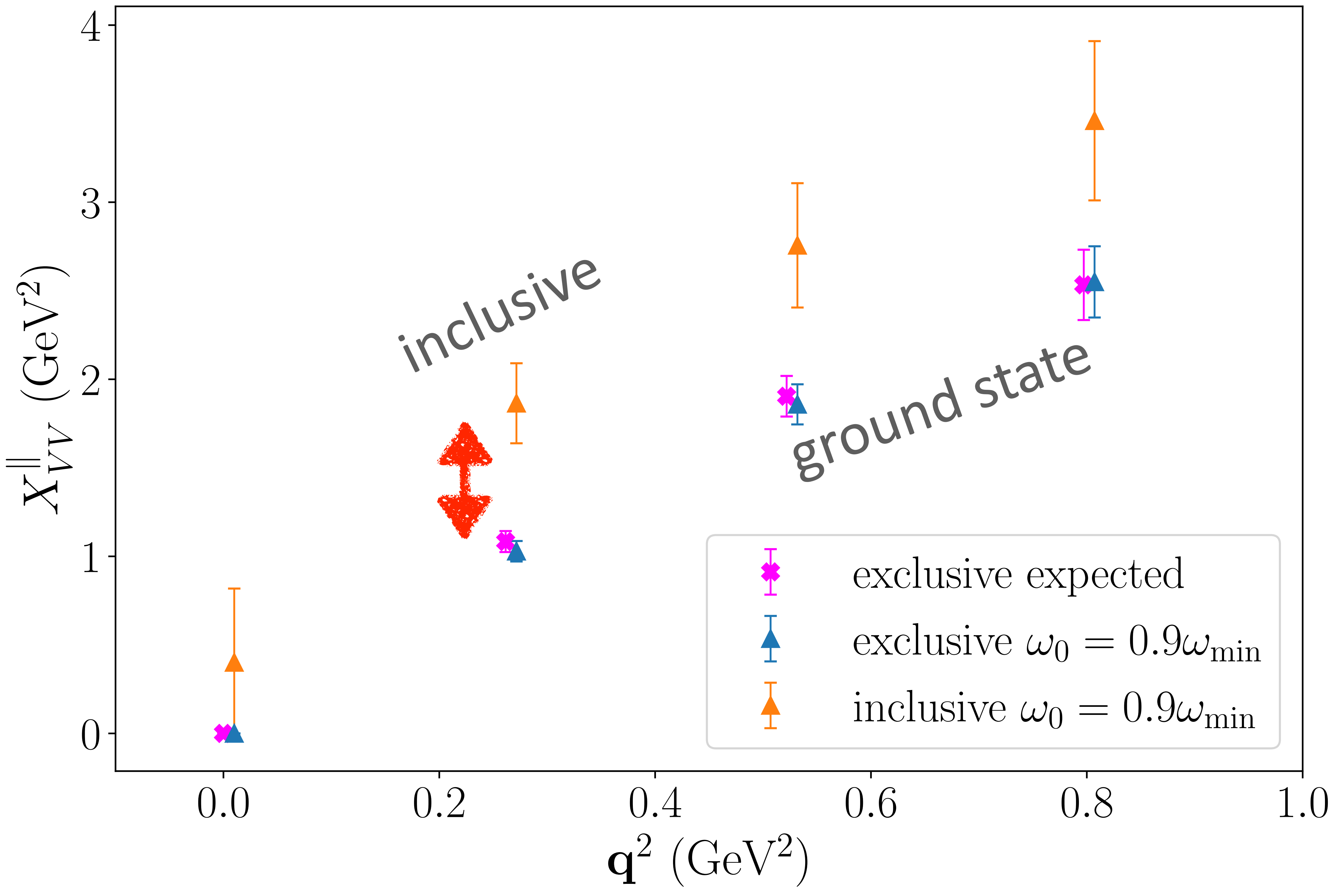
differential rate / |**q**|



JLQCD data from Gambino et al., 2203.11762

Excited states are visible

Barone et al., 2305.14092



excited-state contribution;
so certainly inclusive.

Sum over states: dangerous game?

Sum over states with a kernel $K(s)$: $\int_0^\infty ds\, K(s)\rho(s)$

Crucially depends on our ability to approximate the energy integral.

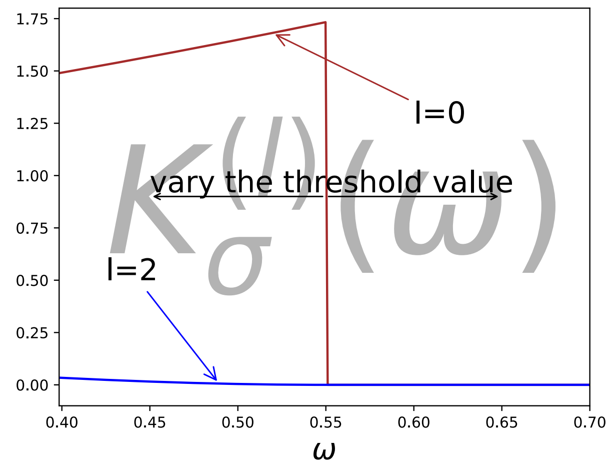
- Possible to treat any $K(s)$?
- **No.** We know $K(s) = \delta(s)$ gets back to the ill-posed problem (= reconstruction of full spectral function from lattice data!)

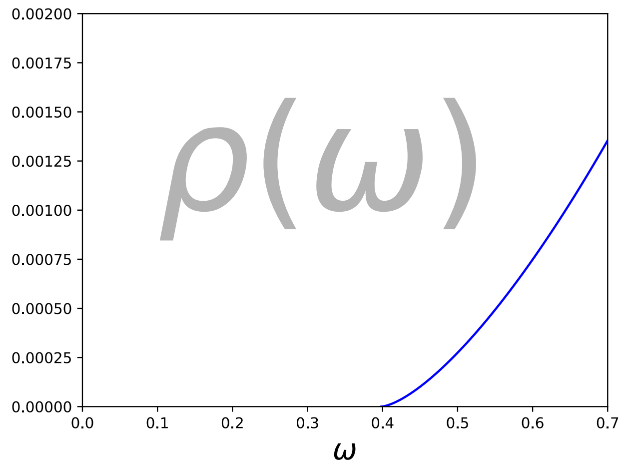
- Then, what is the limitation or potential systematic effect?

Approx: hard or easy?

- Kernel approximation.

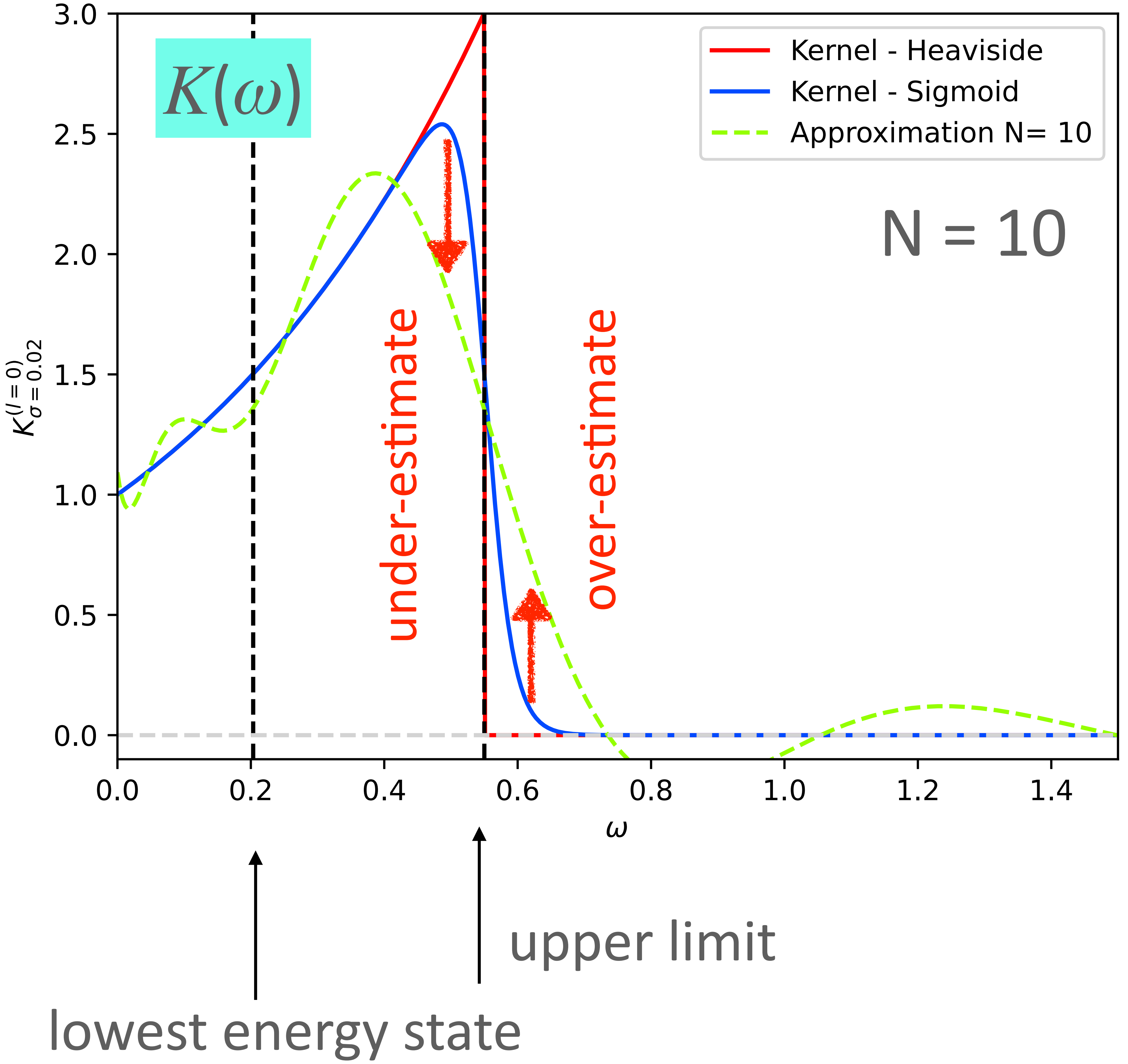
$$\bar{X}^{(l)}(\omega_{th}) \sim \int_0^{\omega_{th}} d\omega \; K_\sigma^{(l)}(\omega) \; \rho(\omega)$$





We don't know the spectrum a priori.

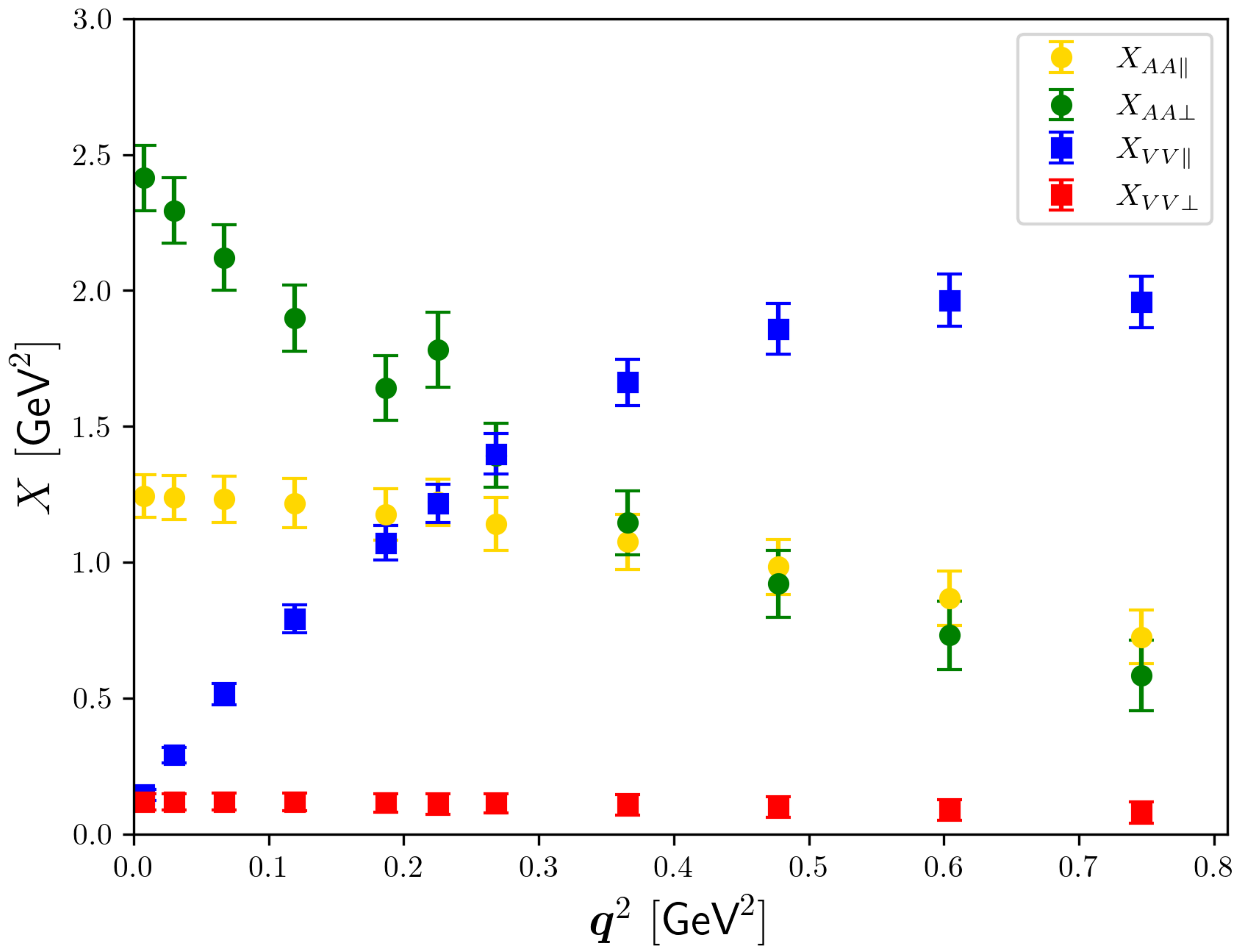
narrow smearing (σ = 0.02)



- Also, potential error from finite volume.

Details are important, … but skipped

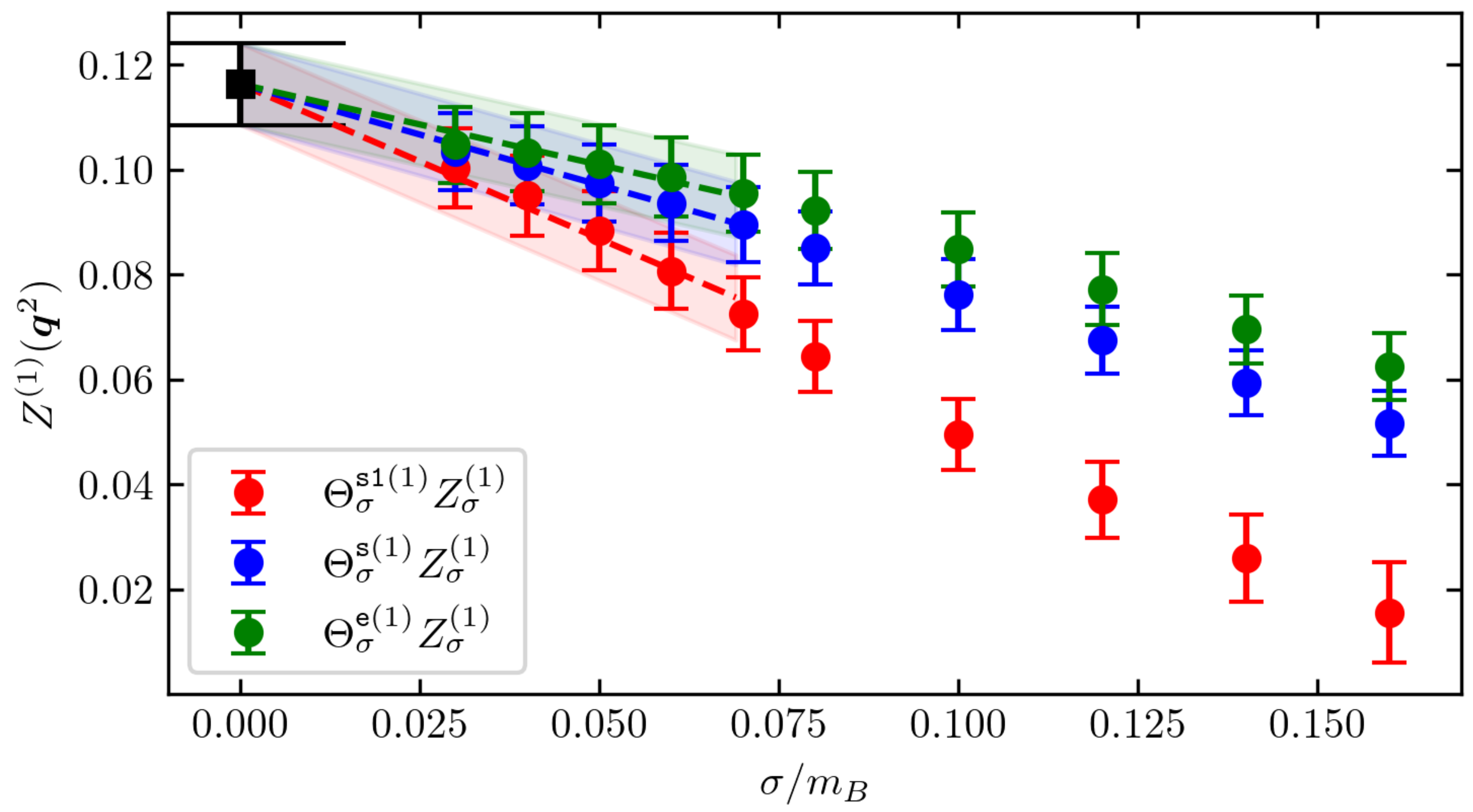
Inclusive decay rate



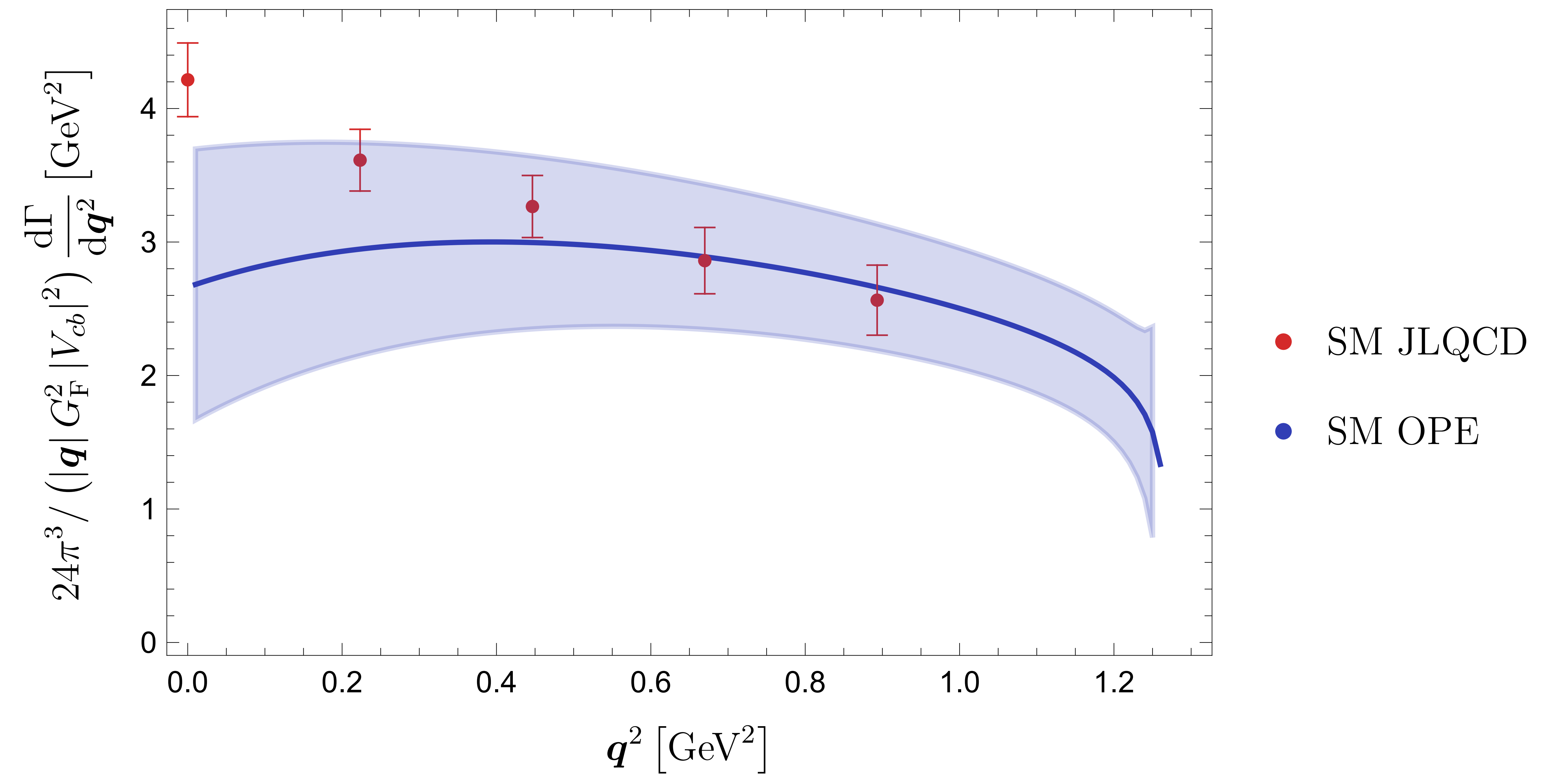
ETMC data from
Gambino et al., 2203.11762

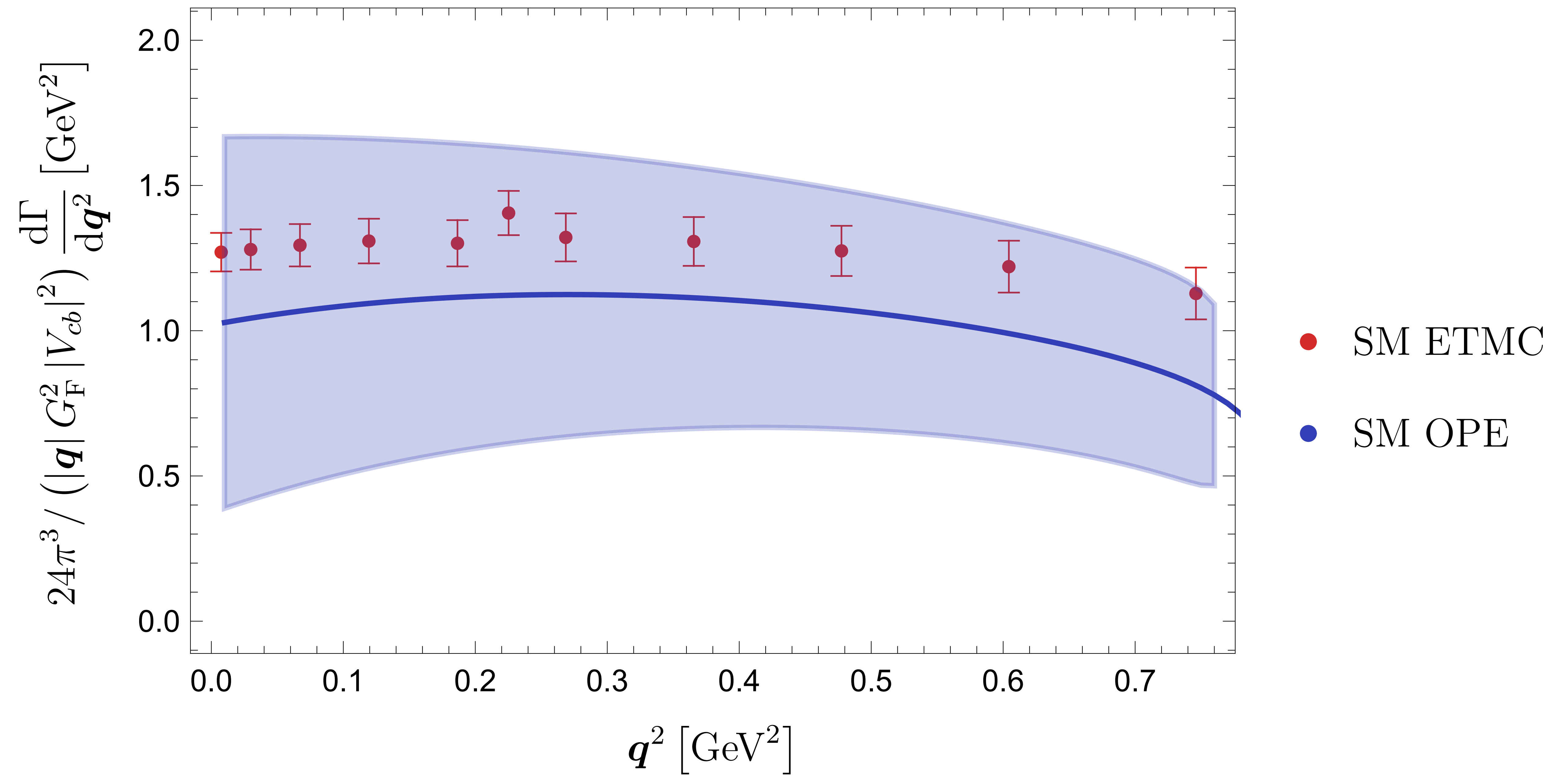
From 2203.11762
Analysis with Backus-Gilbert (by Smecca et al)

- Backus-Gilbert works equally well
- σ→0 limit is taken (with different smearings)



- calculated at many $\mathbf{q}^2$ points
- lighter b quark





From 2203.11762

OPE calculation by Gambino and Machler

- PT including $O(\alpha_s)$, OPE up to $O(1/m^3)$
- Hadronic parameters μ_π^2 etc are taken from the phono analysis.
- b quark mass is adjusted to match the lattice calculations.
- OPE breaks down near the $\mathbf{q}^2$ endpoint.

✓ Good agreement.

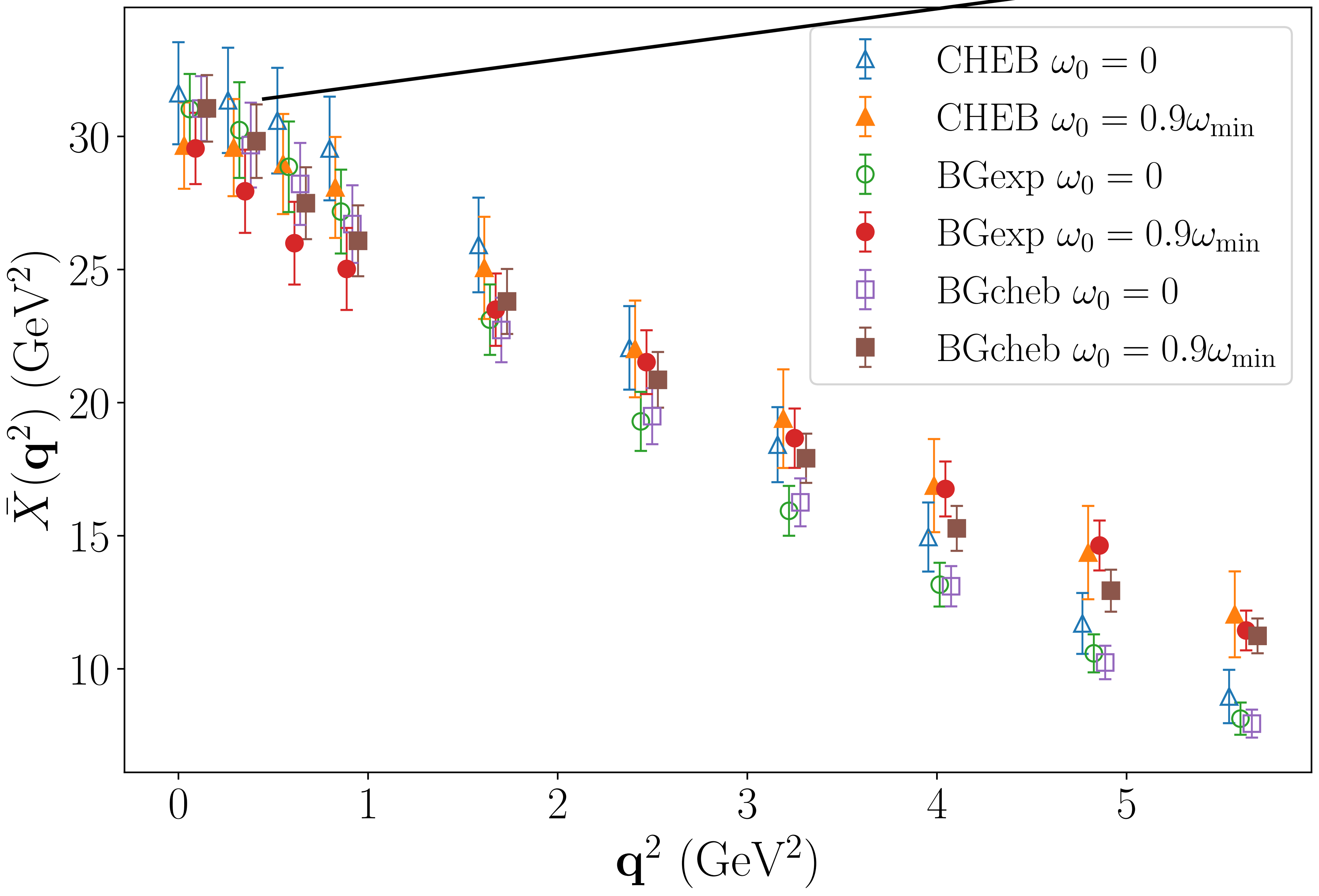
✓ Error of OPE is from the hadronic parameters. Large because of small m_b.

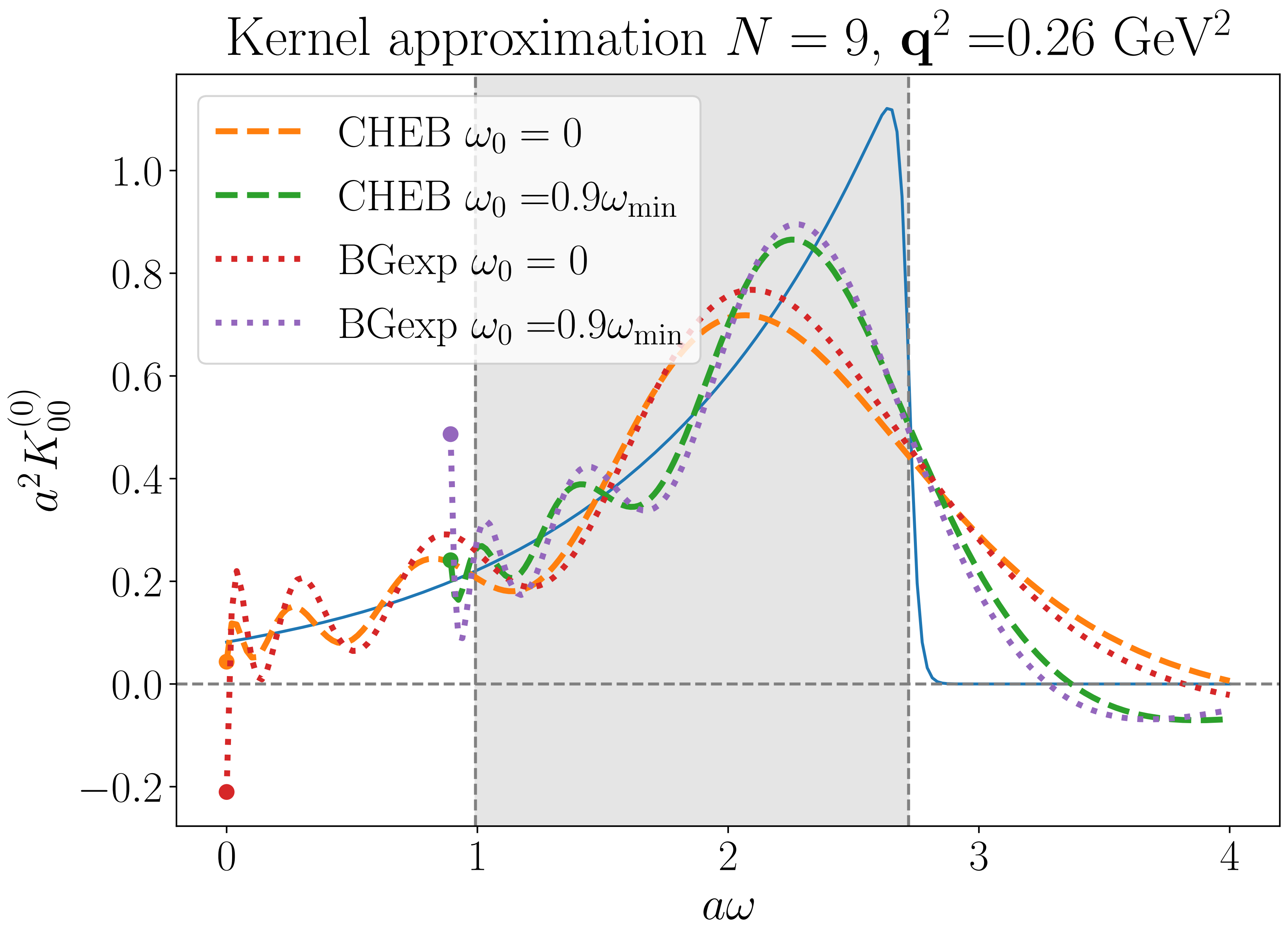
✓ Better for moments $\langle M_X^2\rangle$, $\langle E_l\rangle$, …

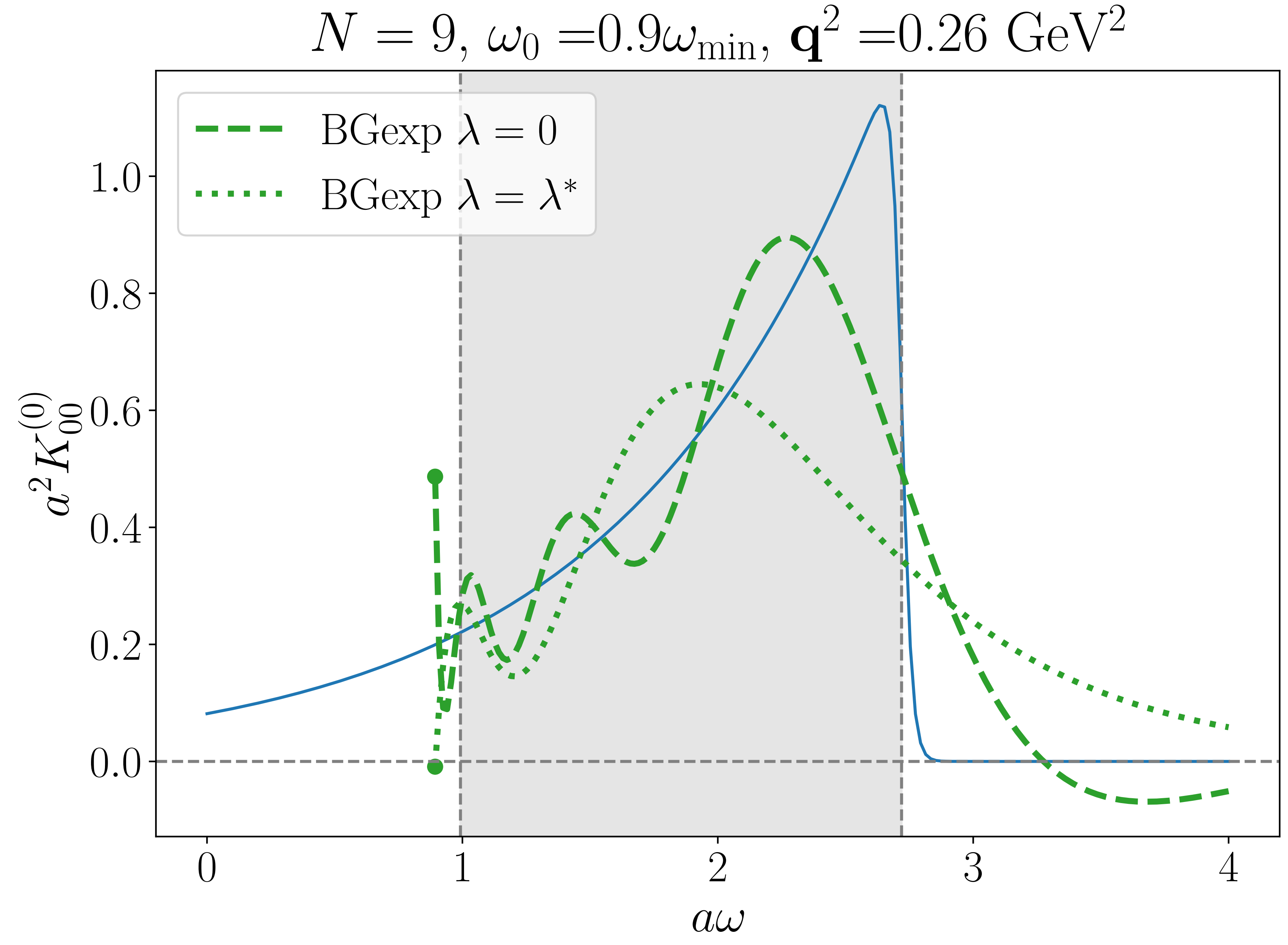
More recent works

Barone et al., 2305.14092

Further detailed study at physical b quark mass







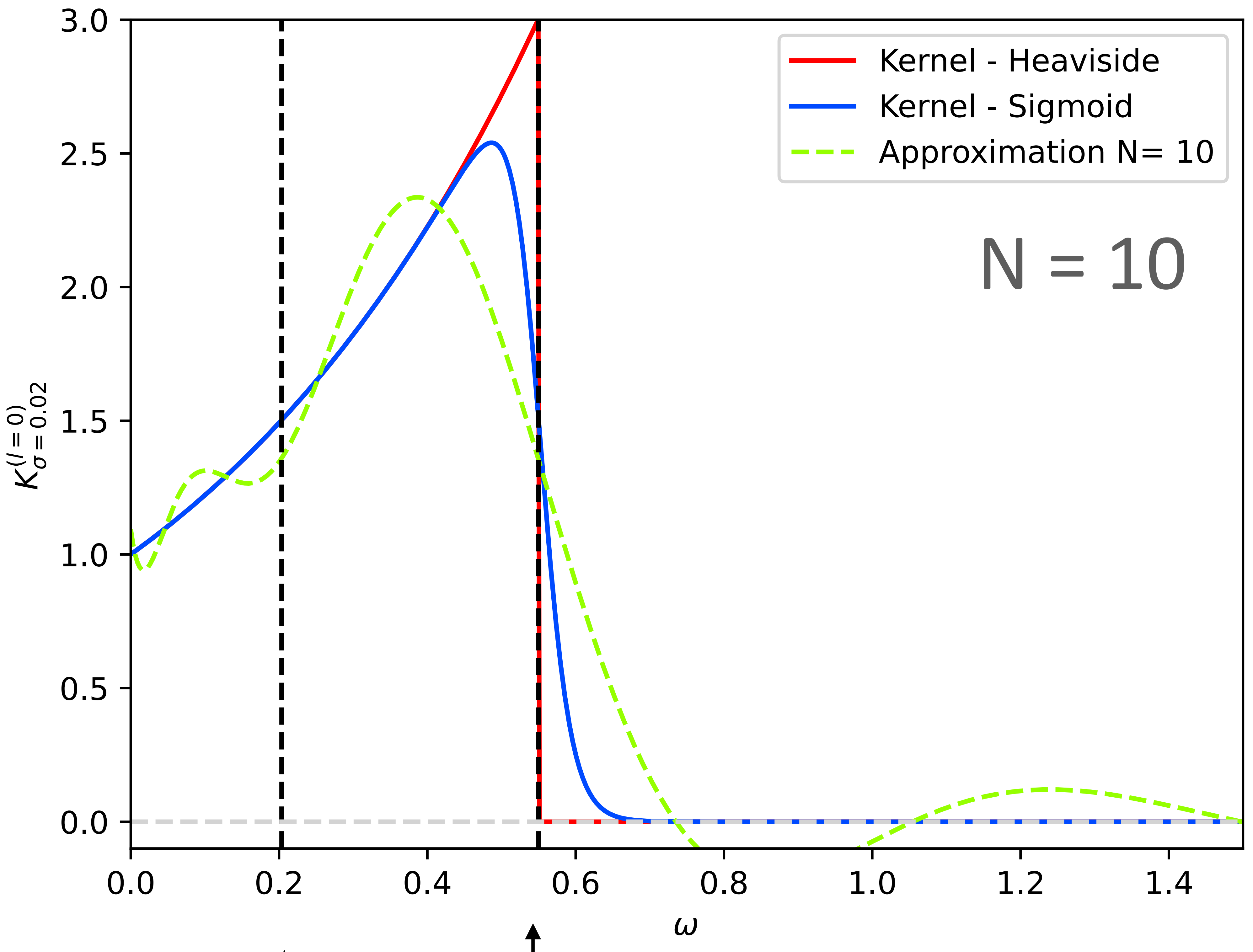
Chebyshev vs Backus-Gilbert:

- consistent at a certain value of σ

Kernel approximation: an example

$$K(\omega) \sim e^{2\omega t_0}(m_B-\omega)^l\theta(m_B-|\mathbf{q}|-\omega)$$

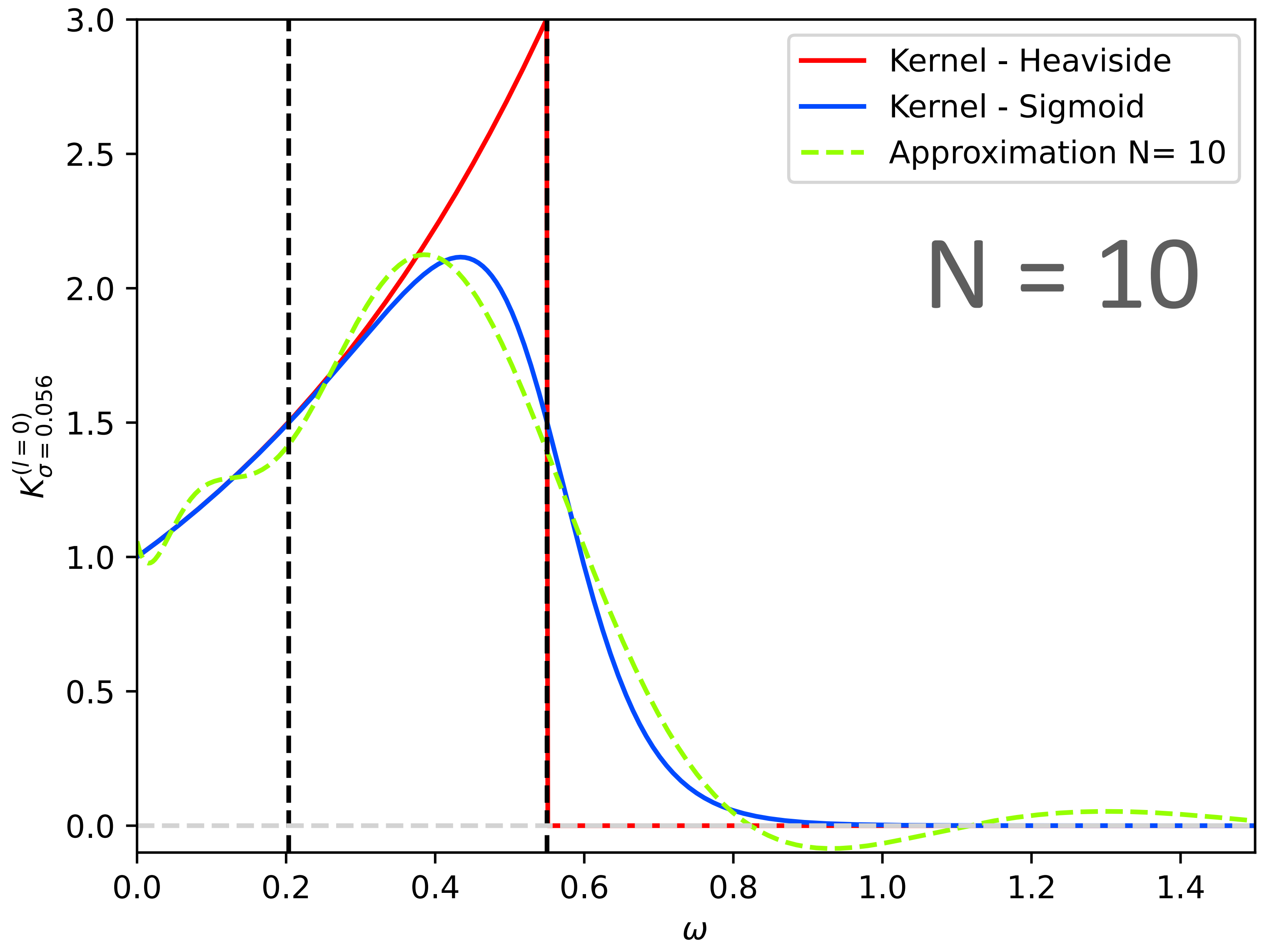
narrow smearing (σ = 0.02)



lowest energy state

upper limit

medium (σ = 0.056)

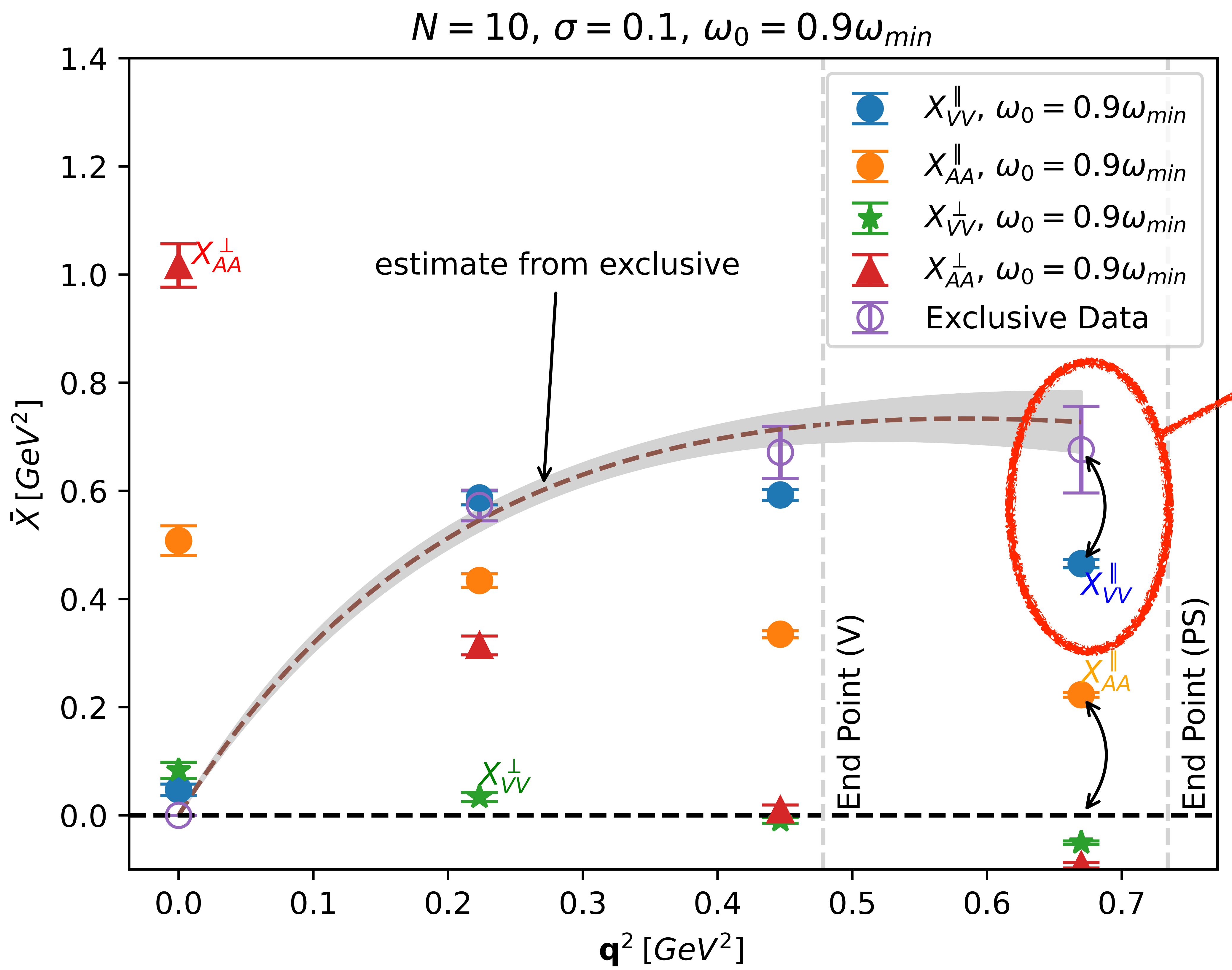


Smearing:

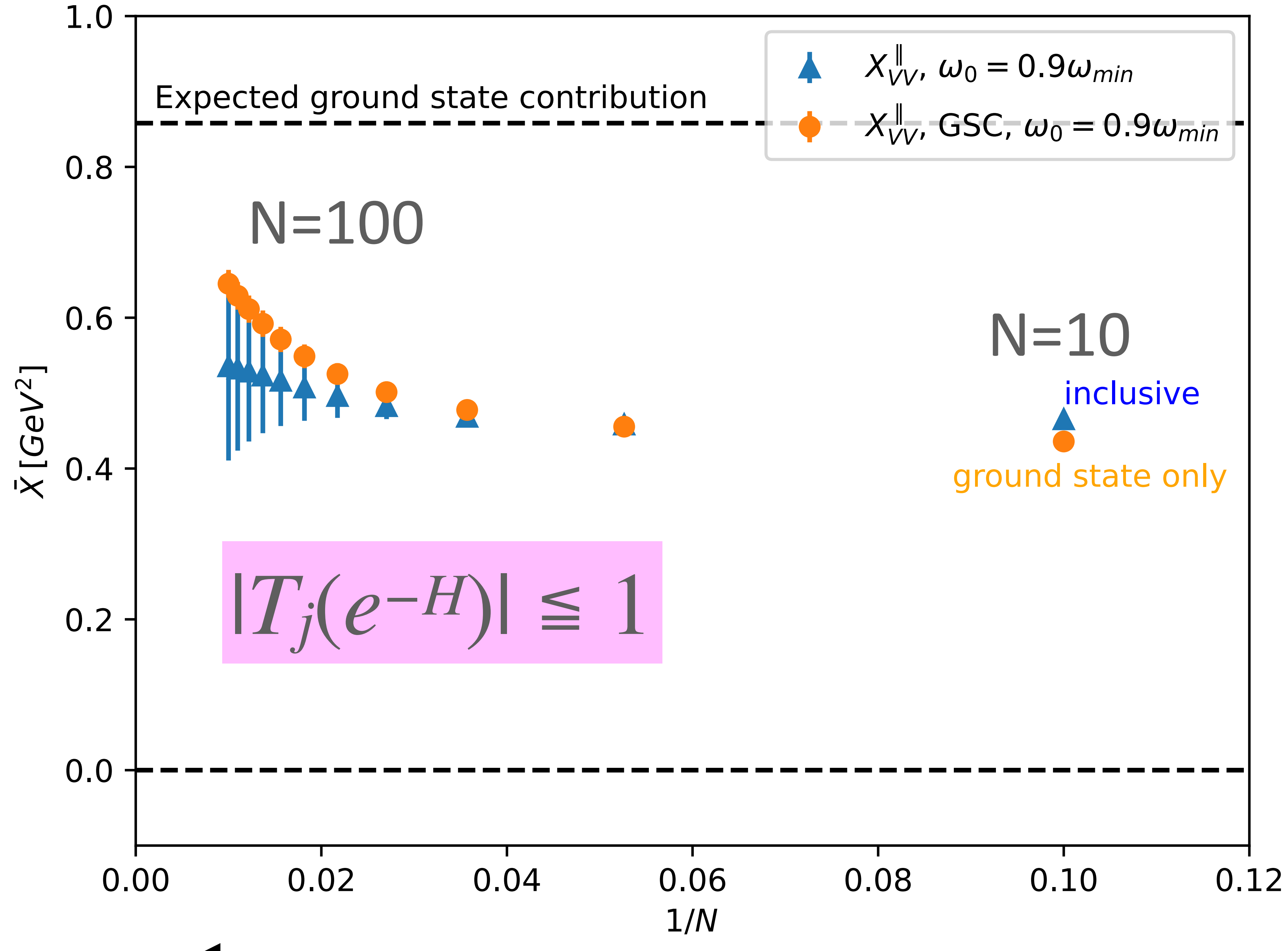
- Too wide = away from the true func
- Too narrow = bad approx

Significance of the error: the worst case

Ds decays:
Kellermann @ Lattice 2022



Error bound (Chebyshev)



increasing order of poly with σ = 1/N

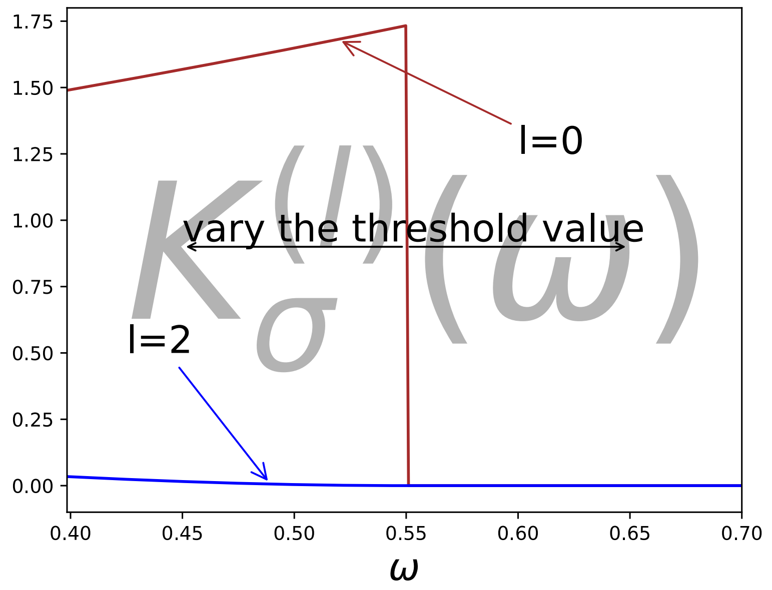
Don't worry. This region is exclusively given by the ground state, anyway.

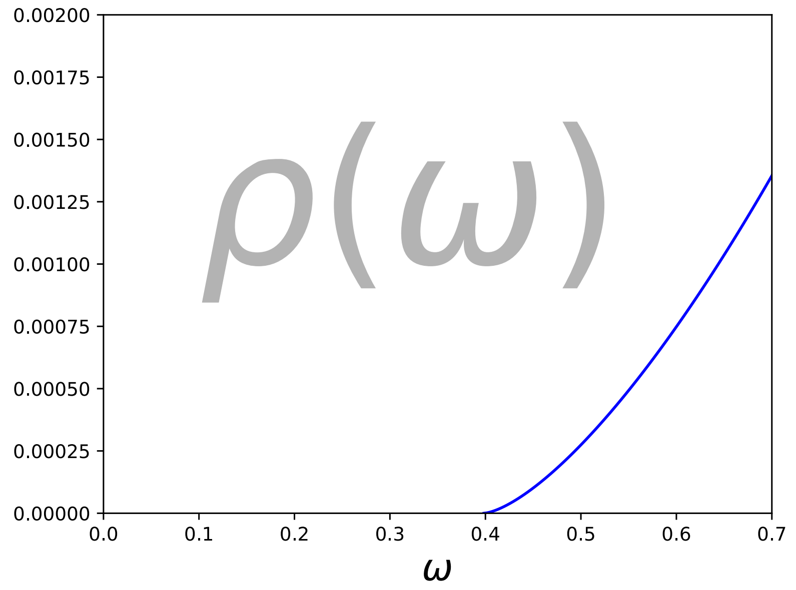
Finite volume effect

Kellermann @ Lattice 2023

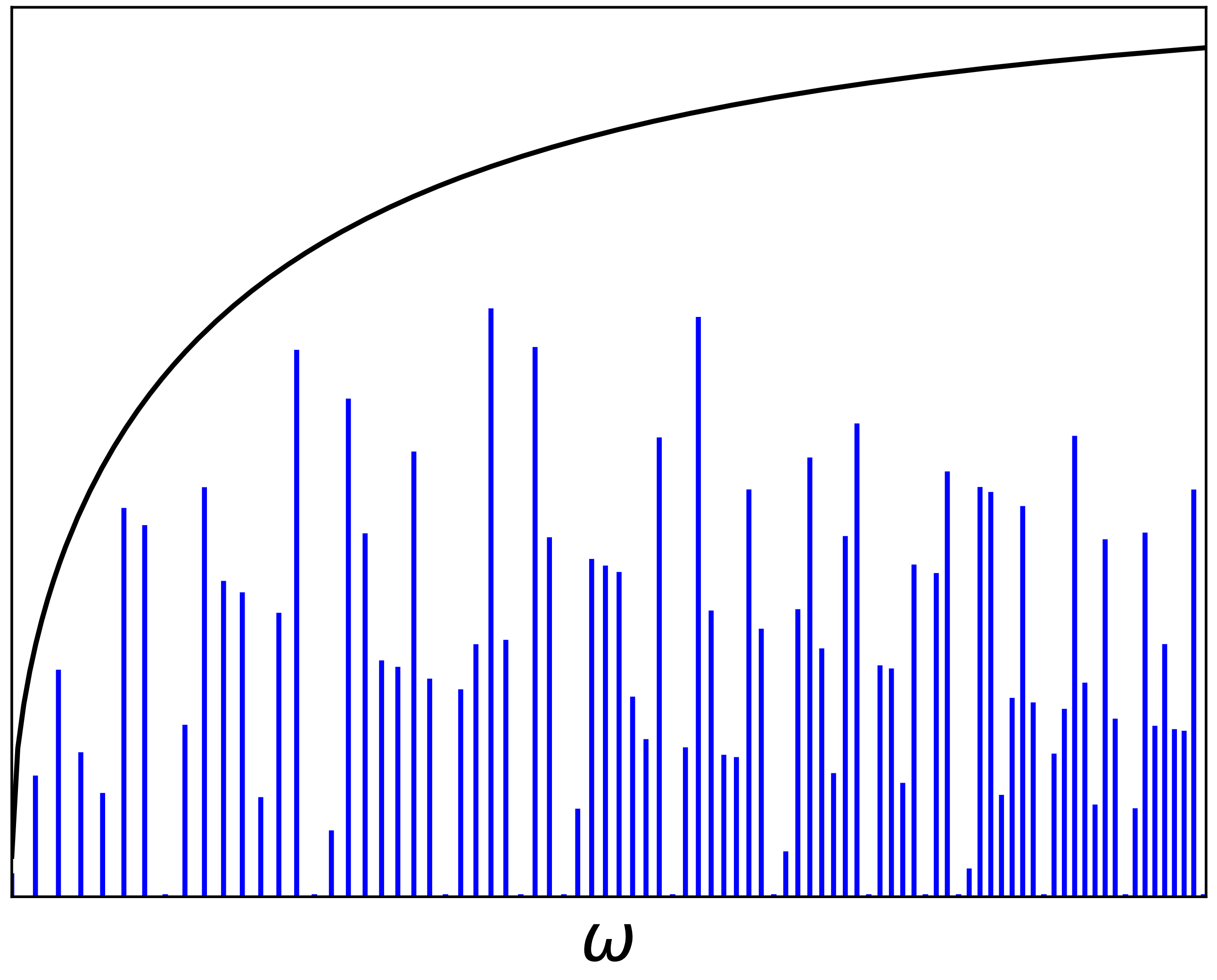
Study with varying upper limit

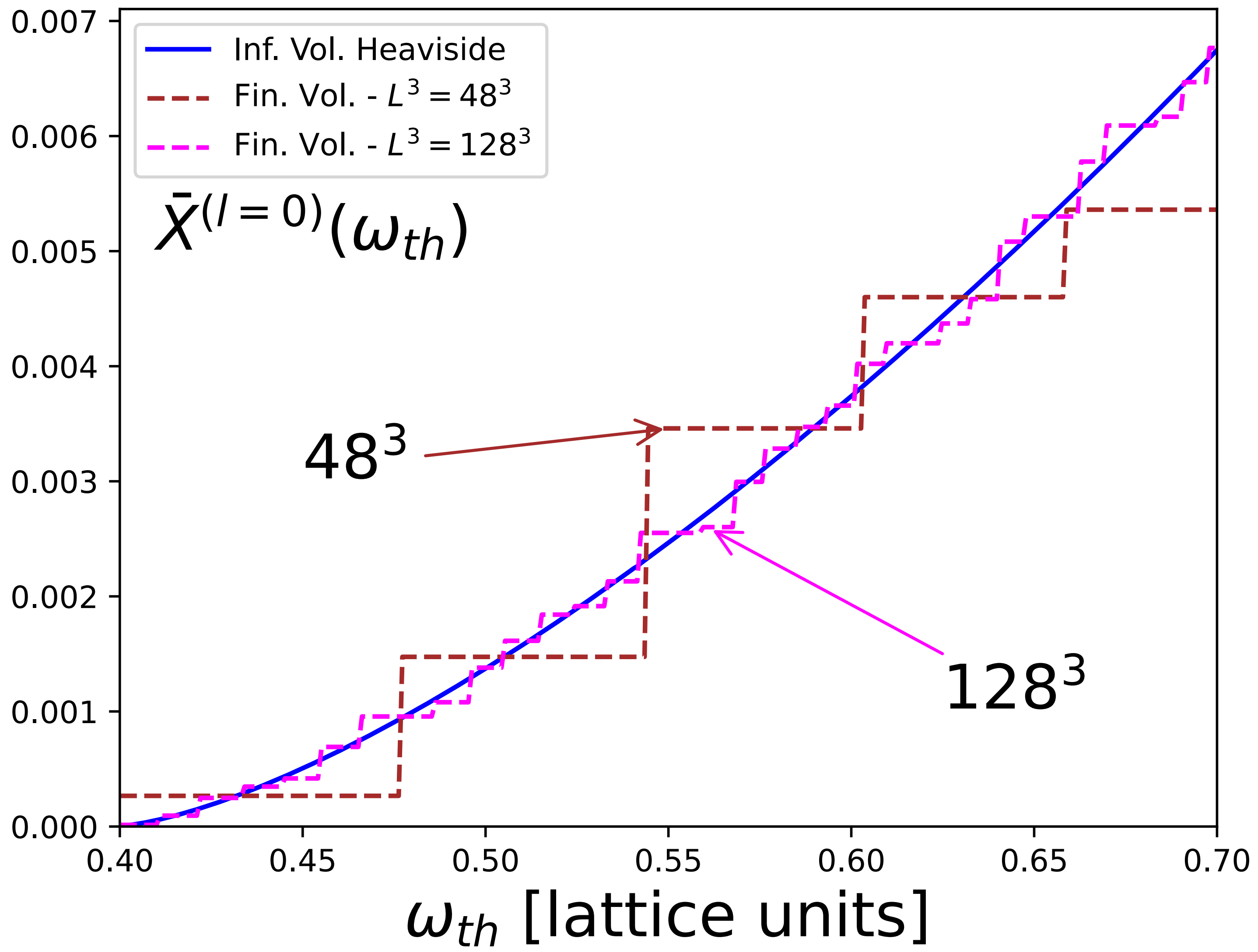
$$\bar{X}^{(l)}(\omega_{th}) \sim \int_0^{\omega_{th}} d\omega$$





(two-body) spectrum is discrete → Integral may depend strongly on the volume





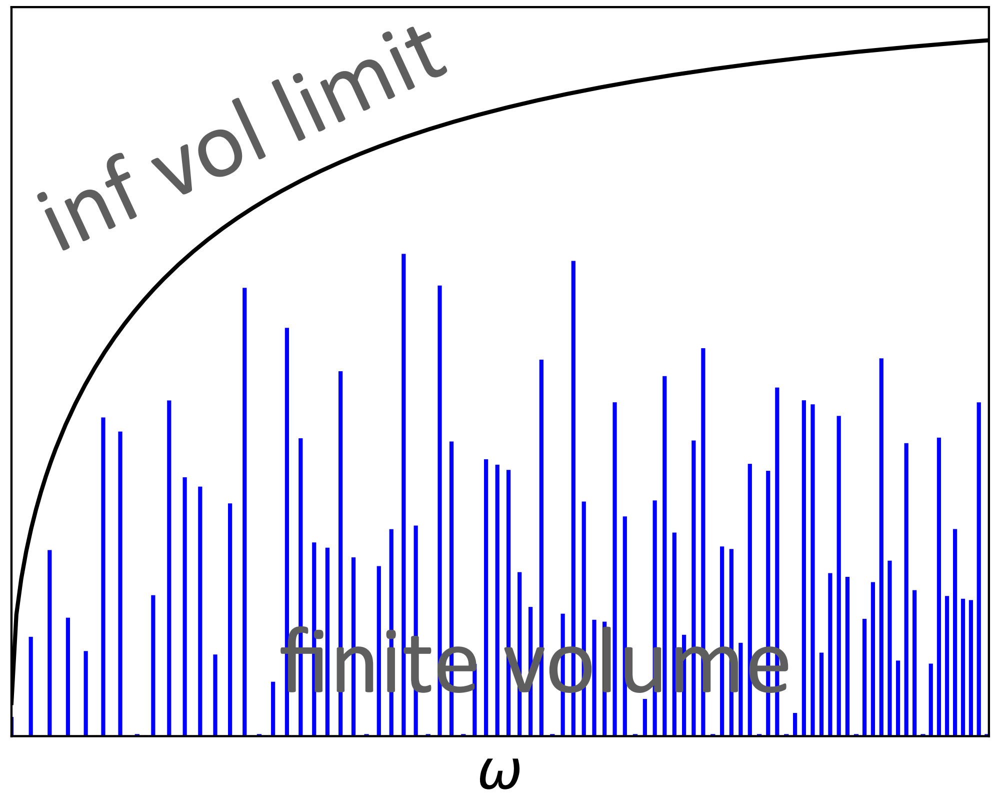
A model for two-body states:

$$[\,]_{\text{Diagramm}} \sim i\int \frac{d^4q}{(2\pi)^4}\,\frac{1}{(p+q)^2-m^2+i\epsilon}\,\frac{1}{q^2-m^2+i\epsilon}$$

Im

$$\pi\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\,\frac{1}{\left(2\sqrt{m^2+\boldsymbol{q}^2}\right)^2}\,\delta\!\left(p_0-2\sqrt{m^2+\boldsymbol{q}^2}\right)$$

$$\rho(\omega)=\frac{1}{16\pi}\sqrt{1-\frac{4m^2}{\omega^2}}$$

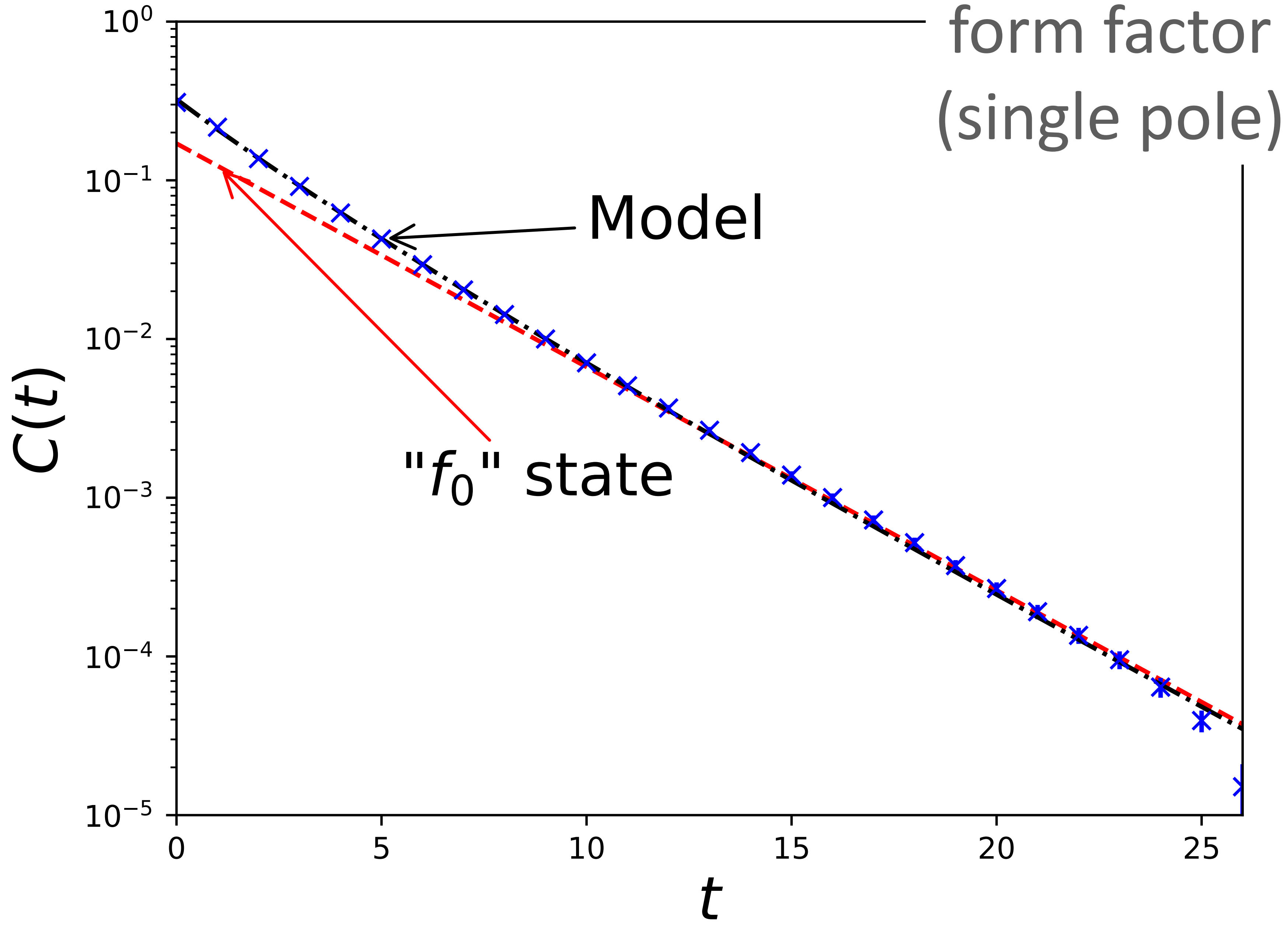


finite volume

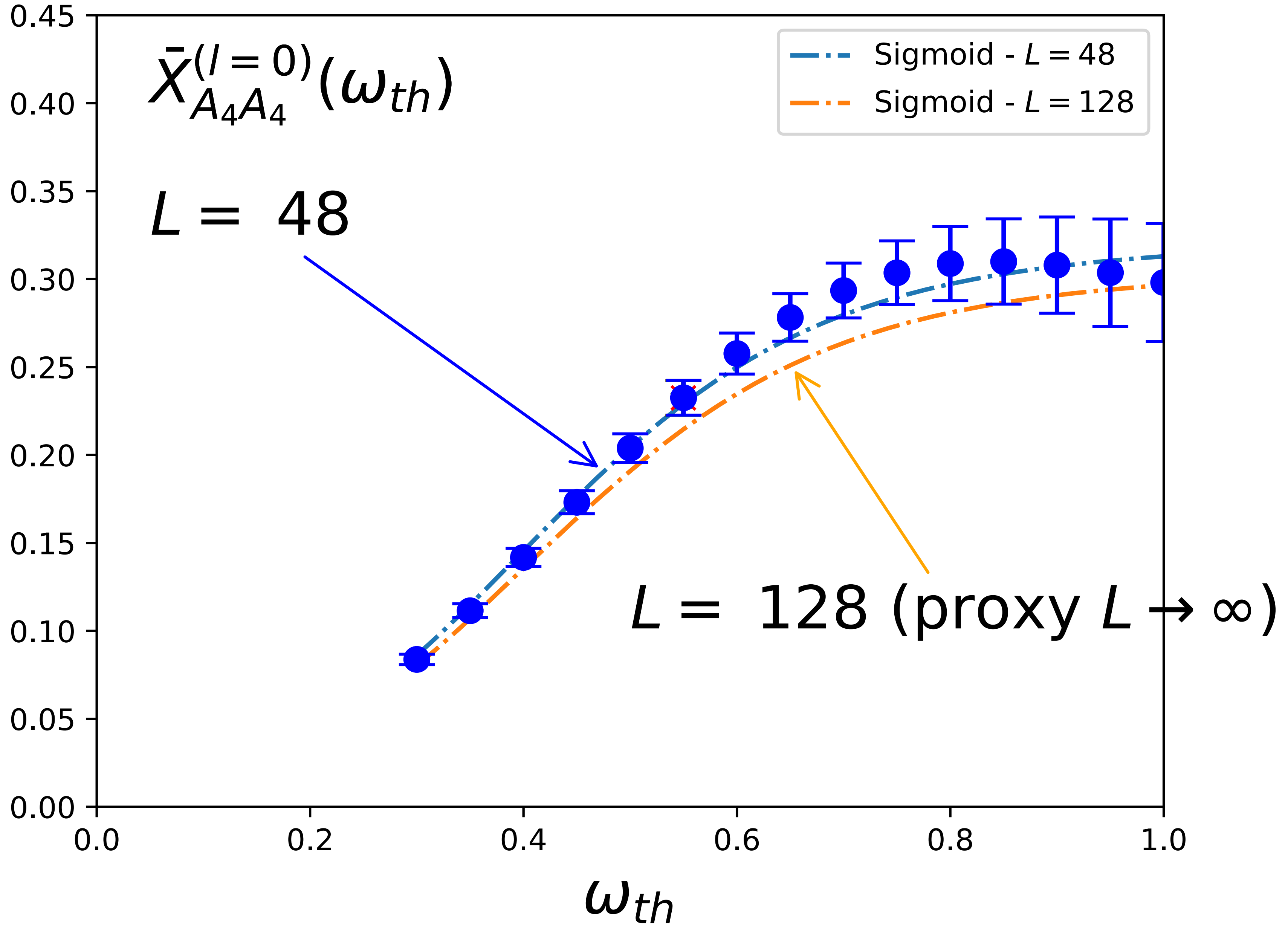
model

$$C(t)=A_0e^{-E_0t}+s\left[\sum_i A_ie^{-E_it}F(E_i)\right]$$

form factor (single pole)



estimate of L → ∞

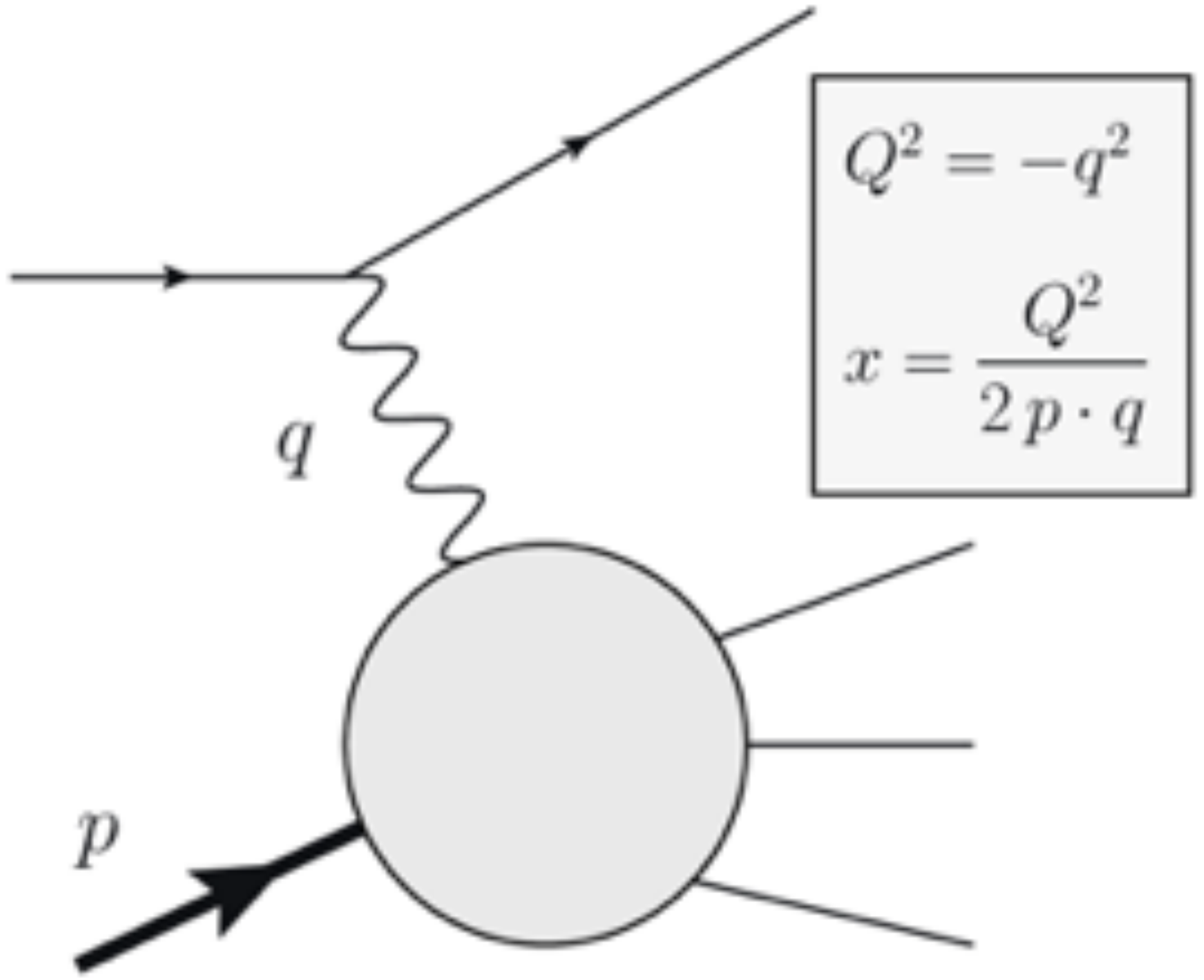


finite volume effect marginal

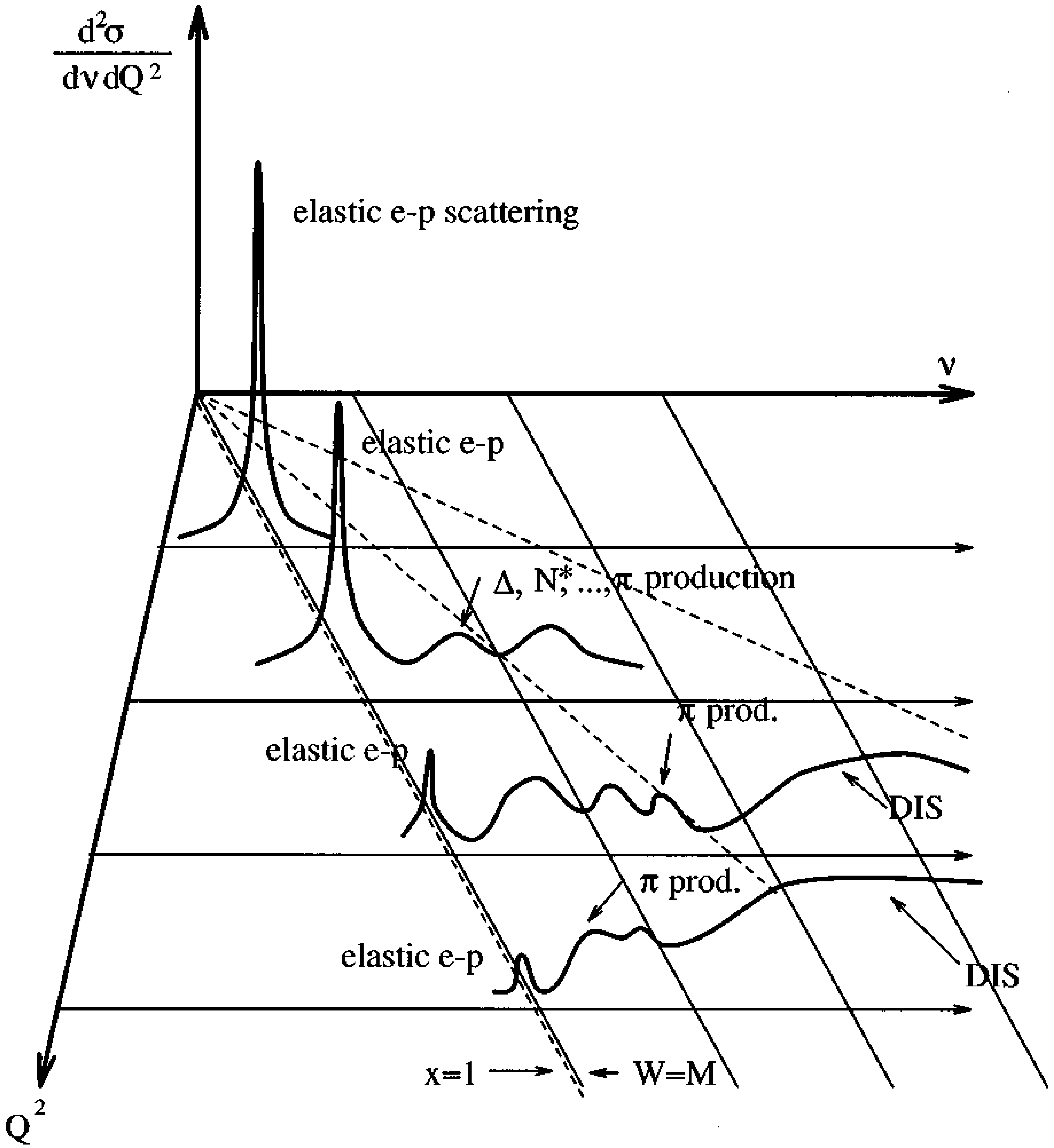
Another application:
(deep) inelastic scattering

see also, QCDSF, PRL 118, 242001 (2017)

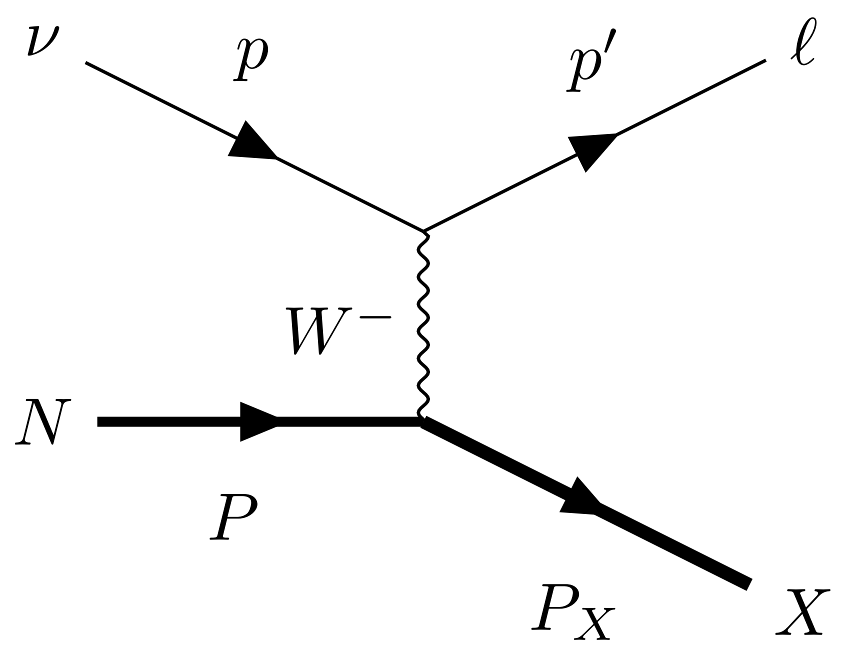
$$e^- N \to e^- X$$

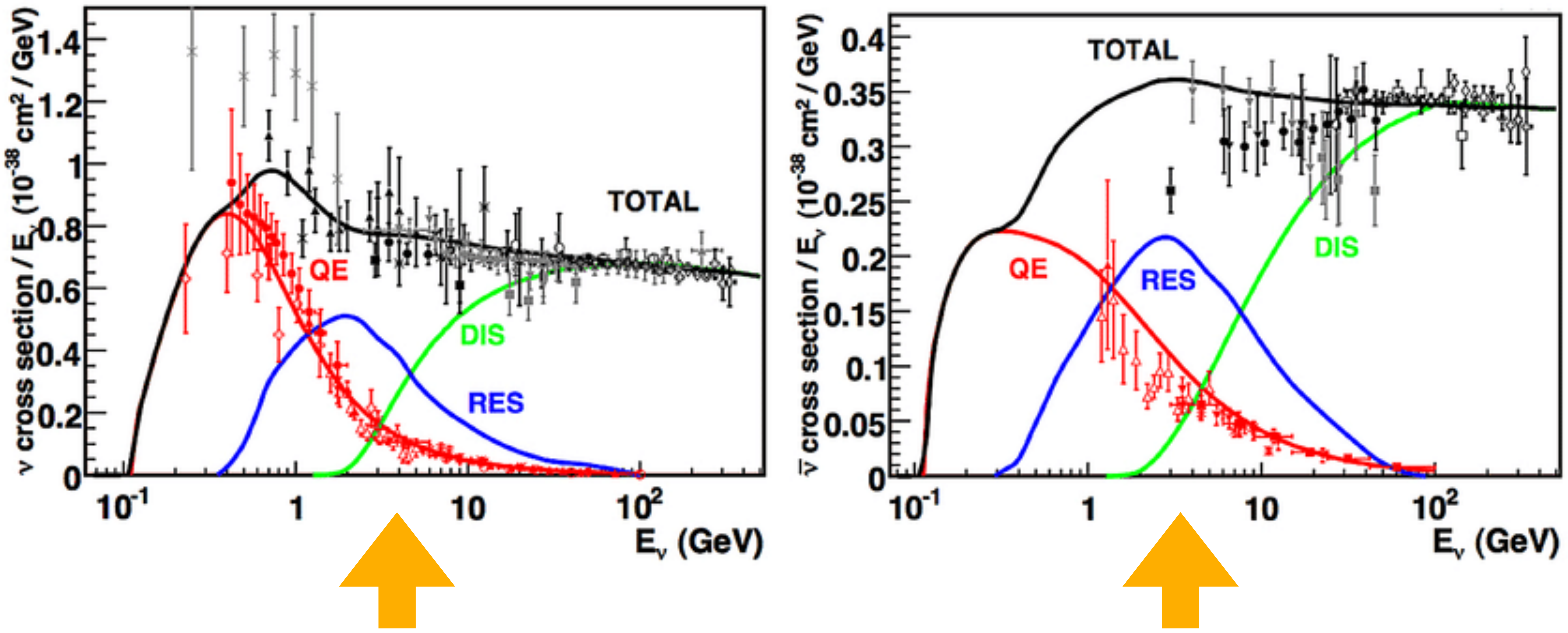


$Q^2 = -q^2$
$x = \frac{Q^2}{2\,p \cdot q}$
q
p



$\frac{d^2\sigma}{d\nu\, dQ^2}$
ν
Q^2
elastic e-p scattering
elastic e-p
Δ, N*, ..., π production
π prod.
elastic e-p
DIS
π prod.
elastic e-p
DIS
x=1
W=M





The energy region relevant for T2K.

Not simply elastic, nor DIS.

(deep) (in)elastic scattering

structure function:

$$W_{\mu\nu} = \frac{1}{2}\sum_{\text{pol}}\sum_{X}\langle N(p)|J_\mu|X(p_X)\rangle\langle X(p_X)|J_\nu|N(p)\rangle$$

$$\times\,(2\pi)^3\delta^{(4)}(p - p_X + q)$$

optical theorem $W_{\mu\nu} = \frac{1}{\pi}\mathrm{Im}T_{\mu\nu}$

forward-scattering amplitude

$$T_{\mu\nu} = \frac{1}{2}\sum_{\text{pol}}\int d^4x\, e^{iqx}\langle N(p)|T\{J_\mu(x)J_\nu(0)\}|N(p)\rangle$$

Calculate on the lattice?

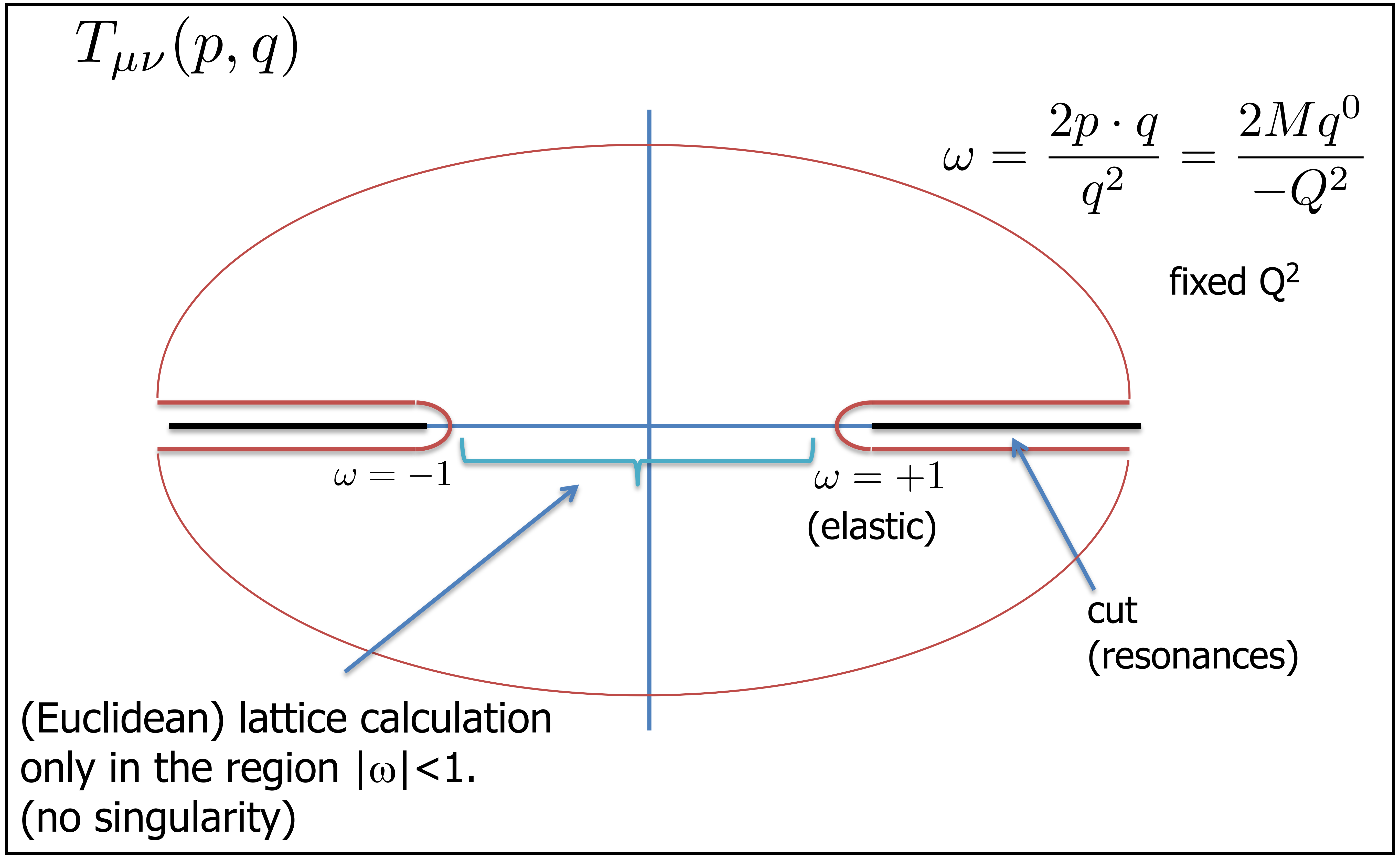
Accessible on the lattice:

$$M_{\mu\nu}(t) \equiv \int d^3\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}} \langle N(\mathbf{p})|J_\mu(\mathbf{x},t)J_\nu(\mathbf{0},0)|N(\mathbf{p})\rangle$$

then,

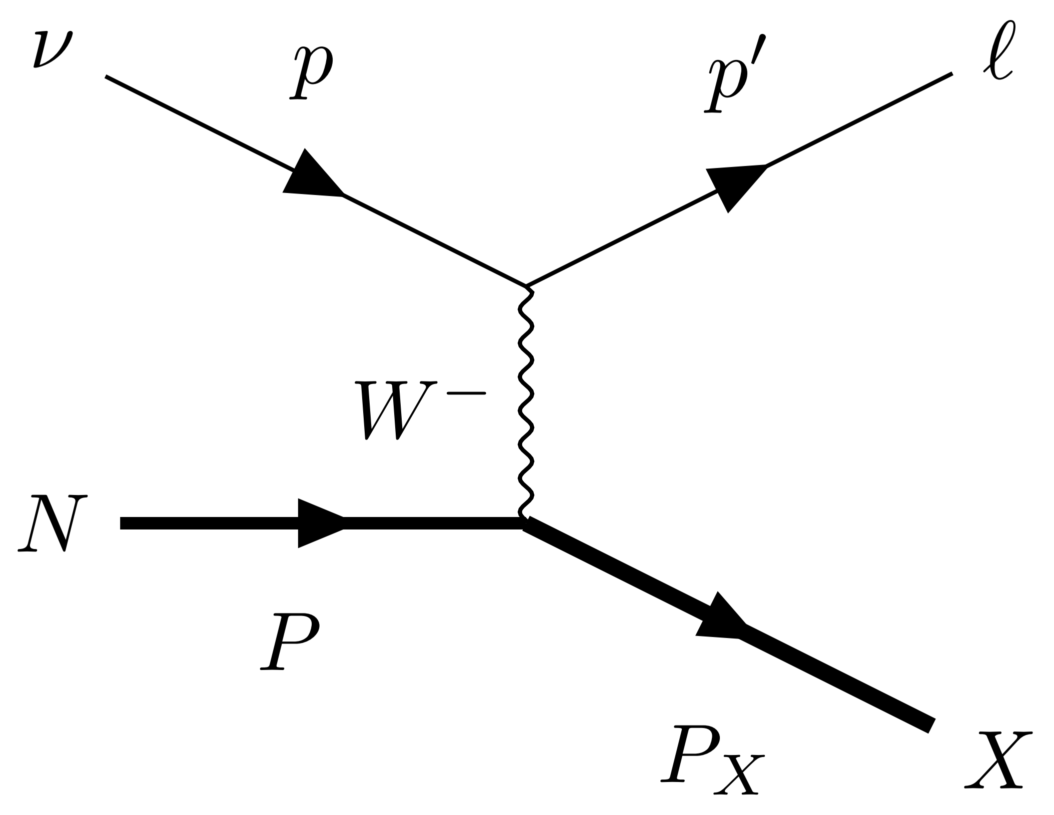
$$T_{\mu\nu}(p,q) = \int_0^\infty dt\, e^{q^0 t} M_{\mu\nu}(t)$$

Possible only for unphysical kinematics. May be related to exp through Cauchy's integral. Or, … (see below)



Total cross section = smeared spectrum

H. Fukaya, T. Kaneko, SH, H. Ohki, Phys. Rev. D102, 11 (2010); arXiv:2010.01253.



Total cross section:

kinematical factor determined by the leptonic part

$$\sigma \propto \int_0^{E^2} d\mathbf{q}^2 \int_{\sqrt{m_N^2+\mathbf{q}^2}}^{m_N+\sqrt{\mathbf{q}^2}} d\omega\, K(\mathbf{q}^2,\omega)\langle N|\tilde{J}^\dagger(\mathbf{q})\delta(\omega-\hat{H})\tilde{J}(\mathbf{q})|N\rangle$$

matrix element for a state with a fixed energy

integral over energy and momentum of X

the same story…

The devil is in the details

- Still in the early stage. Concerning the errors, I am optimistic, but more studies are necessary for various kinematical setups.

- Real calculation of $B \to X_c$, X_u at physical masses still to be done.

- Many potential applications

  - D and B. Not just total rate, but moments, e.g. $\langle M_X^2 \rangle$, $\langle E_l \rangle$

  - Comparison with OPE, then to determine MEs (see 2203.11762)

  - Borel sum (as in the SVZ sum rule; see Ishikawa-SH, 2103.06539)

  - lepton-nucleon scattering, not-so-deep inelastic scattering

So, what happened to duality?

Not an assumption

- Rather, a question of the ability to calculate reliably.
- pQCD + OPE is useful once sufficiently smeared (like the Borel transform). Question remains: truncation?
- Fully non-perturbative by LQCD. Systematic errors can be controlled rigorously. The smearing is arbitrary in principle. (In practice? Need detailed studies.)

Jets, hadronization, … for LHC?

- Without smearing, the assumption is back.
- Large momentum is a stumbling block on the lattice, yet. Go quantum?