Non-Linearities in Black Hole Ringdowns

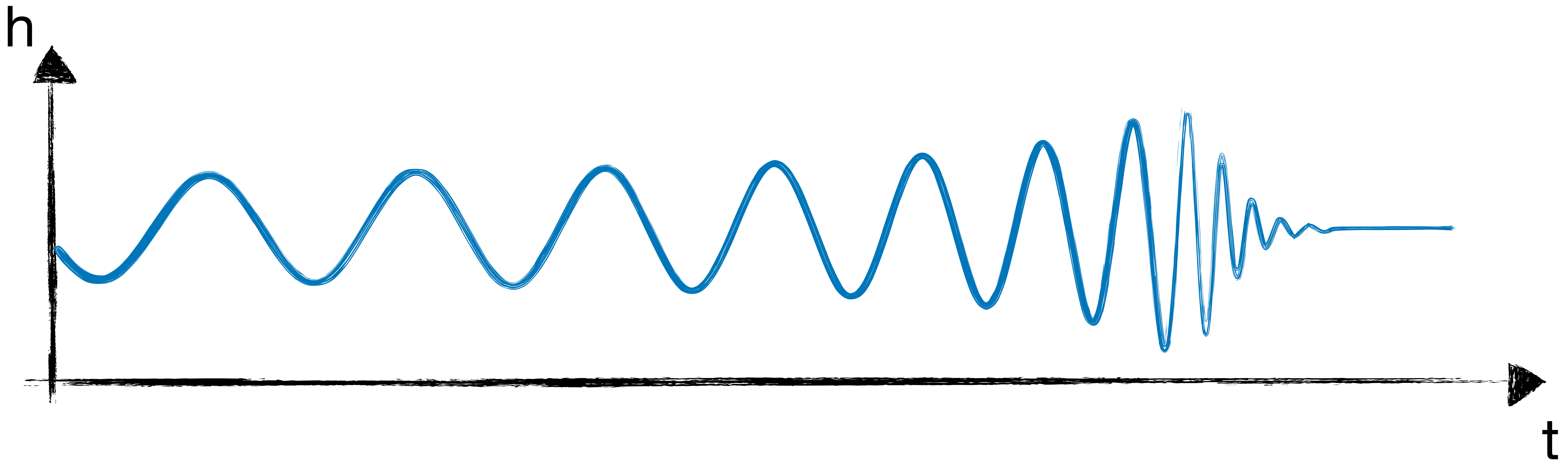
Explained from symmetry

Based on the work arXiv:2301.09345,
In collaboration with A. Kehagias, A. Riotto, F. Riva

CAGE BSM Workshop, CERN, 13 February 2024
Davide Perrone, University of Geneva

Gravitational Wave signal

What we see from a coalescence event



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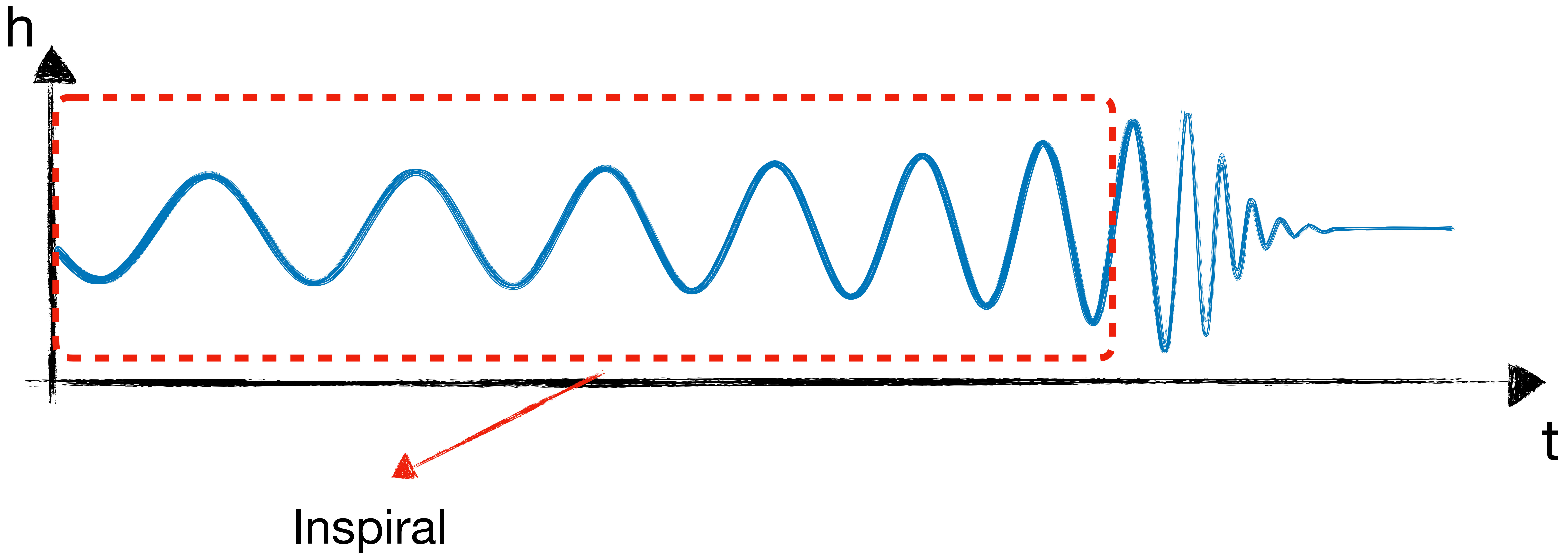

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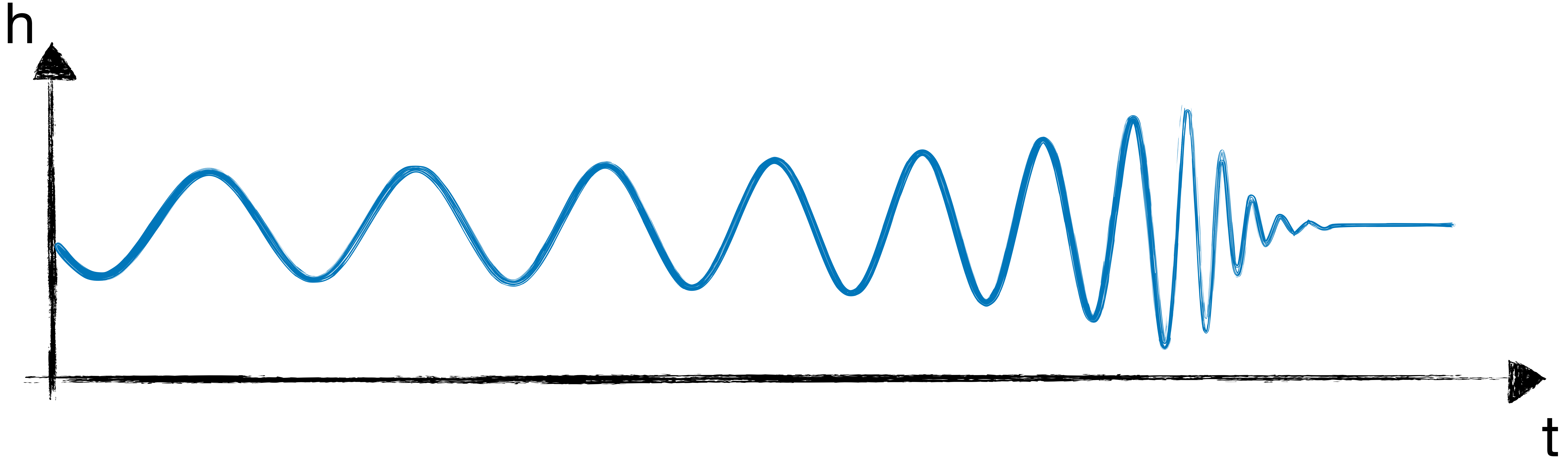

Inspiral

Gravitational Wave signal

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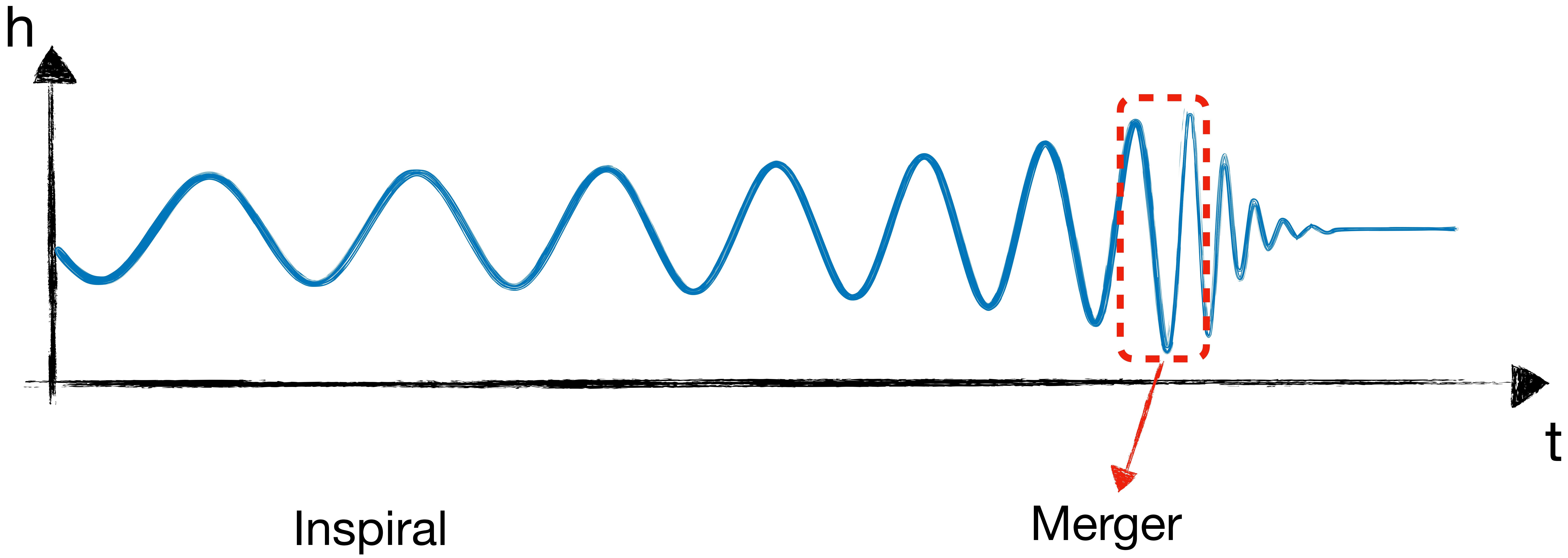

Gravitational Wave signal

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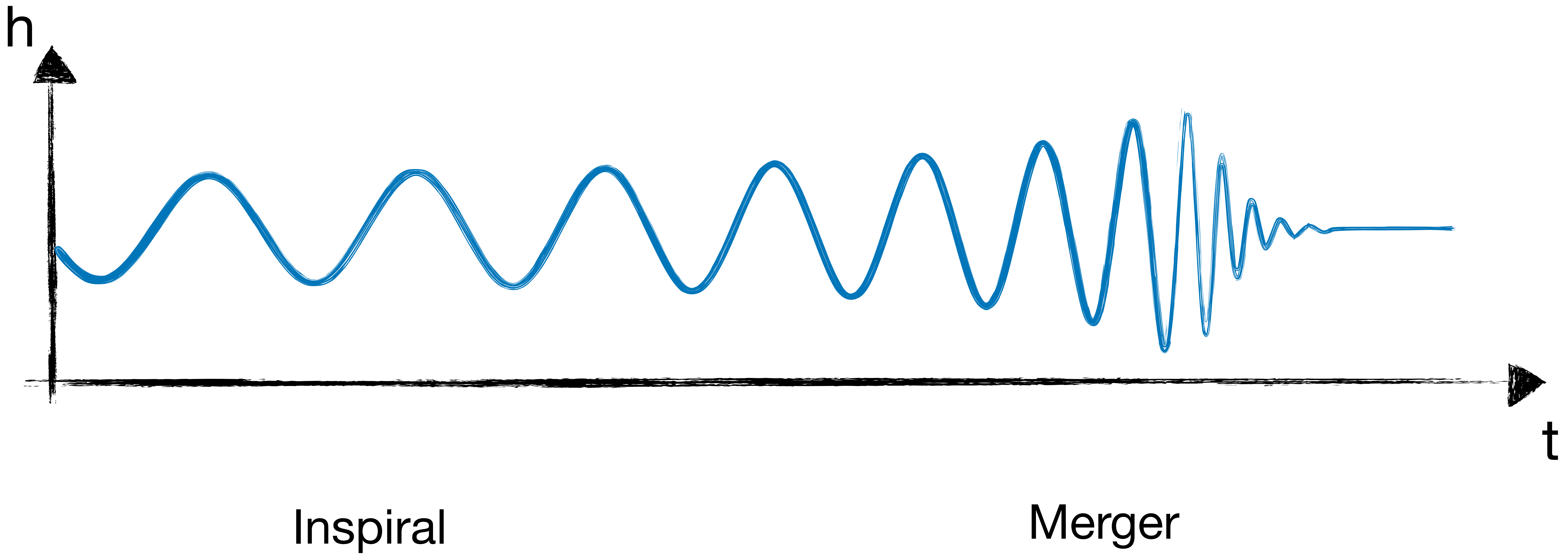

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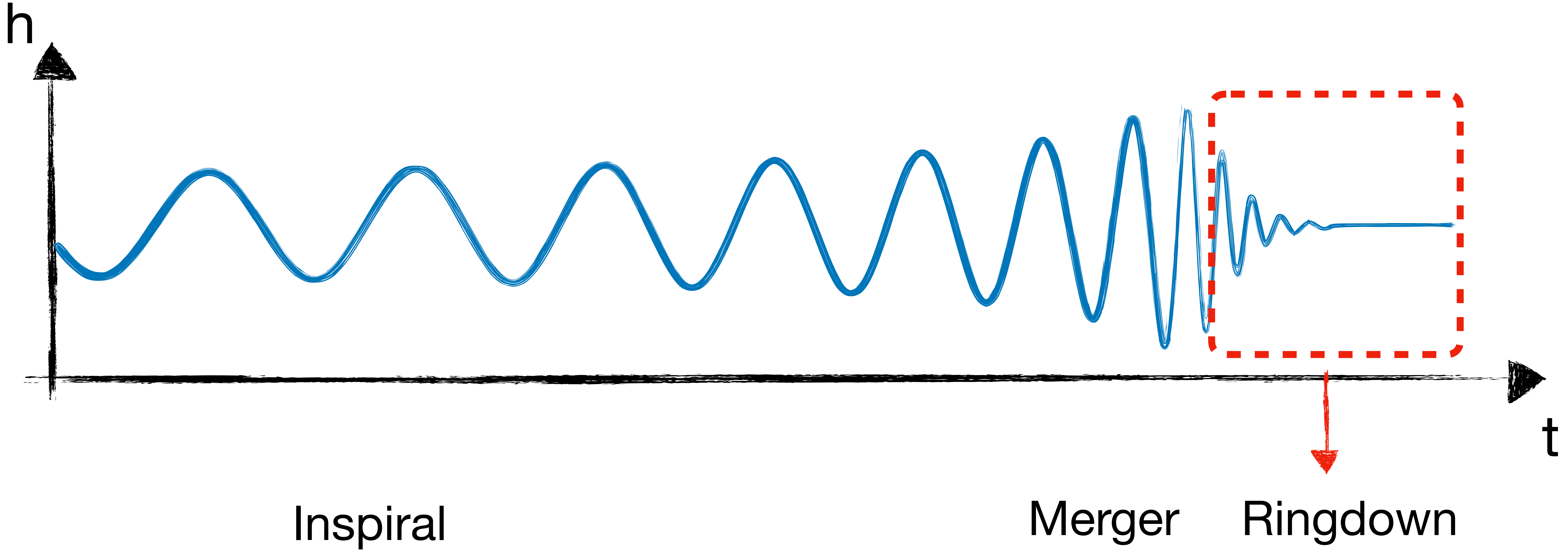

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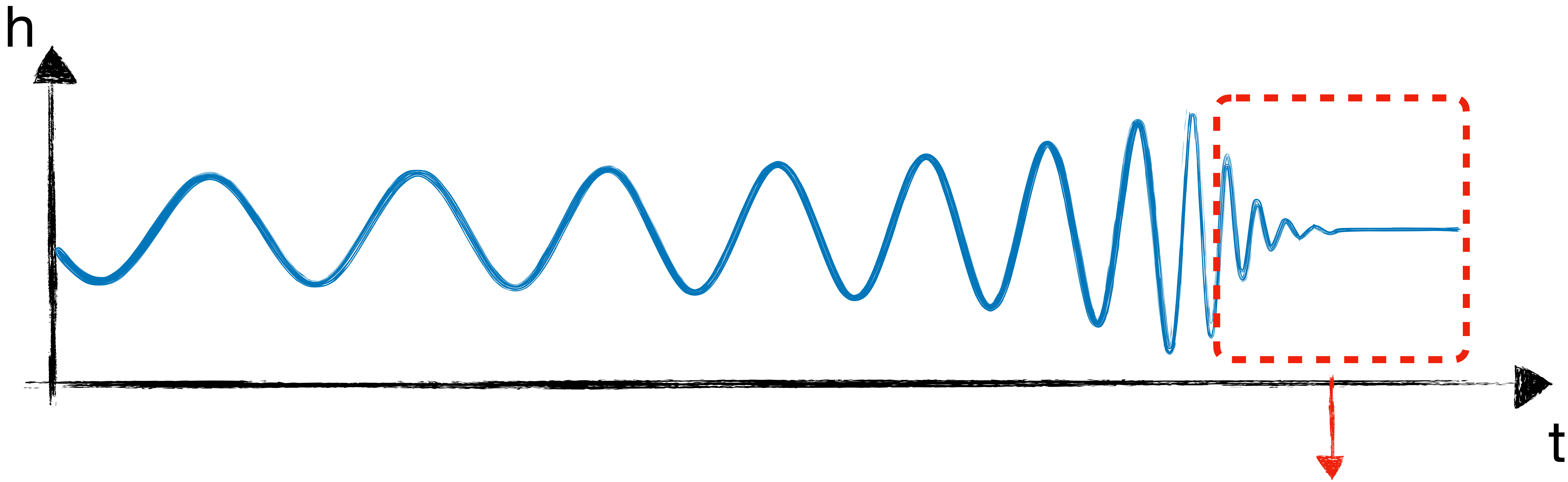


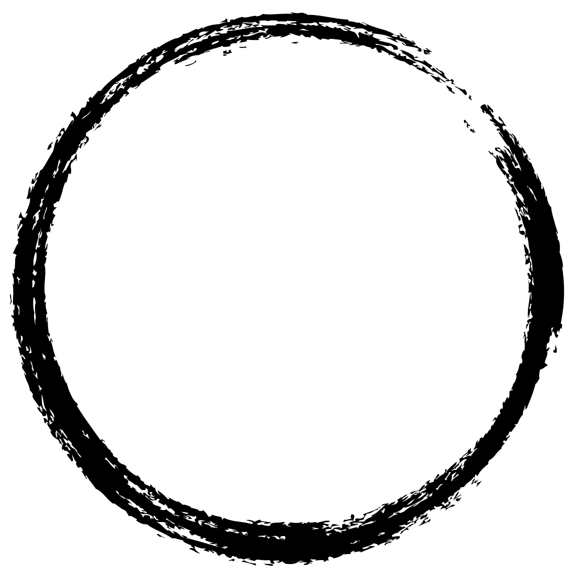
Inspiral
Dimitrios Talk from CAGE 1

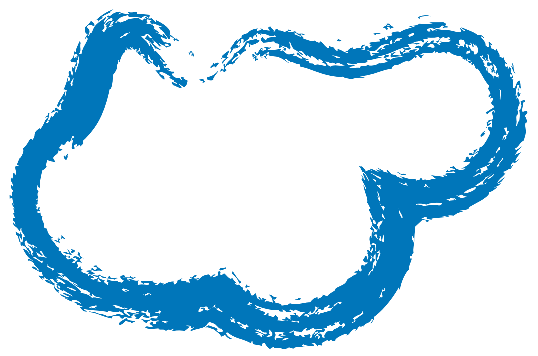
Merger

Ringdown
Our target

Ringdown, sum of Quasi-Normal Modes

Main component of the signal





$$\text{Ringdown} \sim \sum_n \text{QNM}_n$$

They dominate the Ringdown phase

Ringdown, sum of Quasi-Normal Modes

Main component of the signal

$$\text{Ringdown} \sim \sum_n \text{QNM}_n$$

They dominate the
Ringdown phase

Kerr Black Hole

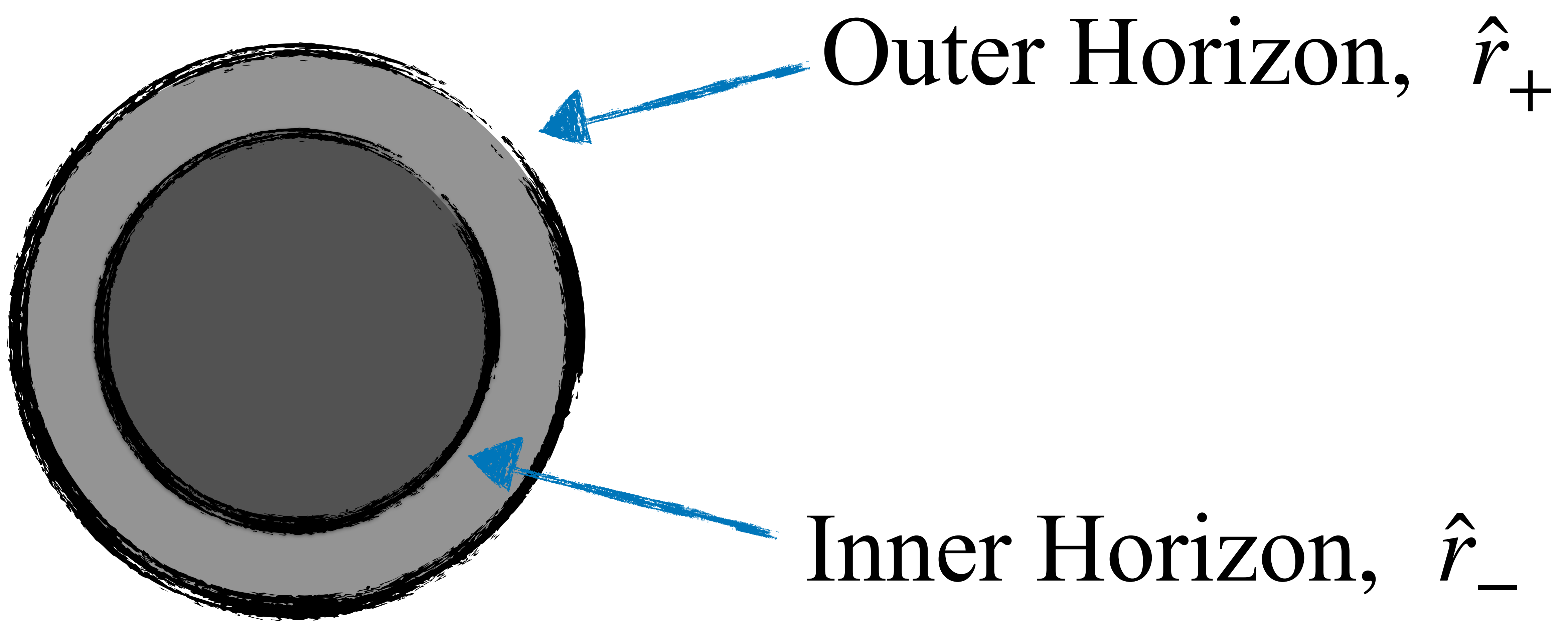
Spin J and mass M

$$a = J/M$$

Horizons $\hat{r}_{\pm} = M \pm \sqrt{M^2 - a^2}$

Temperature $T_H = \dfrac{\hat{r}_+ - \hat{r}_-}{8\pi M \hat{r}_+}$

Angular velocity $\Omega_{BH} = \dfrac{a^2}{2M\hat{r}_+}$



Linear Quasi-Normal modes

Perturbing around the background

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad |h| \ll 1$$

→ Write the perturbation in scalar variables

$$\left(\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \left(\omega^2 - V_{eff}(x)\right)\right)\psi_{l,m}(x) = 0$$

$$+$$

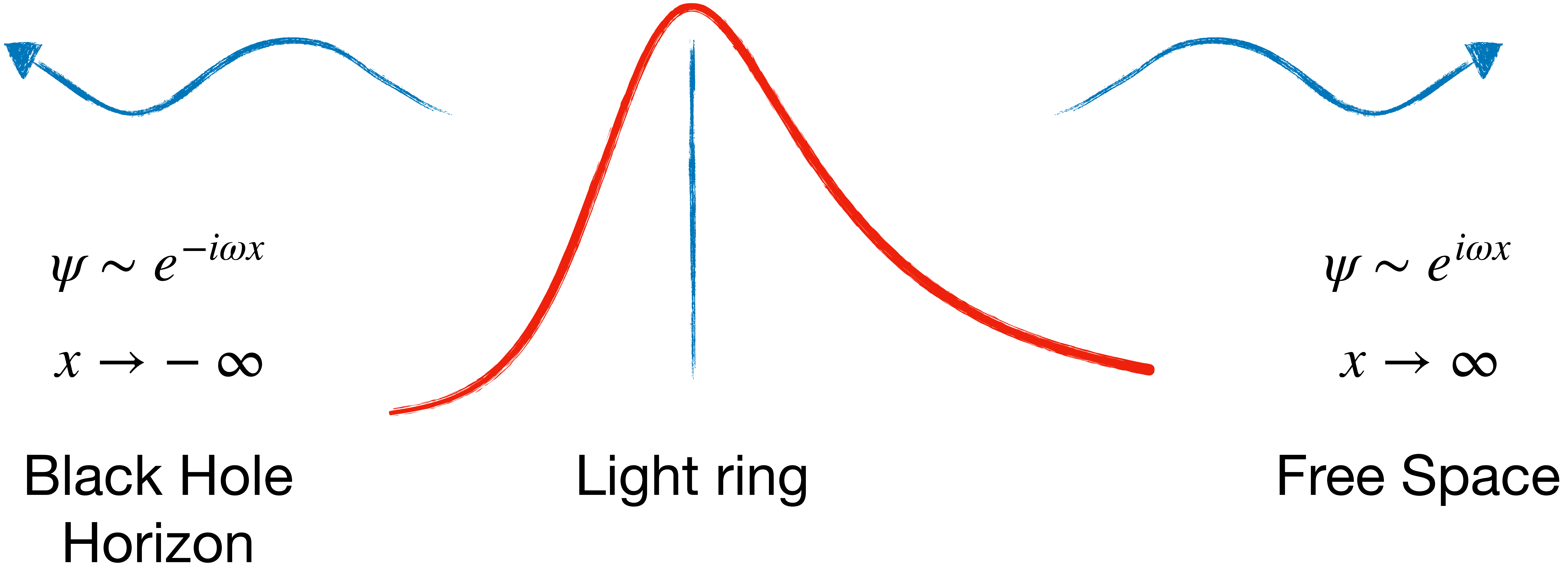
Angular Equation $= 0$

→ Equations decouple if written as Weyl scalars

$x = r + r_+ \log(r - r_+)$ Tortoise Coordinate

Quasi-Normal modes boundary conditions

Ingoing at horizon and outgoing at infinity



Reconstructing the metric

For specific values of angular momentum

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\omega^2 - V_{eff}(x)\right)\right)\psi_{l,m}(x) = 0$$

→ Known frequencies!

$$\omega_{l,m,n} \sim \Omega_{BH} m \; - iT_H^{-1}\,(n + l)$$

$$h_{l,m}(t-r) = \frac{1}{r}\sum_n A_{l,m,n}\, e^{-i\omega_{l,m,n}(t-r)}$$

Ringdown, linear

We target the amplitudes $A_{l,m,n}$

Reconstructing the metric

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Ringdown, linear

We target the amplitudes $A_{l,m,n}$

Depend on initial conditions

General Relativity is Non-Linear!

Does linear perturbation theory suffice?

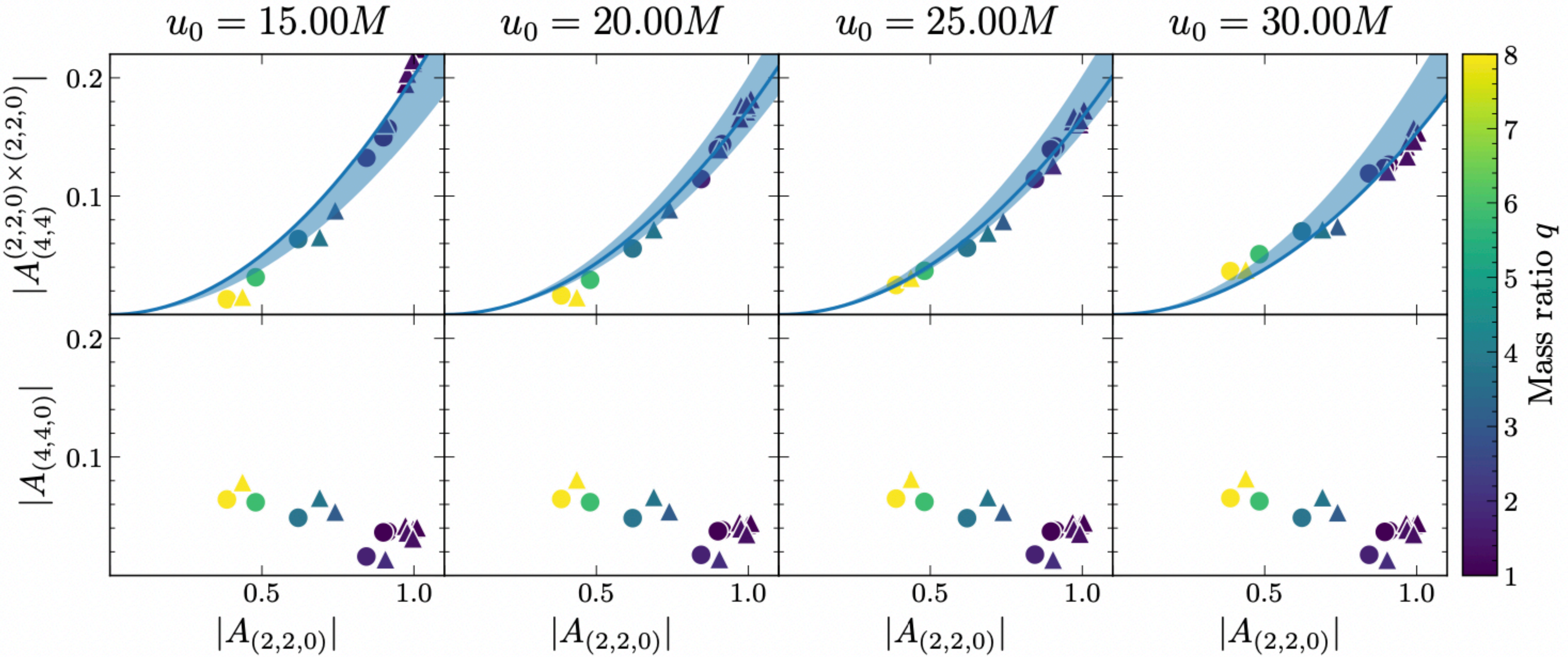
$$\left(\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \left(\omega^2 - V_{eff}\right)\right)\psi^{(1)} = 0,$$

Linear order

$$\left(\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \left(\omega^2 - V_{eff}\right)\right)\psi^{(2)} = S(\psi^{(1)}, \psi^{(1)})$$

Non-Linear order

Correlation between amplitudes

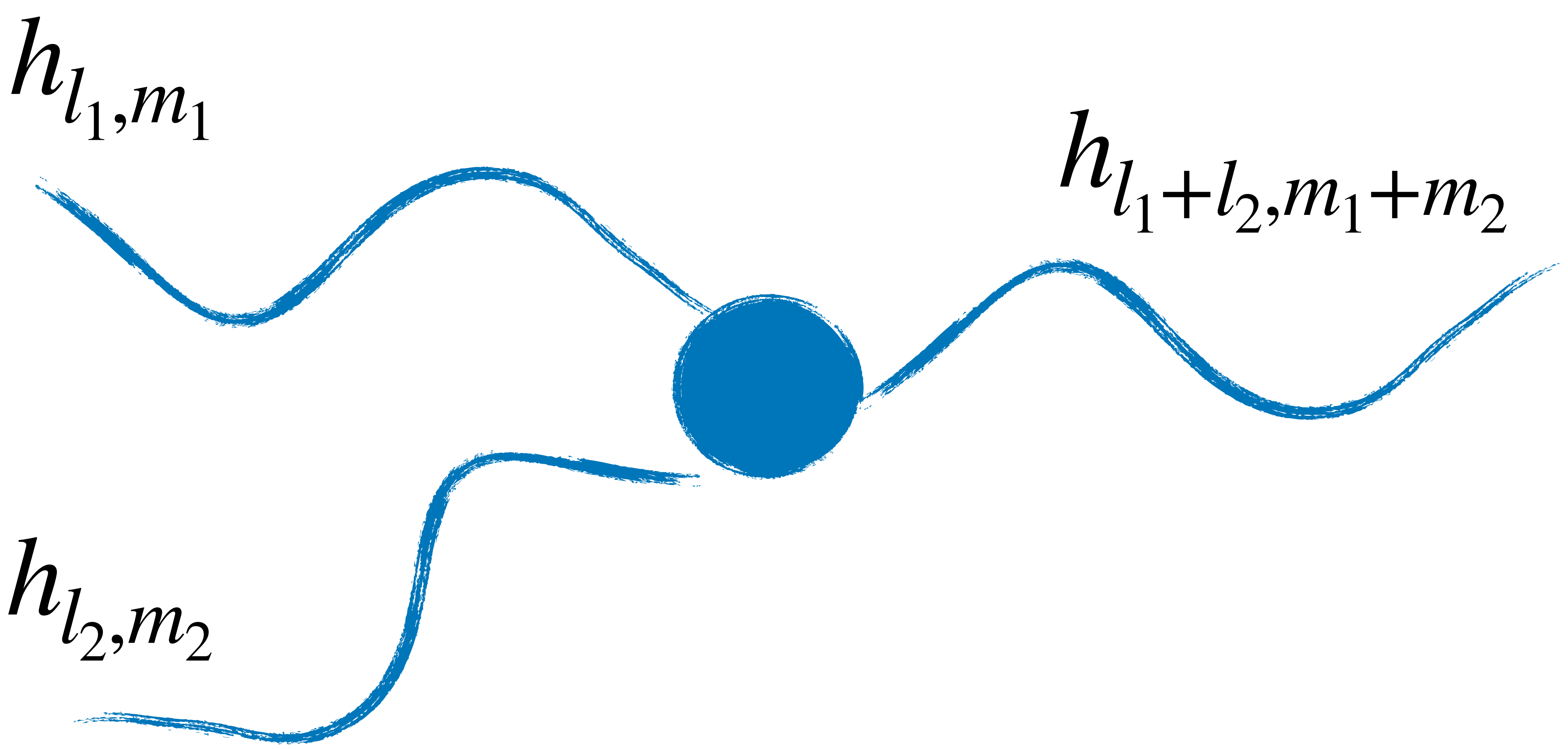


Second-order amplitudes from linear amplitudes are sizeable!

K. Mitman et al. (2022), M. Cheung et al. (2022)

How to find the Amplitude of Non-Linearity

Three point normalised with linear amplitude square



$$\rightarrow \quad \frac{\langle h_{l_1,m_1}\, h_{l_2,m_2}\, h_{l_1+l_2,m_1+m_2} \rangle}{\langle h^2_{l_1,m_1} \rangle \langle h^2_{l_2,m_2} \rangle}$$

$$\frac{\left| A^{(2,2,0)\times(2,2,0)}_{(4,4)} \right|}{\left| A_{(2,2,0)} \right| \, \left| A_{(2,2,0)} \right|} = 0.1637 \pm 0.0018$$

$$\frac{\left| A^{(2,2,0)\times(3,3,0)}_{(5,5)} \right|}{\left| A_{(2,2,0)} \right| \, \left| A_{(3,3,0)} \right|} = 0.4735 \pm 0.0062$$

Numerical values, a=0.7M

K. Mitman et al. (2022), M. Cheung et al. (2022)

QNM Nonlinearities are generated near the horizon

Near horizon limit of Kerr metric

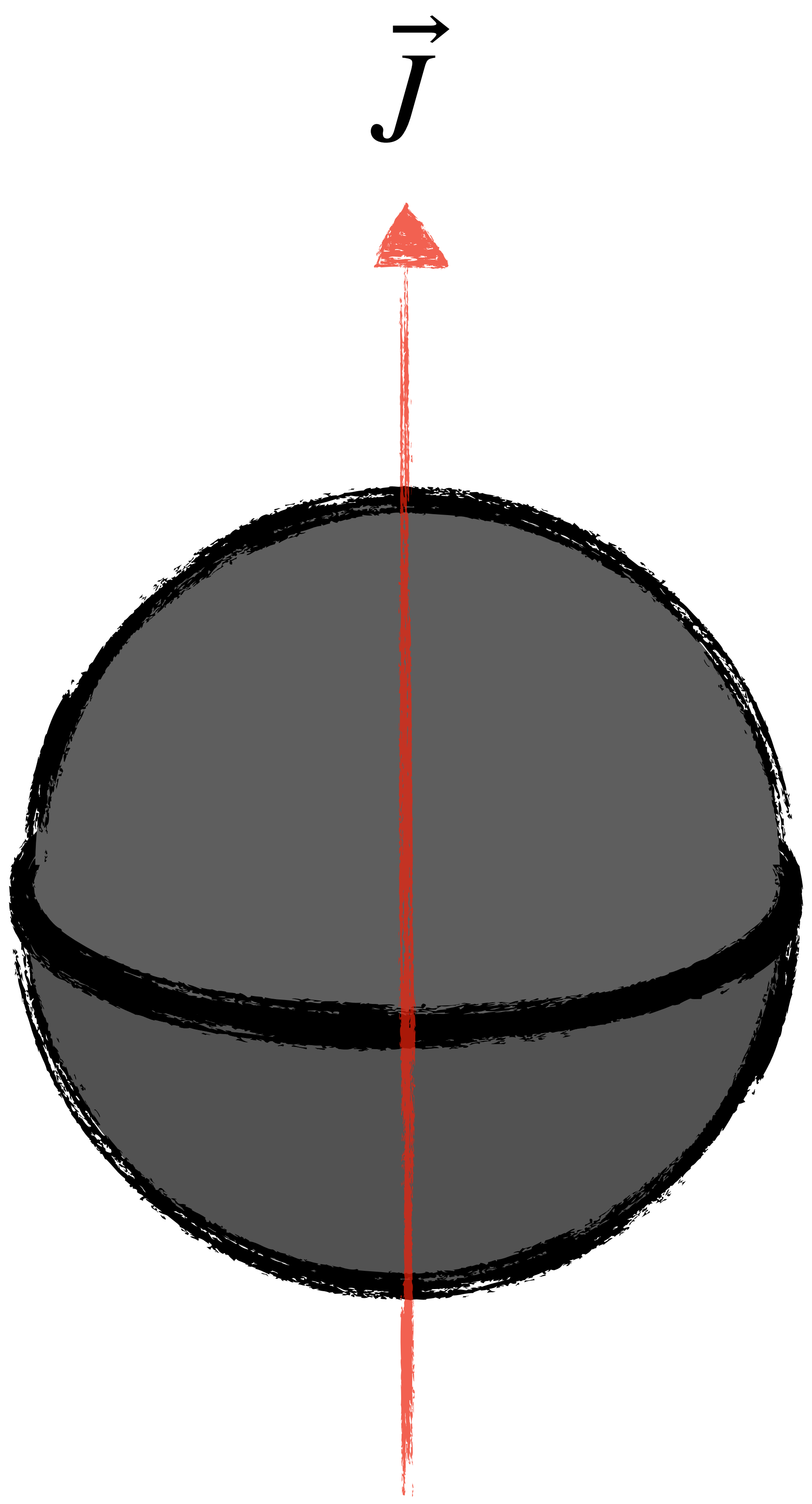
$$J = M^2 \qquad \hat{r}_+ = \hat{r}_- = M \qquad T_H = 0$$

We take the extremal limit for simplicity

A. Kehagias, D.P., F. RIva, A. Riotto (2023)

Near Horizon Extreme Kerr Metric

Squashed AdS for each fixed θ



J. M. Bardeen et al. (1999)

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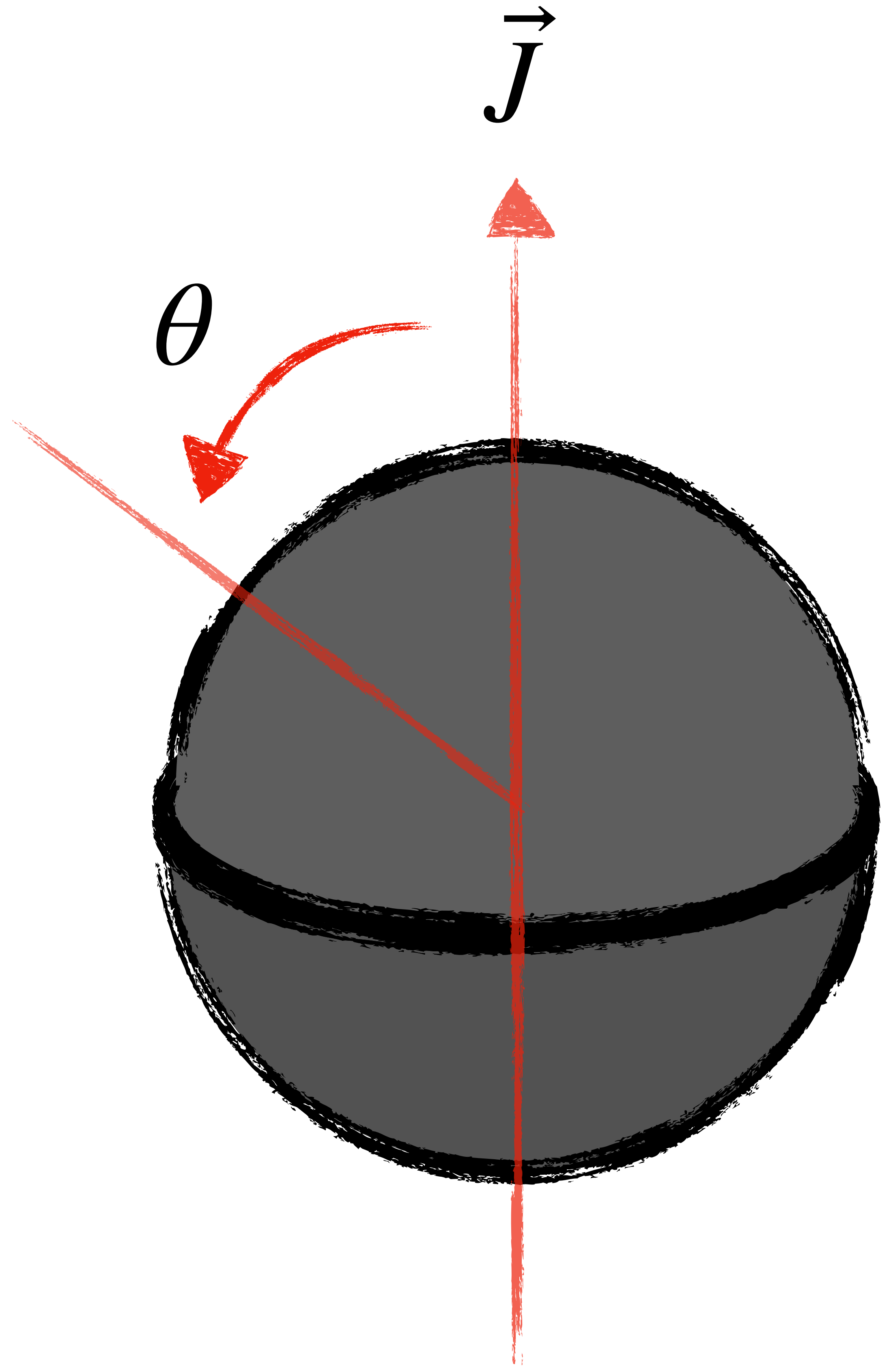

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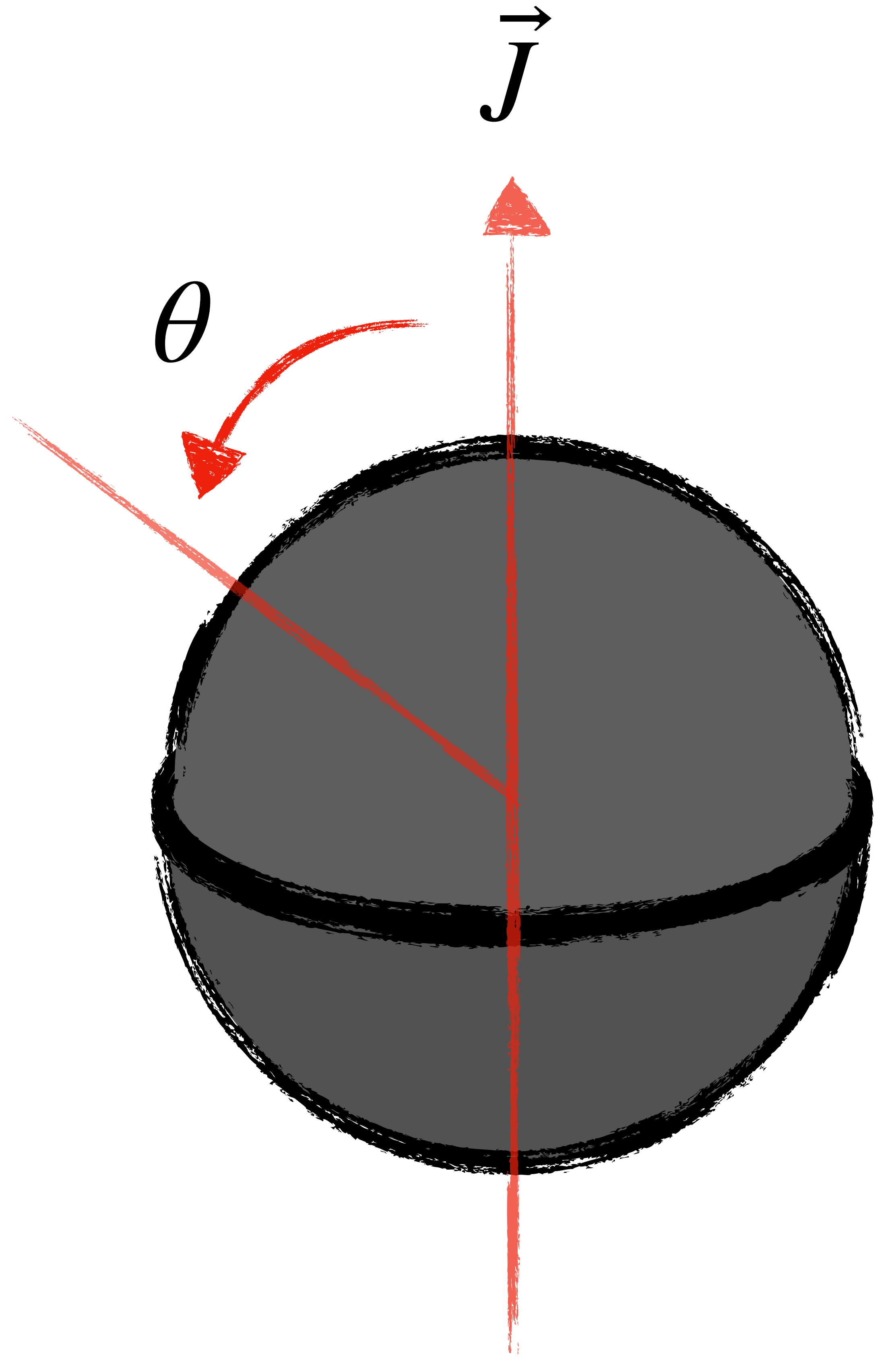

Take a fixed θ slice

J. M. Bardeen et al. (1999)

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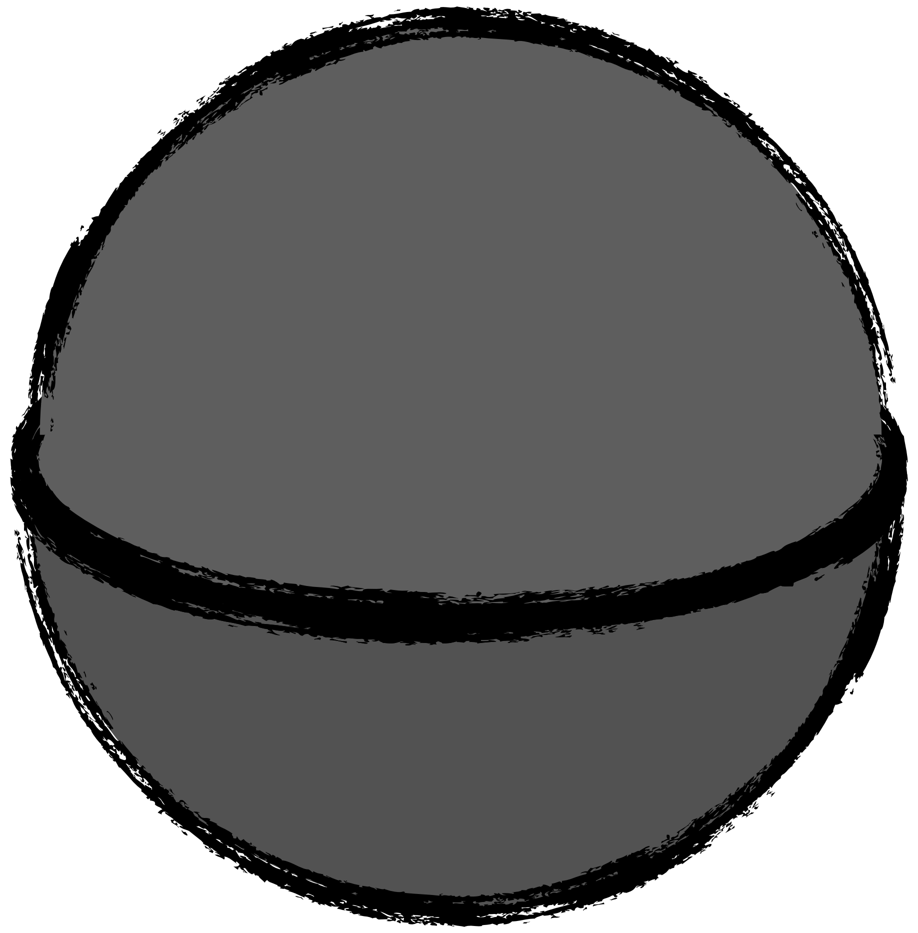

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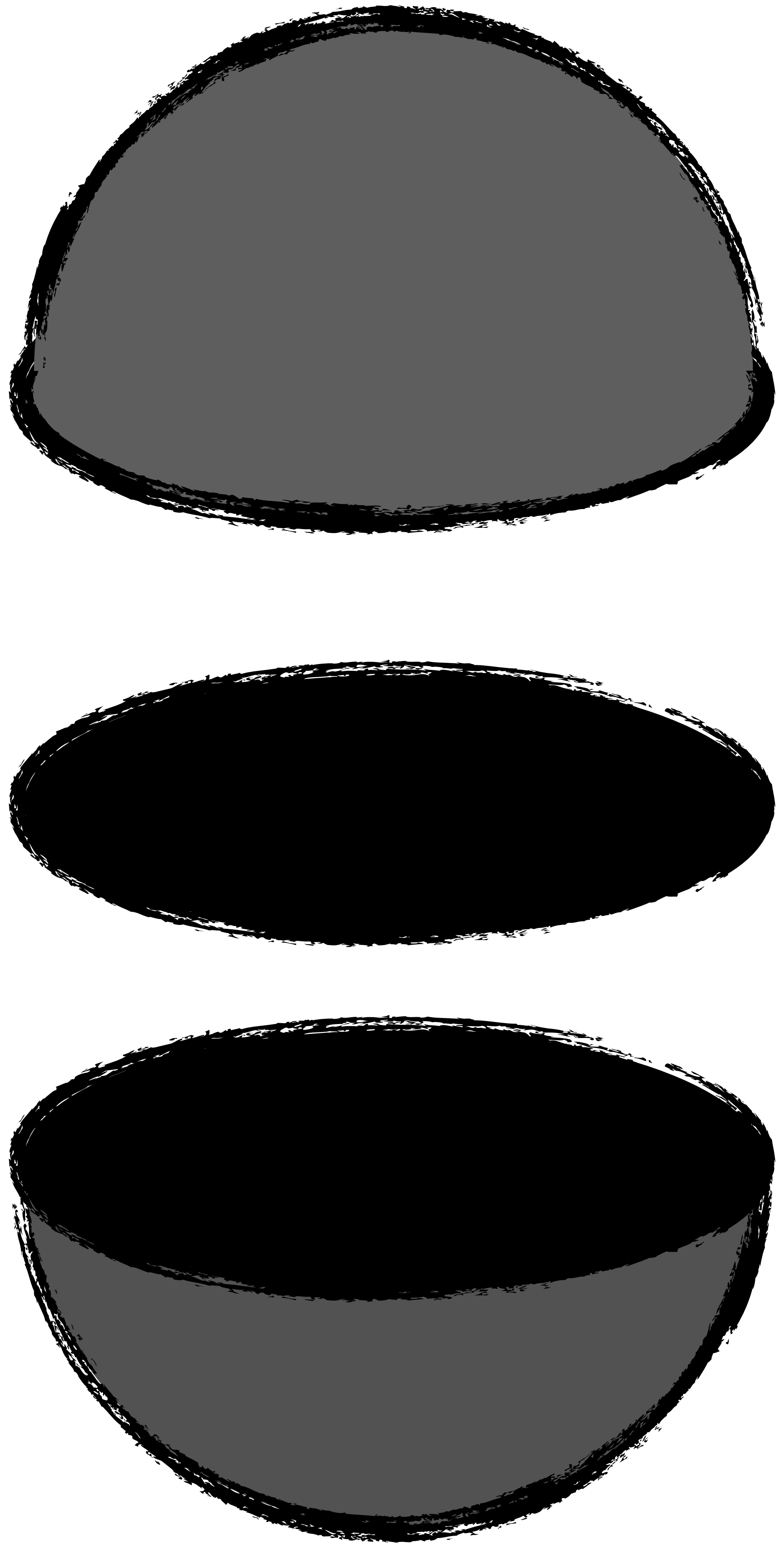

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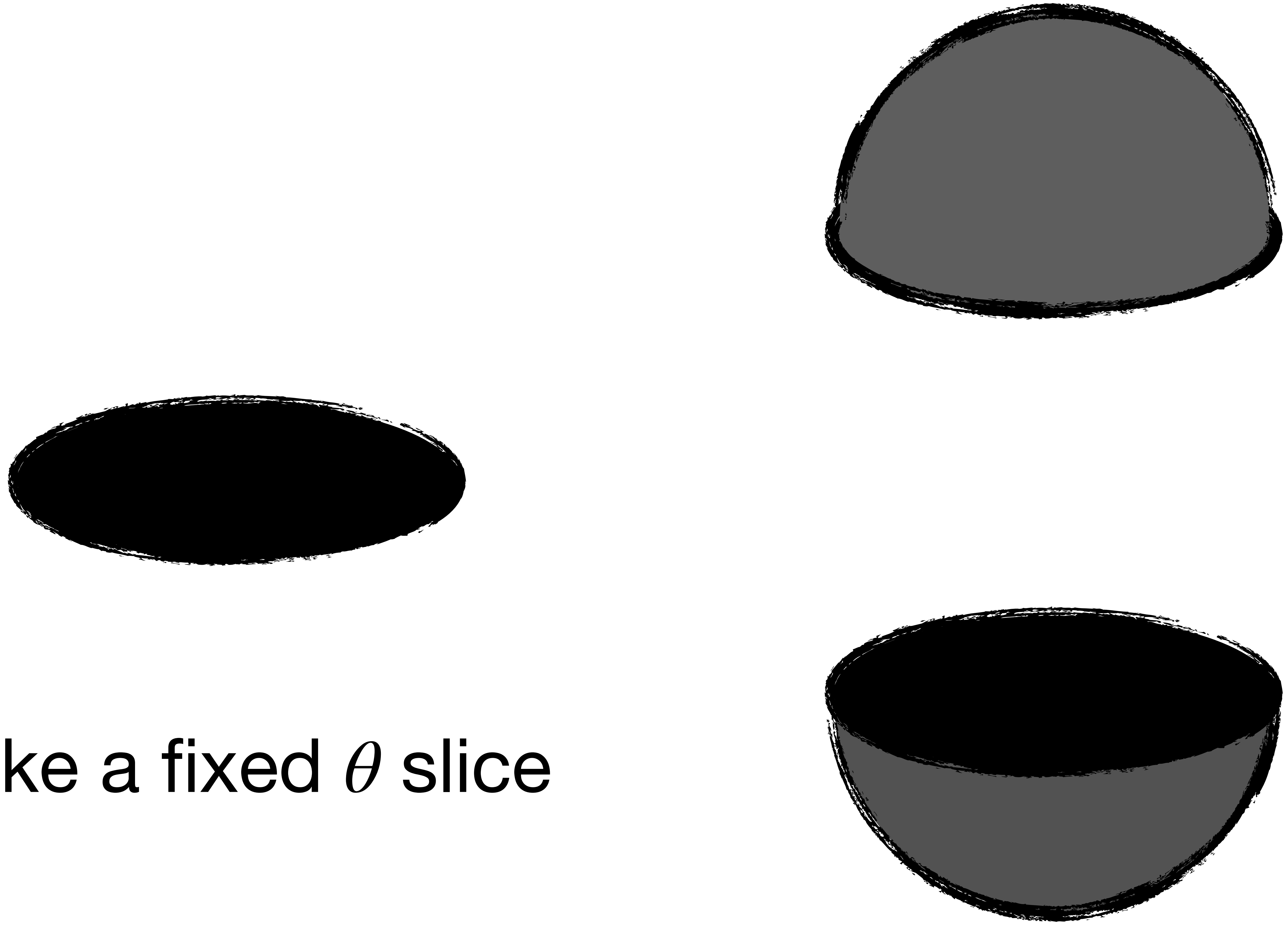

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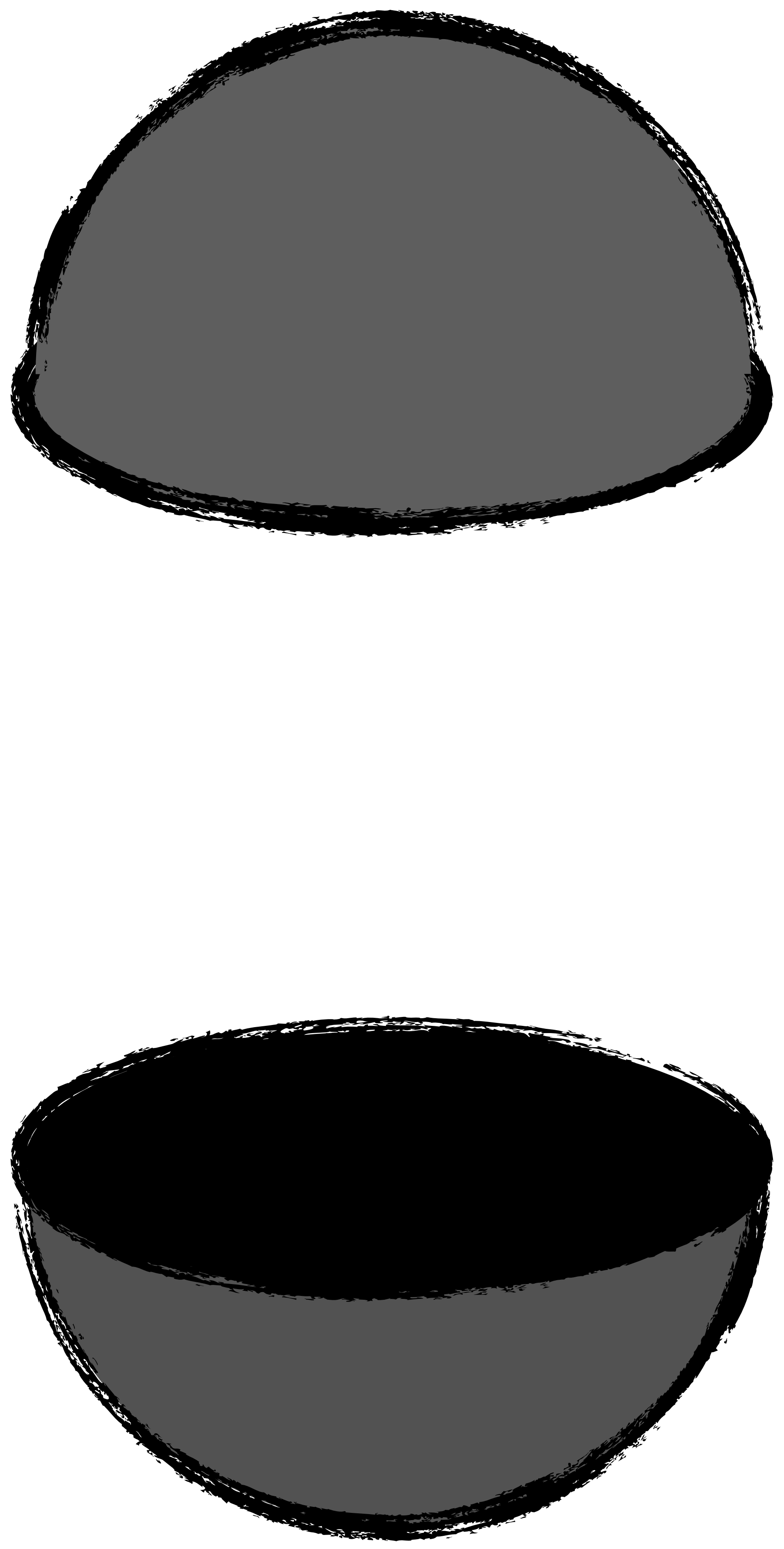


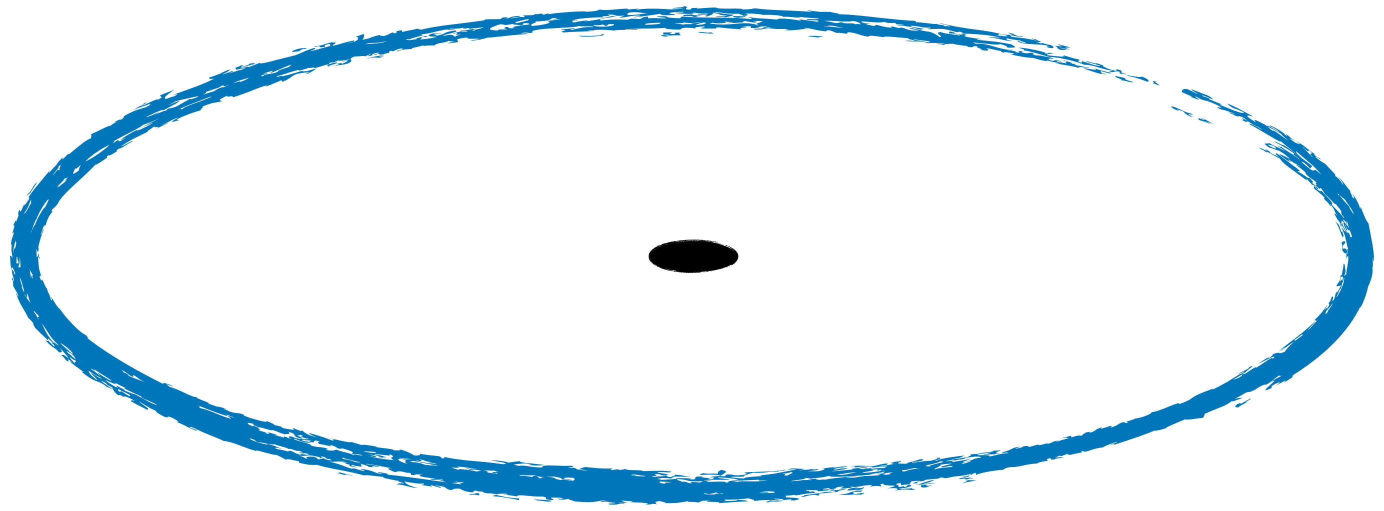
Take a fixed θ slice

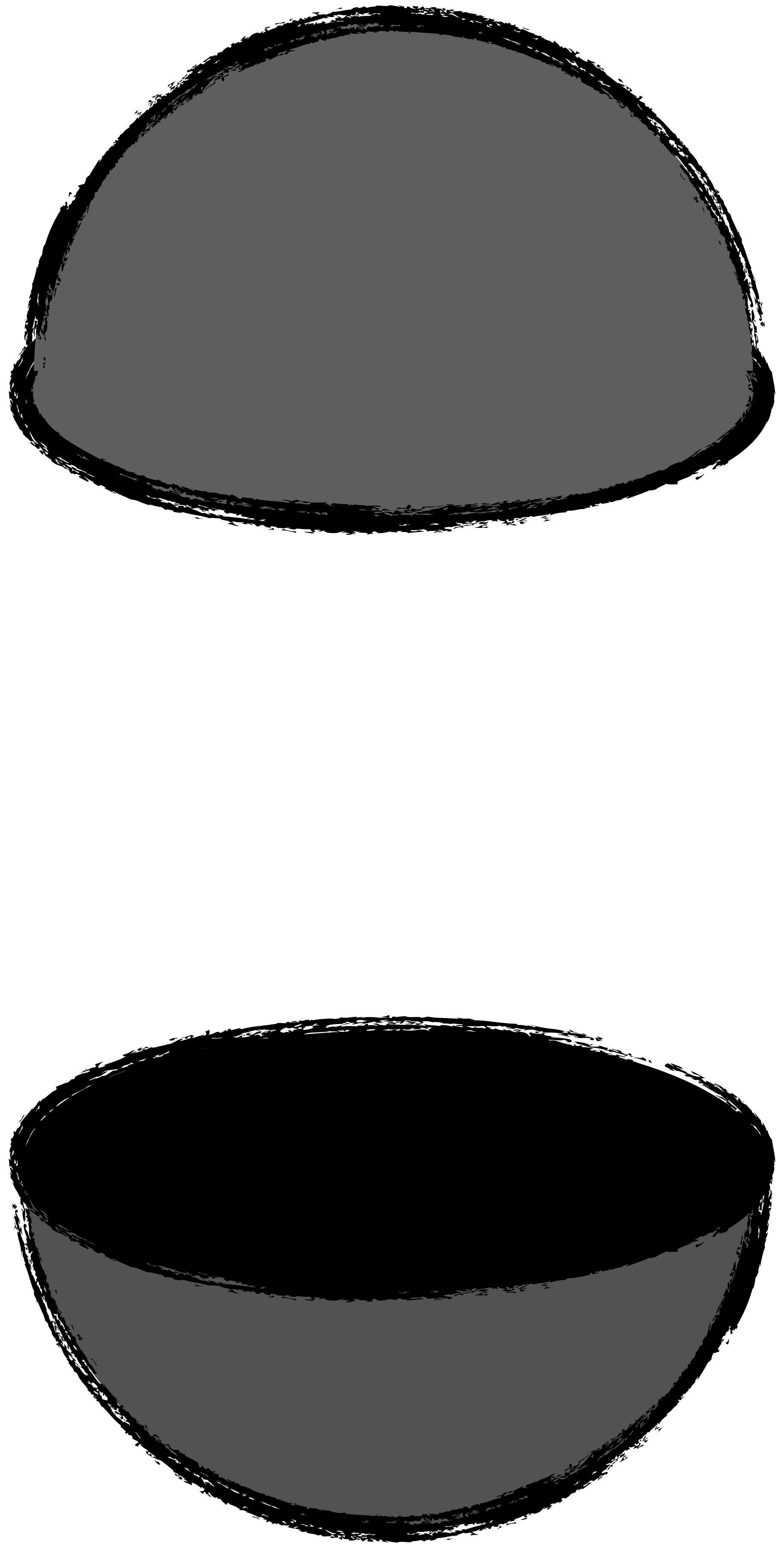
ε outside the horizon

J. M. Bardeen et al. (1999)

Near Horizon Extreme Kerr Metric

Squashed AdS for each fixed θ





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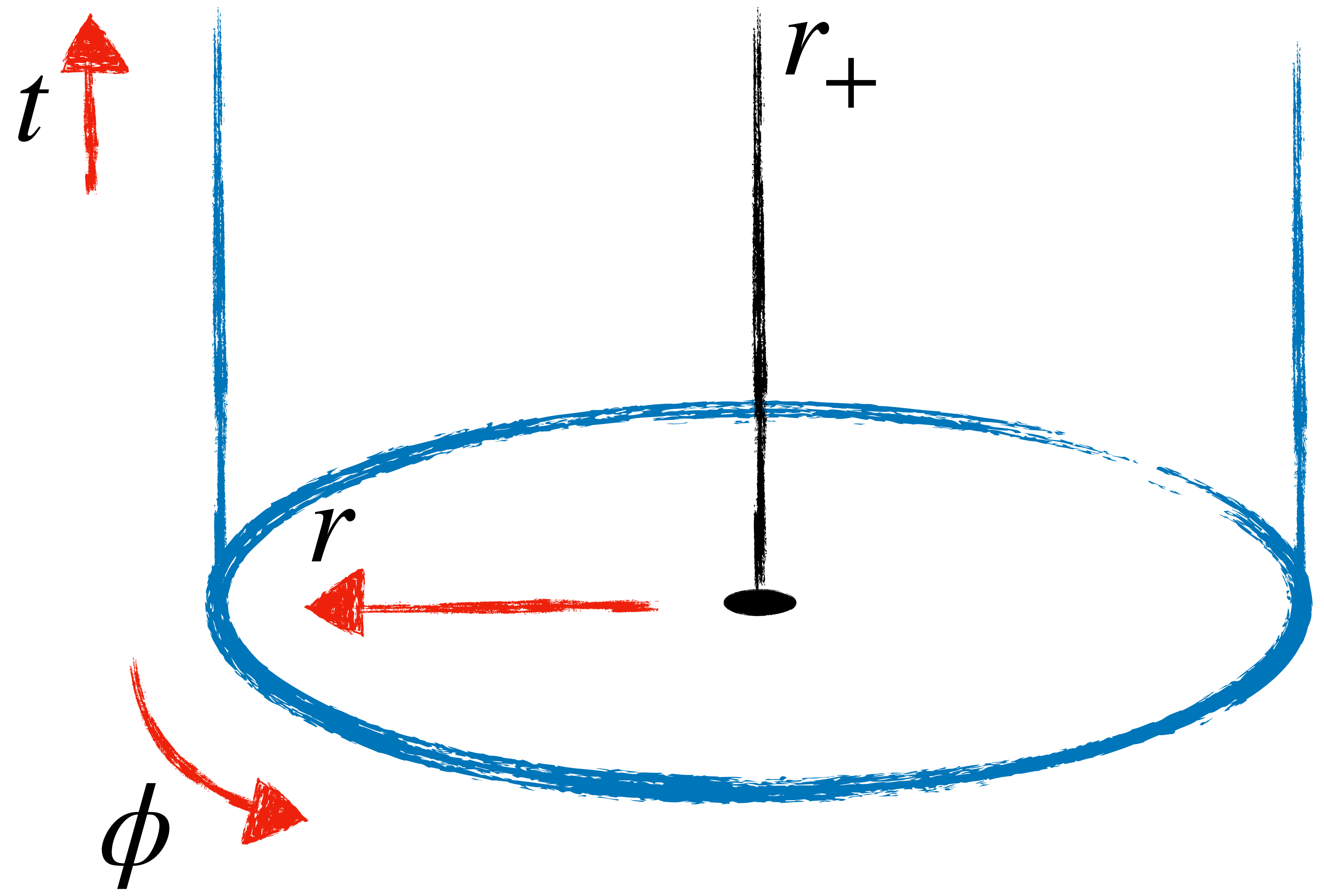

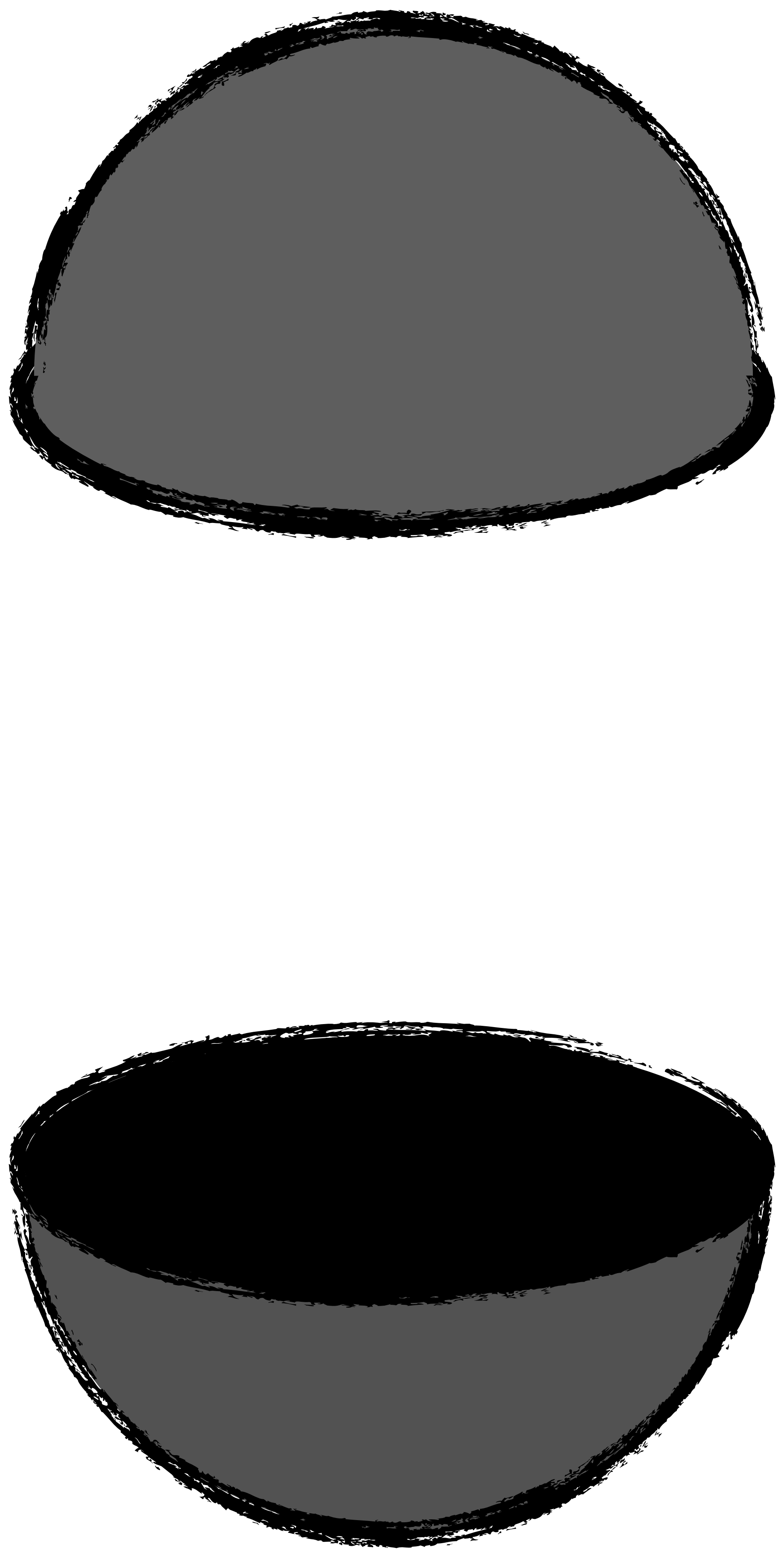

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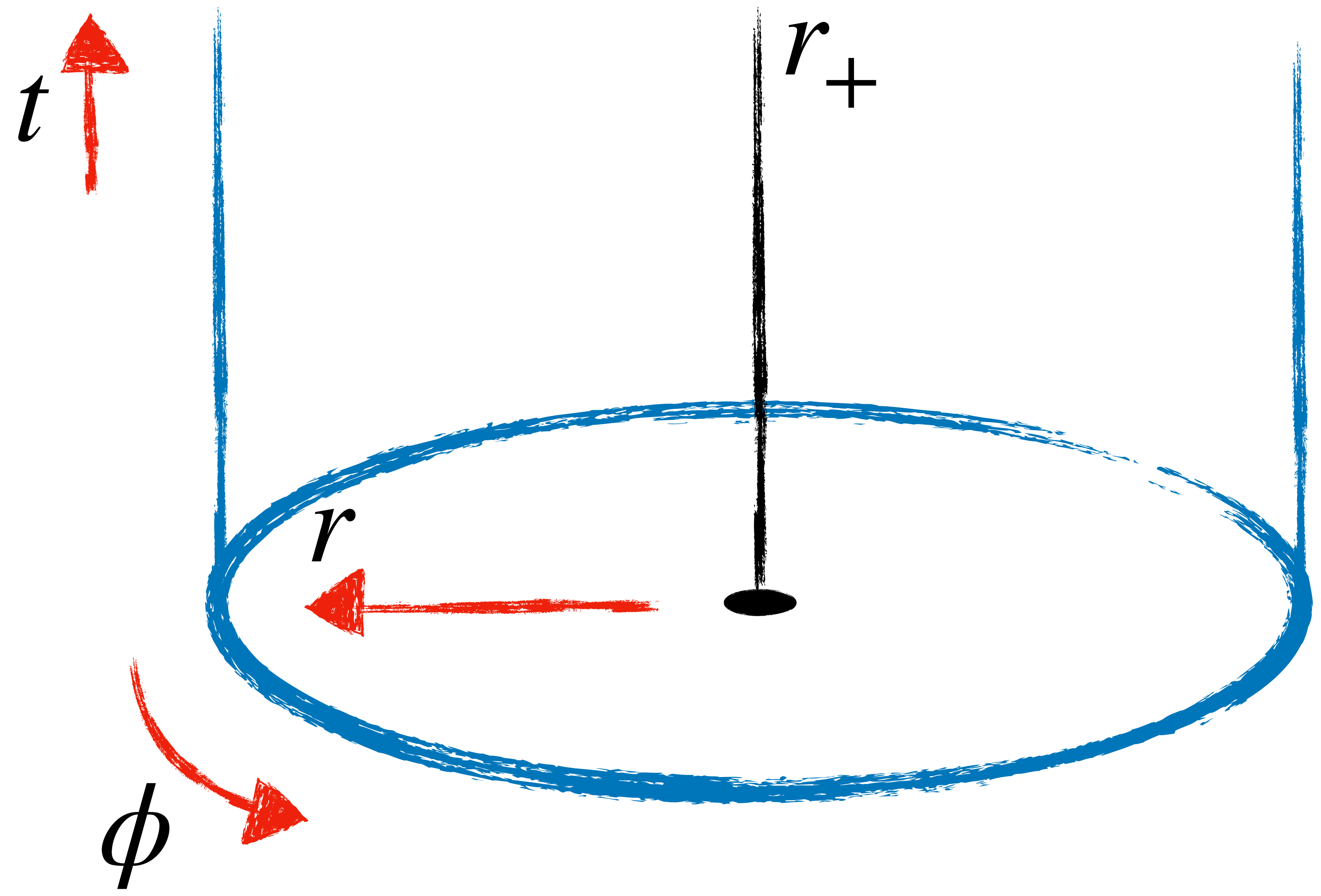
Near Horizon Extreme Kerr Metric

Squashed AdS for each fixed θ

AdS_3 metric

$\mathrm{SL}(2,\mathbb{R}) \otimes U(1)$



Take a fixed θ slice

ε outside the horizon

J. M. Bardeen et al. (1999)

Kerr/CFT correspondence

Evaluating the Asymptotic Symmetry Group

$$\mathrm{AdS}_3 \ \text{metric}$$

$$\mathrm{SL}(2, \mathbb{R}) \otimes U(1)$$

Boundary CFT

M. Guica et al. (2008)

Kerr/CFT correspondence

Evaluating the Asymptotic Symmetry Group

AdS_3 metric

$\mathrm{SL}(2,\mathbb{R}) \otimes U(1)$

Boundary CFT

Not extended

Half Virasoro Algebra, $c = 12J$

M. Guica et al. (2008)

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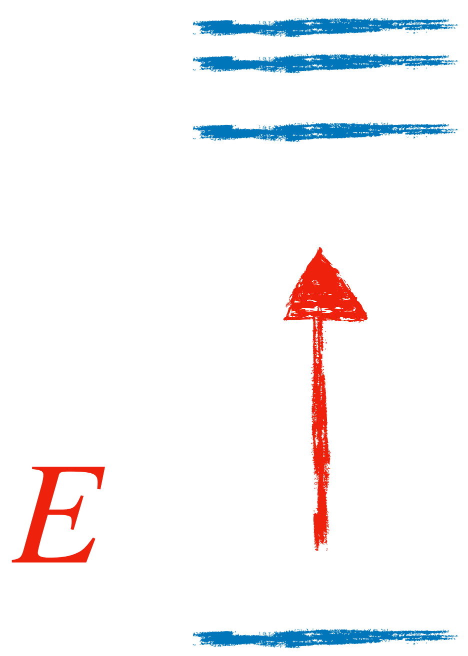


Infinite Gap
in extremal limit

M. Guica et al. (2008)

Kerr/CFT correspondence

Evaluating the Asymptotic Symmetry Group

AdS_3 metric

$\text{SL}(2,\mathbb{R}) \otimes U(1)$

Boundary CFT

Not extended

Half Virasoro Algebra, $c = 12J$

E

Infinite Gap
in extremal limit

Left movers excited
with $T_L = \dfrac{1}{2\pi}$

M. Guica et al. (2008)

2D CFT correlators

Exploiting the duality

$$Z_{\mathrm{AdS,eff}}[h] = e^{iS[h]} = \langle\, e^{\int_{\partial \mathrm{AdS}} T^{\mu\nu} h_{\mu\nu}} \rangle_{\mathrm{CFT}}$$

Gravitational strain in bulk
=
Stress-Energy tensor on boundary

↓

$$\langle T(w_1) T(w_2) \rangle, \quad \langle T(w_1) T(w_2) T(w_3) \rangle$$

Known correlators in 2D CFT

↓

$$w_i = \phi_i \to m_i$$

Fourier transform of correlators

A. Kehagias, D.P., F. RIva, A. Riotto (2023)

Prescription

- Calculate correlators of the 2D energy-momentum tensor at finite temperature

$$\langle T_{m_1} T_{m_2} \rangle, \quad \langle T_{m_1} T_{m_2} T_{m_3} \rangle$$

- Find the gravitational strain correlators

$$\langle h_m, h_{-m} \rangle' = -\frac{1}{\mathrm{Re}\langle \mathrm{T_m T_{-m}} \rangle'}, \quad \langle h_{m_1} h_{m_2} h_{m_3} \rangle' = \frac{2\mathrm{Re}\langle \mathrm{T_{m_1} T_{m_2} T_{m_3}} \rangle'}{\prod_i^3 \left(-2\mathrm{Re}\langle \mathrm{T_{m_i} T_{-m_i}} \rangle'\right)}$$

- Integrate the spin-weighted spherical harmonics over the polar angle θ

A. Kehagias, D.P., F. RIva, A. Riotto (2023), Maldacena (2002)

Kerr Black Hole Non-Linearity

And its estimate in Extremal limit

$$\frac{\langle h_{(\ell_1,m_1)} h_{(\ell_2,m_2)} h_{(\ell_1+\ell_2,m_1+m_2)}\rangle}{\langle h^2_{(\ell_1,m_1)}\rangle\langle h^2_{(\ell_2,m_2)}\rangle} = \frac{6\sqrt{2}}{2\pi}\, {}_{-2}C^{m_1,m_2,m_1+m_2}_{\ell_1,\ell_2,\ell_1+\ell_2} \frac{G^{3,3}_{3,3}\left(\begin{matrix} -im_1, & 0, & im_2 \\ 1-im_1, & 1, & 1+im_2 \end{matrix} \middle| e^{i\pi}\right)}{\left|\Gamma\left(2-i(m_1+m_2)\right)\right|^2}$$

$$\frac{\langle h_{(2,2)} h_{(2,2)} h_{(4,4)}\rangle}{\langle h^2_{(2,2)}\rangle^2} \simeq 0.62 \cdot \frac{5}{24}\sqrt{\frac{7}{\pi}} \simeq 0.19 \quad \text{vs} \quad \frac{\left|A^{(2,2,0)\times(2,2,0)}_{(4,4)}\right|}{\left|A_{(2,2,0)}\right|^2} = 0.1637 \pm 0.0018$$

$$\frac{\langle h_{(2,2)} h_{(3,3)} h_{(5,5)}\rangle}{\langle h^2_{(2,2)}\rangle\langle h^2_{(3,3)}\rangle} \simeq 1.57 \cdot \frac{2}{3}\sqrt{\frac{7}{11\pi}} \simeq 0.47 \quad \text{vs} \quad \frac{\left|A^{(2,2,0)\times(3,3,0)}_{(5,5)}\right|}{\left|A_{(2,2,0)}\right|\left|A_{(3,3,0)}\right|} = 0.4735 \pm 0.0062$$

A. Kehagias, D.P., F. RIva, A. Riotto (2023)

Non-Linearities for non-extremal

Kerr/CFT works only in the extremal case

Summary and outlook

Non-Linearities are large but their magnitude can be understood

- Crucial for the future of Gravitational Wave analysis
- A lot of techniques are being developed to evaluate them
- Could be captured by symmetry, but still unclear outside extremal case

Results with exact definition of C

$$\frac{\langle h_{(\ell_1,m_1)} h_{(\ell_2,m_2)} h_{(\ell_1+\ell_2,m_1+m_2)}\rangle}{\langle h^2_{(\ell_1,m_1)}\rangle\langle h^2_{(\ell_2,m_2)}\rangle} = \frac{6\sqrt{2}}{2\pi}\, {}_{-2}C^{m_1,m_2,m_1+m_2}_{\ell_1,\ell_2,\ell_1+\ell_2} \frac{G^{3,3}_{3,3}\left(\begin{array}{ccc|c} -im_1, & 0, & im_2 & \\ 1-im_1, & 1, & 1+im_2 & e^{i\pi}\end{array}\right)}{\left|\Gamma\left(2-i(m_1+m_2)\right)\right|^2}$$

$$\begin{aligned} {}_{-2}C^{m_1,m_2,m_3}_{\ell_1,\ell_2,\ell_3} &= 2\pi\int_0^\pi \mathrm{d}\theta\,\sin\theta\; {}_{-2}Y_{(\ell_1,m_1)}\, {}_{-2}Y_{(\ell_2,m_2)}\, {}_{2}\overline{Y}_{(\ell_3,m_3)} \\ &= \frac{\Gamma\left(-2+\sum_{i=1}^{3}\frac{|m_i|}{2}\right)\Gamma\left(4+\sum_{i=1}^{3}\frac{|m_i|}{2}\right)}{2\sqrt{\pi}\,\Gamma\left(2+\sum_{i=1}^{3}|m_i|\right)}\left(\prod_{i=1}^{3}\frac{(2|m_i|+1)!}{(|m_i|+2)!\;(|m_i|-2)!}\right)^{1/2} \end{aligned}$$

A. Kehagias, D.P., A.Riotto and F. Riva, gr-qc/2301.09345

Full Teukolsky equations

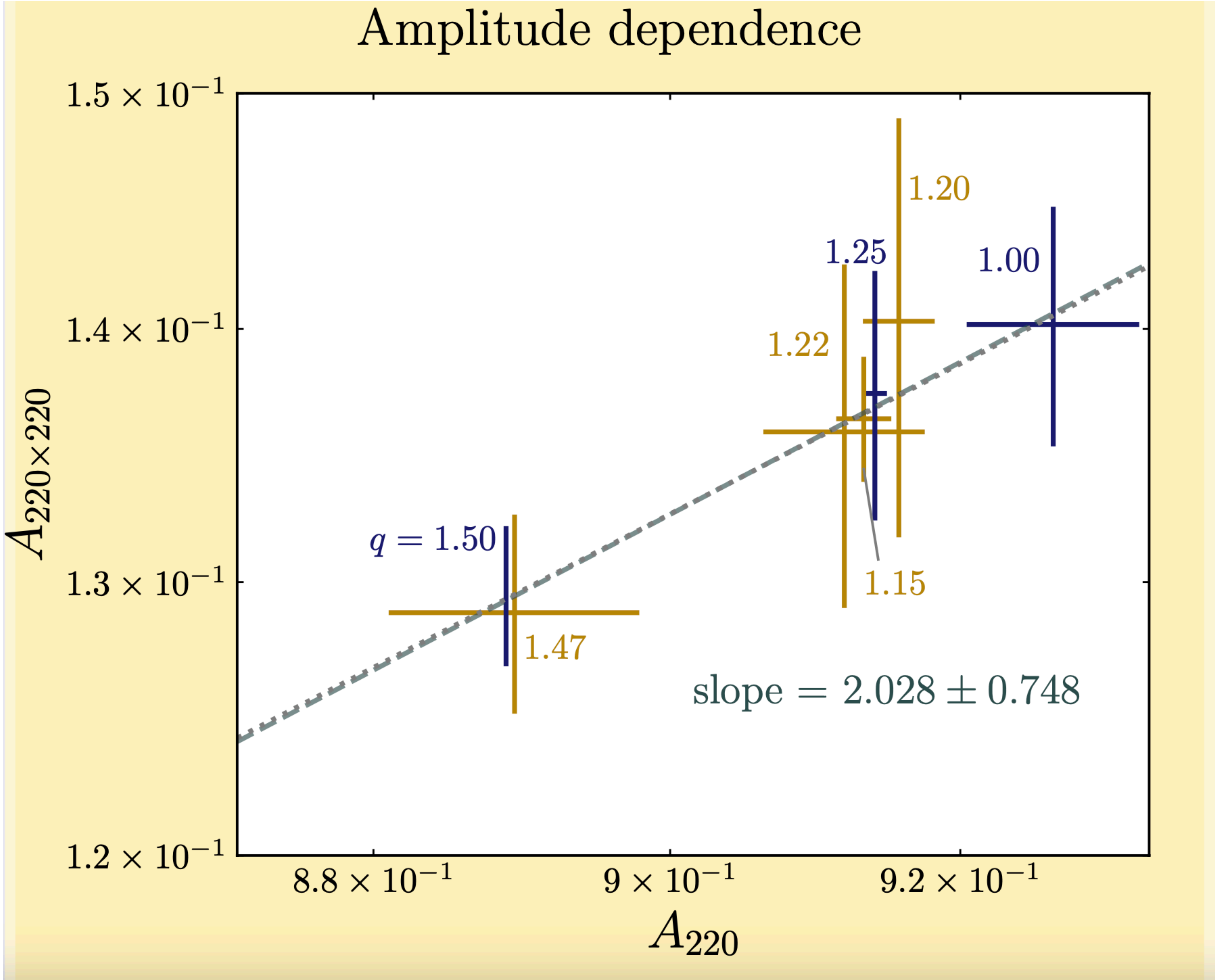
$$\Psi_s(t,r,\theta,\phi) = e^{-i\omega t} e^{im\phi} S(\theta) R(r)$$

$$\Delta^{-s}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Delta^{s+1}\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \left(\frac{K^2 - is(\mathrm{d}\Delta/\mathrm{d}r)K}{\Delta} + 4is\omega r - \lambda_\omega\right) R = 0$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}S}{\mathrm{d}\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2m\,s\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A\right) S = 0 \quad (1)$$

$$K = (r^2 + a^2)\omega - am$$

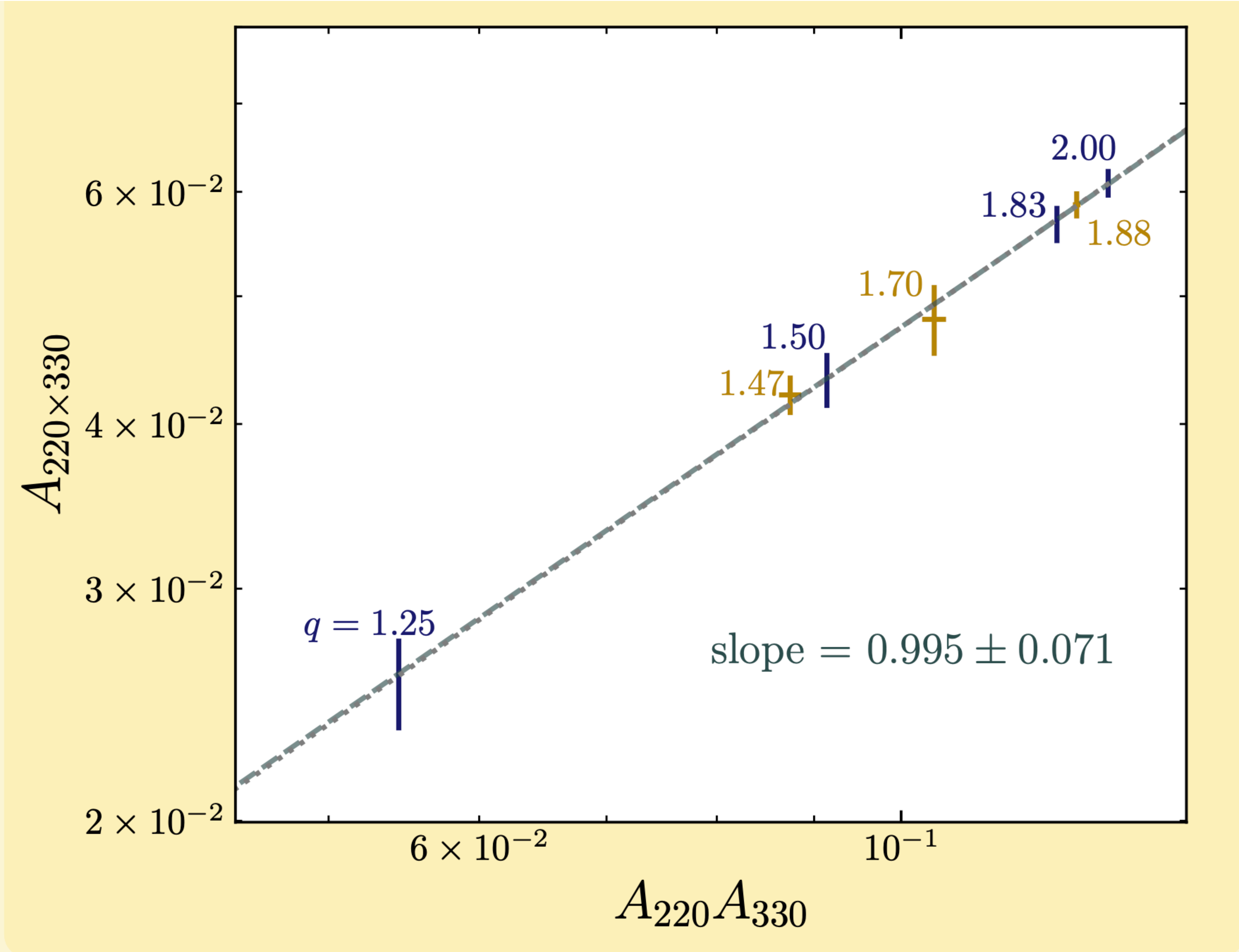
$$\lambda_\omega = A + a^2\omega^2 - 2am\omega$$



$$\frac{\left|A^{(2,2,0)\times(2,2,0)}_{(4,4)}\right|}{\left|A_{(2,2,0)}\right|^2} = 0.1637 \pm 0.0018$$

Second-order amplitudes from linear amplitudes are sizeable

M. Cheung et al. (2022)



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M. Cheung et al. (2022)

Near Horizon Extremal Kerr geometry

Zooming close to the Horizon

$$\mathrm{ds}^2 = 2\mathrm{M}^2\Gamma(\theta)\left[-\mathrm{r}^2\mathrm{dt}^2 + \frac{\mathrm{dr}^2}{\mathrm{r}^2} + \mathrm{d}\theta^2 + \Lambda^2(\theta)(\mathrm{d}\phi + \mathrm{rdt})^2\right]$$

$$\Gamma(\theta) = \frac{1+\cos^2\theta}{2}, \quad \Lambda(\theta) = \frac{2\sin\theta}{1+\cos^2\theta}$$

NHEK is a warped AdS_3 geometry

Isometry group $\mathrm{SL}(2,\mathbb{R}) \otimes U(1)$

J. M. Bardeen et al. (1999)

Asymptotic symmetry group

Non-trivial symmetries at the boundary of AdS

Boundary conditions on AdS_3

$$h_{\mu\nu} = \begin{pmatrix} h_{tt} = \mathcal{O}(r^2) & h_{t\phi} = \mathcal{O}(1) & h_{t\theta} = \mathcal{O}(r^{-1}) & h_{tr} = \mathcal{O}(r^{-2}) \\ h_{\phi t} = h_{t\phi} & h_{\phi\phi} = \mathcal{O}(1) & h_{\phi\theta} = \mathcal{O}(r^{-1}) & h_{\phi r} = \mathcal{O}(r^{-1}) \\ h_{\theta t} = h_{t\theta} & h_{\theta\phi} = h_{\phi\theta} & h_{\theta\theta} = \mathcal{O}(r^{-1}) & h_{\theta r} = \mathcal{O}(r^{-2}) \\ h_{rt} = h_{tr} & h_{r\phi} = h_{\phi r} & h_{r\theta} = h_{\theta r} & h_{rr} = \mathcal{O}(r^{-3}) \end{pmatrix}$$

Diffeomorphism which preserve the boundary conditions, w/o the trivial ones

$$\xi_m = \epsilon_m(\phi)\partial_\phi - r\epsilon'_m(\phi)\partial_r$$

$$\epsilon_m(\phi) = -\, e^{-im\phi}$$

M. Guica et al. (2008)

Kerr NHEK transformation

$$t = \frac{\lambda \hat{t}}{2M}, \quad r = \frac{\hat{r} - M}{\lambda M}, \quad \phi = \hat{\phi} - \frac{\hat{t}}{2M},$$

$$\lambda \to 0 \quad \text{keeping } (t, r, \phi, \theta) \text{ fixed}$$

$$H_{-1} = \partial_t,$$
$$H_0 = t\partial_t - r\partial_r,$$
$$H_1 = \left(\frac{1}{2r^2} + \frac{t^2}{2}\right)\partial_t - tr\partial_r - \frac{1}{r}\partial_\phi,$$
$$Q_0 = -\partial_\phi$$

Generators of isometry group

$$\mathrm{SL}(2, \mathbb{R}) \otimes U(1)$$

M. Guica et al. (2008)

Typical values for Black Hole QNMs

$$M\omega$$

	$\ell = 2,\ n = 0$				
a	$m = 2$	$m = 1$	$m = 0$	$m = -1$	$m = -2$
0.00	.3737,.0890	.3737,.0890	.3737,.0890	.3737,.0890	.3737,.0890
0.10	.3870,.0887	.3804,.0888	.3740,.0889	.3678,.0890	.3618,.0891
0.20	.4021,.0883	.3882,.0885	.3751,.0887	.3627,.0889	.3511,.0892
0.30	.4195,.0877	.3973,.0880	.3770,.0884	.3584,.0888	.3413,.0892
0.40	.4398,.0869	.4080,.0873	.3797,.0878	.3546,.0885	.3325,.0891
0.50	.4641,.0856	.4206,.0862	.3833,.0871	.3515,.0881	.3243,.0890
0.60	.4940,.0838	.4360,.0846	.3881,.0860	.3489,.0876	.3168,.0890
0.70	.5326,.0808	.4551,.0821	.3941,.0845	.3469,.0869	.3098,.0887
0.80	.5860,.0756	.4802,.0780	.4019,.0822	.3454,.0860	.3033,.0885
0.90	.6716,.0649	.5163,.0698	.4120,.0785	.3444,.0849	.2972,.0883
0.98	.8254,.0386	.5642,.0516	.4223,.0735	.3439,.0837	.2927-.0881

E. Berti, V. Cardoso and C.M. Will (2006)