The functorial approach to chiral 2D CFT

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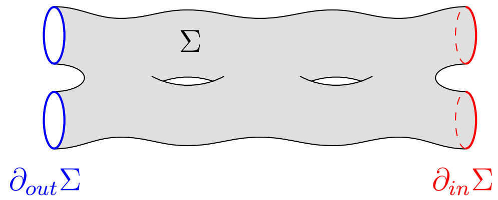
Definiton of χCFT

Complex cobordisms:



Take a closed Riemann surface $\hat{\Sigma}$, and a collection $D_1, \ldots, D_n \subset \hat{\Sigma}$ of disjoint closed discs with smooth boundary. A complex cobordism is something of the form $\Sigma = \hat{\Sigma} \setminus (\mathring{D}_1 \cup \ldots \cup \mathring{D}_n)$, along with a partition of its boundary components into some labelled "in" and some labelled "out".

We equip Σ with the sheaf of function that are continuous, smooth on the boundary, and holomorphic in the interior. Composition of cobordisms = pushout of topological spaces, along with the sheaf

$$\mathcal{O}_{\Sigma_1 \cup \Sigma_2}(U) = \{ f : U \to \mathbb{C} \text{ s.t. } f|_{U \cap \Sigma_1} \in \mathcal{O}_{\Sigma_1}(U \cap \Sigma_1) \text{ and } f|_{U \cap \Sigma_2} \in \mathcal{O}_{\Sigma_2}(U \cap \Sigma_2) \}$$

Technical point. We allow $\partial_{in}\Sigma$ and $\partial_{out}\Sigma$ to touch: if D_i corresponds to an incoming circle and D_j corresponds to an outgoing circle, then we only require their interiors to be disjoint in $\hat{\Sigma}$.

To define a complex cobordism *from* S_1 *to* S_2, also keep track of diffeomorphisms $\varphi_{in} : S_1 \to \partial_{in}\Sigma$ and $\varphi_{out} : S_2 \to \partial_{out}\Sigma$.

The semigroup of annuli $\mathrm{Ann}(S)$**:**

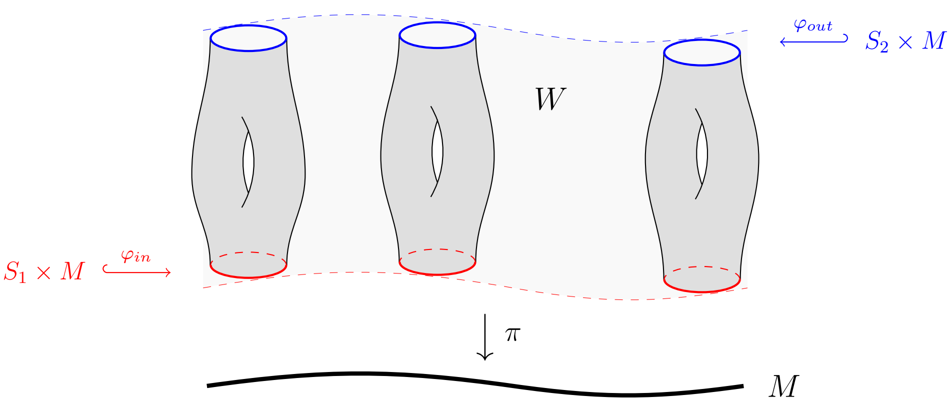
Let S be a connected 1-manifold. A pair of smooth embeddings $S \hookrightarrow \mathbb{C}$ that are one 'inside' the other specifies an element of $\mathrm{Ann}(S)$. One can think of $\mathrm{Ann}(S)$ as a quotient of the space of such pairs of embeddings:

$$\mathrm{Ann}(S) \subset \mathrm{Hom}_{\mathrm{Cob}^{\mathrm{Conf}}}(S, S)$$

There's an embedding $\mathrm{Diff}(S) \hookrightarrow \mathrm{Ann}(S)$ which sends a diffeomorphism ψ to the completely thin annulus $(A{=}S, \varphi_{in}{=}\psi, \varphi_{out}{=}\mathrm{id})$. The semigroup $\mathrm{Ann}(S)$ should be thought of as the complexification of the group $\mathrm{Diff}(S)$.

Moduli space of complex cobordisms:

$\mathrm{Hom}_{\mathrm{Cob}^{\mathrm{Conf}}}(S_1, S_2)$ is an infinite dimensional complex manifold. One may encode this by defining what it means to be a family of complex cobordisms, parametrised by a complex manifold M:



The main condition is that for every point x of S_1 or S_2, the section $\varphi|_{\{x\}\times M} : M \to W$ of π should be holomorphic.

Full CFT versus chiral CFT:

They're not the same thing. (It's a bit like associative algebras versus Lie algebras. They're both "algebras", but they're just not the same thing.)

Concrete linear category:

A concrete linear category is a pair $(\mathcal{C}, U)$ consisting of a linear category $\mathcal{C}$ and a faithful functor U from $\mathcal{C}$ to the category of topological vector spaces. *Think:* $\mathcal{C}$ is the category of representations of a group or an algebra, and U is the functor that sends a representation to its underlying vector space.

Chiral CFT:

$\rightarrow$ A χCFT is a symmetric monoidal functor $\text{Cob}^{\text{Conf}} \to \{\text{Concrete linear categories}\}$ $\leftarrow$ satisfying a couple extra conditions. Unpacking, we get:

For every 1-manifold S, a category $\mathcal{C}(S)$.	A 'forgetful' functor $U : \mathcal{C}(S) \to \mathsf{TopVec}$.
For every cpx cobordism Σ, a functor $F_\Sigma : \mathcal{C}(\partial_{in}\Sigma) \to \mathcal{C}(\partial_{out}\Sigma)$.	For every $V \in \mathcal{C}(\partial_{in}\Sigma)$, a *linear map* $Z_\Sigma : V \to F_\Sigma(V)$.

The extra conditions (explained later) convey the ideas that "F is topological", and "Z is holomorphic".

Central extensions:

Let $\mathfrak{g}$ be a Lie algebra. A Lie algebra 2-cocycle is a bilinear map $\omega : \mathfrak{g} \times \mathfrak{g} \to A$ which is antisymmetric, and satisfies $\sum_3 \omega([X,Y],Z) = 0$. Given a 2-cocycle, one can form a central extension $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus A$, with Lie bracket $[(X,a),(Y,b)]_{\tilde{\mathfrak{g}}} := ([X,Y]_{\mathfrak{g}}, \omega(X,Y))$. The 2-cocycle identity guarantees that this new bracket satisfies the Jacobi identity.

The Witt and Virasoro algebras:

The Lie algebra associated to $\text{Diff}(S^1)$ is the set of vector fields on S^1, equipped with the *opposite* of the usual Lie bracket of vector fields:

$$[f(z)\partial/\partial z, g(z)\partial/\partial z] := (gf' - fg')\partial/\partial z.$$

We'll be working with the associated complex Lie algebra, which corresponds to $\text{Ann}(S^1)$. It admits a dense subalgebra, known as the Witt algebra, spanned by $\ell_n := z^{n+1}\partial/\partial z$. The formula

$$\omega_{Vir}(\ell_m, \ell_n) := C/12(m^3 - m)\delta_{m+n,0}$$

defines a 2-cocycle on the Witt algebra with values in $\mathbb{C}\cdot C$ (where C is a formal symbol). The corresponding central extension is called the Virasoro algebra. The underlying vector space of the Virasoro algebra is $\mathit{Witt} \oplus \mathbb{C}{\cdot}C$, and it fits into a central extension

$$0 \to \mathbb{C}{\cdot}C \to \mathit{Vir} \to \mathit{Witt} \to 0.$$

We write L_n for ℓ_n viewed as an element of Vir:

$$[L_m, L_n] = (m-n)L_{m+n} + 1/12(m^3 - m)\delta_{m+n,0} \cdot C$$

This is a universal central extension. This means that one doesn't need to know the formula in order to be able to think about it. In particular, it also exists and makes sense for the Lie algebra of vector fields on any other circle manifold.

Central extension of semigroup of annuli:

The Virasoro central extension integrates to a central extension

$$0 \to \mathbb{C} \oplus \mathbb{Z} \to \widetilde{\mathrm{Ann}}_{\mathrm{univ}}(S) \to \mathrm{Ann}(S) \to 0$$

The theory of integration works well for simply connected Lie groups, and applies with minor changes to the semigorup of annuli.

- $\mathbb{Z}$: comes from taking the universal cover of $\mathrm{Ann}(S)$.
- $\mathbb{C}$: comes from integrating the Virasoro cocycle.

Fix a central charge $c \in \mathbb{R}$. Performing the pushout along the map $z \mapsto e^{zc} : \mathbb{C} \to \mathbb{C}^\times$, we get a new central extension:

$$0 \to \mathbb{C}^\times \oplus \mathbb{Z} \to \widetilde{\mathrm{Ann}}_c(S) \to \mathrm{Ann}(S) \to 0 \tag{1}$$

which depends on $c \in \mathbb{R}$.

End of definition of χCFT:

... of central charge c:

For every $\tilde{A} \in \widetilde{\mathrm{Ann}}_c(S)$, a trivialization $T_{\tilde{A}} : F_A \to \mathrm{id}_{\mathcal{C}(S)}$.	For every $V \in \mathcal{C}(S)$, the map (representation) $\widetilde{\mathrm{Ann}}_c(S) \to \mathrm{LinEnd}(V)$ $\tilde{A} \mapsto [V \xrightarrow{Z_A} F_A(V) \xrightarrow{T_{\tilde{A}}} V]$ is holomorphic.

Let's check that it's a representation:

$$\begin{array}{ccccccc} V & \xrightarrow{Z_A} & F_A(V) & \xrightarrow{T_{\tilde{A}}} & V & & \\ & & \downarrow{\scriptstyle Z_B} & & \downarrow{\scriptstyle Z_B} & & \\ & & F_{B\cup A}(V) = F_B F_A(V) & \xrightarrow{F(T_{\tilde{A}})} & F_B(V) & \xrightarrow{T_{\tilde{B}}} & V \end{array}$$

Examples of χCFT

• WZW models: One per compact simple Lie group G, and level $k \in \mathbb{N}$.
• minimal models: Classified by their central charge of the form $c = 1 - 6/m(m+1)$ for $m \in \{2, 3, 4, ...\}$

$$c \;=\; 0,\;\; 1/2,\;\; 7/10,\;\; 4/5,\;\; 6/7,\;\; ...$$

Affine Lie algebras:

Fix $\mathfrak{g}$ a simple Lie algebra ($\mathfrak{sl}_n$, $\mathfrak{so}_n$, $\mathfrak{sp}_n$, or some exceptional ones G_2, F_4, $E_{6,7,8}$) and $k \in \mathbb{N}$ a level. For $X \in \mathfrak{g}$, we write

$$X_n := Xz^n \in \mathfrak{g}[t, t^{-1}],$$

so that the bracket of $\mathfrak{g}[t, t^{-1}]$ is given by $[X_m, Y_n] = [X, Y]_{m+n}$. The *affine Lie algebra*

$$\hat{\mathfrak{g}} \;=\; \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$$

is the central extension of $\mathfrak{g}[t, t^{-1}]$ associated to the cocycle $\omega(f, g) = \mathrm{Res}\langle df, g\rangle$, where $\langle\;,\;\rangle$ is the basic inner product (given by $\mathrm{Tr}(XY)$ in the case $\mathfrak{g} = \mathfrak{sl}_n$). Its bracket is given by

$$[X_m, Y_n] \;=\; [X, Y]_{m+n} + m\delta_{m+n,0}\langle X, Y\rangle K,$$

and K spans its center.

Classification of $\mathfrak{g}$-reps:

– Λ : weight lattice.
– $\alpha_1, \ldots, \alpha_r \in \Lambda$: simple roots.
– $\Lambda_+ := \{\lambda \in \Lambda : \langle \lambda, \alpha_i\rangle \geq 0\}$: dominant weights.

Main Theorem of Lie theory: simple finite dimensional $\mathfrak{g}$-reps $\leftrightarrow$ dominant weights.

We write L_λ for the $\mathfrak{g}$-rep corresponding to $\lambda \in \Lambda_+$.

In the case $\mathfrak{g} = \mathfrak{sl}_2$:
$\Lambda = \mathbb{Z}$
$\Lambda_+ = \mathbb{N}$
and $L_\lambda = \mathrm{Sym}^\lambda(\mathbb{C}^2)$ = irrep of dim $\lambda + 1$.

Rep($\hat{\mathfrak{g}}$), construction and classification:

Take $\lambda \in \Lambda_+$. Then we let $V_\lambda :=$ simple quotient of $\mathrm{Ind}_{\hat{\mathfrak{g}}_{\geq 0}\hat{\mathfrak{g}}} L_\lambda$ (quotient by the maximal proper submodule). Here, $\mathrm{Ind}\colon \mathrm{Rep}(\hat{\mathfrak{g}}_{\geq 0}) \to \mathrm{Rep}(\hat{\mathfrak{g}})$ is the adjoint to the forgetful functor, $\hat{\mathfrak{g}}_{\geq 0}$ acts on L_λ via the projection map $\hat{\mathfrak{g}}_{\geq 0} \twoheadrightarrow \mathfrak{g}$, and K acts by k.

Not all these are good to keep:

• We only want the *unitary* ones = the ones that admit a non-degenerate inner product.
• Equivalently, the *integrable* ones = the ones that exponentiate to a projective action of $LG := Map_{C^\infty}(S^1, G)$ on the Hilbert space completion H_λ of V_λ.

$\mathrm{Rep}^k(\hat{\mathfrak{g}}) :=$ level k integrable positive energy representation of $\hat{\mathfrak{g}}$.

Here, "level k" means K acts by k.

The set of irreducible level k integrable positive energy representations of the affine Lie algebra $\hat{\mathfrak{g}}$ is in canonical bijection with the finite set

$$A_k := kA \cap \Lambda = \big\{\lambda \in \Lambda_+ : \langle \Lambda, \alpha_{\max} \rangle \leq k\big\},$$

where $\alpha_{\max}$ = highest root.

In the case $\mathfrak{g} = \mathfrak{sl}_2$:
$A_k = \{0, 1, ..., k\} \subset \mathbb{N}$
$\mathrm{Rep}^k(\hat{\mathfrak{sl}}_2)$ has $k+1$ simple objects $V_0, V_1, \ldots, V_k$.

In general, A_k is the set of integer points of some simplex.

Coordinate-free affine Lie algebras:

Let $L\mathfrak{g}$ be the completion of $\mathfrak{g}[t, t^{-1}]$ given by $L\mathfrak{g} := Map_{C^\infty}(S^1, \mathfrak{g})$, and let H_λ be the Hilbert space completion of V_λ. Passing to H_λ has the advantage that it's not just $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ that acts, but also its completion $\widetilde{L\mathfrak{g}} := L\mathfrak{g} \oplus \mathbb{C}K$.

Annoyingly, the action of $\widetilde{L\mathfrak{g}}$ on H_λ is an action by *unbounded operators* (i.e., these are only densely defined operators). Nevertheless, all these admits a common dense domain of "smooth vectors" $\check{H}_\lambda \subset H_\lambda$. So we get an honest action of $\widetilde{L\mathfrak{g}}$ on $\check{H}_\lambda$.

If we replace the standard circle S^1 by some arbitrary 1-manifolds S, we let

$$\widetilde{L_S\mathfrak{g}} := Map_{C^\infty}(S, \mathfrak{g}) \oplus \mathbb{C}K$$

where the central extension is given by the same cocycle $K/2\pi i \cdot \int_S \langle df, g \rangle$.

The chiral WZW model:

The linear category associated to a 1-manifold S by the chiral WZW model is

$$\begin{aligned} \mathrm{WZW}_{G,k}(S) &= \mathrm{Rep}^k(\widetilde{L_S\mathfrak{g}}) \\ &= \text{level } k \text{ integrable positive energy representation of } \tilde{L_S}\mathfrak{g}. \end{aligned}$$

Positive energy condition:

So far, all I've told you is that a unitary representation of the affine Lie algebra has positive energy if it's one of the ones obtained by the above construction. But there's also an axiomatic way to define that condition:

Definition: A representation of Vir has positive energy if the operator associated to L_0 has discrete spectrum, the spectrum is bounded from below, and all the eigenspaces are finite dimensional.

Definition: An irreducible representation of $\widetilde{L\mathfrak{g}}$ has positive energy if it extends to a representation of $\widetilde{L\mathfrak{g}} \rtimes Vir$, and the Virasoro action has positive energy. A positive energy representation of $\widetilde{L\mathfrak{g}}$ is a finite direct sum of irreducible positive energy representations.

Segal commutation relations (definition of chiral WZW model):

Let Σ be a complex cobordism from S_{in} to S_{out}.
Let $\mathcal{C}_{in} := \mathrm{Rep}^k\big(\widetilde{L_{S_{in}}\mathfrak{g}}\big)$ and, similarly, let $\mathcal{C}_{out} := \mathrm{Rep}^k\big(\widetilde{L_{S_{out}}\mathfrak{g}}\big)$.

Given an object $(V, \rho_V) \in \mathcal{C}_{in}$, its image $(W, \rho_W) \in \mathcal{C}_{out}$ under the functor F_Σ comes equipped with a linear map $Z_\Sigma : V \to W$ satisfying:

$$\forall f \in \mathcal{O}_{hol}(\Sigma; \mathfrak{g}) \quad : \qquad Z_\Sigma \circ \rho_V(f_{in}) = \rho_W(f_{out}) \circ Z_\Sigma$$

where $f_{in/out} := f|_{\partial_{in/out}\Sigma}$.

Moreover, (W, ρ_W) and Z_Σ should be universal in the sense that for any $(\tilde{W}, \rho_{\tilde{W}}) \in \mathcal{C}_{out}$ and for any linear map $\tilde{Z} : V \to \tilde{W}$ satisfying the same relations as above, there should exist a unique morphism $\kappa : W \to \tilde{W}$ in $\mathcal{C}_{out}$ that makes the following diagram commute:

$$\begin{array}{ccc} & \xrightarrow{Z_\Sigma} & W \\ V & & \downarrow \kappa \\ & \xrightarrow[\tilde{Z}]{} & \tilde{W} \end{array}$$

The above universal property defines a functor $F_\Sigma : \mathcal{C}_{in} \to \mathcal{C}_{out}$. But, sadly, it does not guarantee that it has any good formal properties.

Open problem: Given composable cobordisms Σ_1 and Σ_2, prove that the canonical map $F_{\Sigma_1 \cup \Sigma_2}(V) \to F_{\Sigma_1} F_{\Sigma_2}(V)$ is an isomorphism.

The state-operator correspondence

The vacuum sector and the vacuum vector:

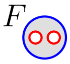
Recall that, the $\otimes$ of linear categories has $\mathsf{Vec}_{\mathrm{fd}}$ as its unit. So, by the compatibility between $\sqcup$ and $\otimes$, we must have $\mathcal{C}(\emptyset) = \mathsf{Vec}_{\mathrm{fd}}$.

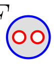
Now, $\mathbb{C} \in \mathsf{Vec}_{\mathrm{fd}}$.
Let $\mathbb{D}$ be the unit disc.

We call $H_0 := F_\mathbb{D}(\mathbb{C}) \in \mathcal{C}(S^1)$ the vacuum sector of the χCFT.
It has a vacuum vector $\Omega := Z_\mathbb{D}(1) \in H_0$, where $Z_\mathbb{D} : \mathbb{C} \to F_\mathbb{D}(\mathbb{C})$.
The other simple objects of $\mathcal{C}(S^1)$ are called charged sectors.

Everything is defined in terms of $\mathbb{D}$ $\Rightarrow$ the Möbius group Möb $:= \mathrm{Aut}(\mathbb{D})$ acts on H_0, fixing Ω.

The fusion product:

The category $\mathcal{C} = \mathcal{C}(S^1)$ is equipped with the fusion product $V \boxtimes W := F$$(V \otimes W)$.
The vacuum sector is the unit of the fusion product:

$H_0 \boxtimes H_0 := F$$(H_0 \otimes H_0) = F_{\mathrm{Pants}}(F_\mathbb{D}(\mathbb{C}) \otimes F_\mathbb{D}(\mathbb{C}))$

$$= F_{\mathrm{Pants}}(F_{\mathbb{D} \sqcup \mathbb{D}}(\mathbb{C})) = F_{\mathrm{Pants} \cup (\mathbb{D} \sqcup \mathbb{D})}(\mathbb{C}) = F_\mathbb{D}(\mathbb{C}) = H_0$$

Moreover,

$$\begin{array}{ccc} H_0 \otimes H_0 & \xrightarrow{Z_{\text{Pants}}} & H_0 \boxtimes H_0 = H_0 \\ \Downarrow & & \Downarrow \\ \Omega \otimes \Omega & \mapsto & \Omega \end{array}$$

Univalent maps:

Given a holomorphic embedding $\psi : \mathbb{D} \to \mathbb{D}$ (a univalent map), we let $A_\psi := \mathbb{D} \setminus \psi(\mathring{\mathbb{D}}) \in \text{Ann}(S^1)$.

$$\text{Univ} := \{A_\psi \,|\, \psi : \mathbb{D} \to \mathbb{D} \text{ a holomorphic embedding}\}$$

The subsemigroup $\text{Univ} \subset \text{Ann} := \text{Ann}(S^1)$ is a bit like a Borel subgrorup of an algebraic group (like upper triangular matrices in GL_n).

Let $\widetilde{\text{Univ}}$ be the universal cover of Univ:

$$0 \to \mathbb{Z} \to \widetilde{\text{Univ}} \to \text{Univ} \to 0$$

The Lie algebra of Univ is the span of $\ell_n = z^{n+1}\partial/\partial z$ for $n \geq -1$.

The Virasoro cocycle vanishes on those $\Rightarrow$ $\widetilde{\text{Univ}} \subset \widetilde{\text{Ann}}_c$ and $\text{Univ} \subset \text{Ann}_c$, where $\text{Ann}_c := \widetilde{\text{Ann}}_c/\mathbb{Z}$ is the quotient of $\widetilde{\text{Ann}}_c$ by the central $\mathbb{Z}$ which appears in (1).

The action of Möb on H_0 extends to an action of Univ, again fixing Ω.
The action of $\psi \in \text{Univ}$ on H_0 is given by:

$$H_0 = F_{\mathbb{D}}(\mathbb{C}) \xrightarrow{Z_{A_\psi}} F_{A_\psi} F_{\mathbb{D}}(\mathbb{C}) \cong F_{A_\psi \cup \mathbb{D}}(\mathbb{C}) \cong F_{\mathbb{D}}(\mathbb{C}) = H_0, \tag{2}$$

and it indeed fixes the vacuum vector: $\Omega = Z_{\mathbb{D}}(1) \overset{Z_{A_\psi}}{\longmapsto} Z_{A_\psi} Z_{\mathbb{D}}(1) \mapsto Z_{A_\psi \cup \mathbb{D}}(1) \mapsto Z_{\mathbb{D}}(1) = \Omega$.

Recall that $\widetilde{\text{Ann}}_c$ acts on all the sectors of the χCFT by the formula:

$$\widetilde{\text{Ann}}_c \to \text{End}(V) : \tilde{A} \mapsto T_{\tilde{A}} \circ Z_A. \tag{3}$$

Lemma. *The restriction of the action* (2) *of* Univ *on* H_0 *along the projection map* $\widetilde{\text{Univ}} \to \text{Univ}$ *equals the restriction of the action* (3) *of* $\widetilde{\text{Ann}}_c$ *on* H_0 *to the subsemigroup* $\widetilde{\text{Univ}} \subset \widetilde{\text{Ann}}_c$.

Corollary: The central $\mathbb{Z} \subset \widetilde{\text{Ann}}_c$ acts trivially on H_0, and the action of $\widetilde{\text{Ann}}_c$ on H_0 descends to an action of Ann_c.

Local operators:

A *primary operator* of conformal dimension Δ is a gadget φ that assigns to every complex cobordism Σ equipped with:

- distinct interior points $z_1, \ldots, z_n \in \mathring{\Sigma}$, and
- non-zero tangent vectors $v_i \in T_{z_i}\Sigma$,

and to every object $V \in \mathcal{C}(\partial_{in}\Sigma)$, a linear map

$$Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)} : V \to F_\Sigma(V).$$

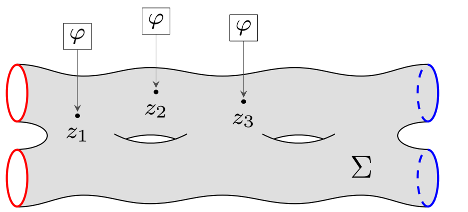
These maps are homogeneous of degree Δ in the v_i's:

$$Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_i;av_i),\ldots,\varphi(z_n;v_n)} = a^\Delta Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)} \qquad \forall a \in \mathbb{C}^\times, \tag{4}$$

and agree with Z_Σ when $n = 0$. Moreover, they satisfy the same axioms that the Z_Σ satisfy (naturality in V and in Σ, compatibility with disjoint union, and with composition of cobordisms).

We abbreviate the relation (4) by writing: $\varphi(z; av) = a^\Delta\varphi(z; v)$.

The map $Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)}$ is called the *evolution operator with point insertions*:



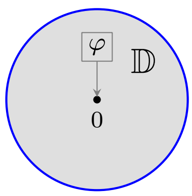
$$Z_{\Sigma,\varphi(z_1;v_1),\ldots,\varphi(z_n;v_n)} : V \longrightarrow F_\Sigma(V)$$

Theorem. (State-operator correspondence) *There is a natural bijection*

$$\left\{ \begin{array}{c} \text{Primary operators of} \\ \text{conformal dimension } \Delta \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{States } \xi \in H_0 \text{ such that} \\ L_0(\xi) = \Delta\xi \text{ and } L_n(\xi) = 0\, \forall n > 0 \end{array} \right\}.$$

Proof. Given an operator φ, the corresponding state $\xi \in H_0$ is given by

$\xi :=$



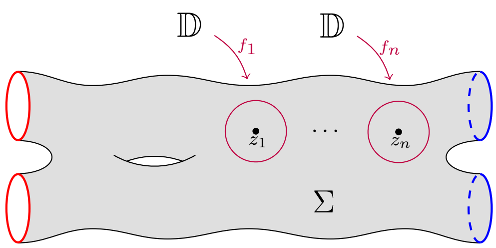
$= Z_{\mathbb{D},\varphi(0;1)}(1) \in H_0.$

$1 \in \mathbb{C} \in \mathsf{Vec} = \mathcal{C}(\emptyset)$

$1 \in T_0\mathbb{D} = \mathbb{C}$

Conversely, starting from a vector $\xi \in H_0$ that satisfies the equations $L_0(\xi) = \Delta\xi$ and $L_n(\xi) = 0\, \forall n > 0$, we proceed as follows. Given a complex cobordism Σ together with points $z_1, \ldots, z_n$ and tangent vectors $v_i \in T_{z_i}\Sigma$, choose disjoint embeddings $f_i : \mathbb{D} \to \Sigma$, $f_i(0) = z_i$, and let $\Sigma^0 := \Sigma \setminus (f_1(\mathring{\mathbb{D}}) \sqcup \ldots \sqcup f_n(\mathring{\mathbb{D}}))$

Σ^0 :



We then define

$$Z_{\Sigma,\varphi(z_1),\ldots,\varphi(z_n)} := \prod \left(\frac{v_i}{f_i'(0)}\right)^{\Delta} Z_{\Sigma^0}\big(\xi \otimes \ldots \otimes \xi \otimes -\big). \tag{5}$$

Let's check that this map lands in the right place:

$$\begin{array}{l} Z_{\Sigma^0} : \underset{\xi}{\underset{\Cup}{H_0}} \otimes \ldots \otimes \underset{\xi}{\underset{\Cup}{H_0}} \otimes V = F_{\mathbb{D}}(\mathbb{C}) \otimes \ldots \otimes F_{\mathbb{D}}(\mathbb{C}) \otimes V \\ \qquad\qquad\qquad\qquad \longrightarrow F_{\Sigma^0}\big(F_{\mathbb{D}}(\mathbb{C}) \otimes \ldots \otimes F_{\mathbb{D}}(\mathbb{C}) \otimes V\big) \\ \qquad\qquad\qquad\qquad = F_{\Sigma^0 \cup (\mathbb{D} \sqcup \ldots \sqcup \mathbb{D})}(\mathbb{C} \otimes \ldots \otimes \mathbb{C} \otimes V) = F_{\Sigma}(V). \quad \checkmark \end{array}$$

□

Given a bunch of primary operators $\varphi_1, \ldots, \varphi_n$, with corresponding vectors $\xi_1, \ldots, \xi_n \in H_0$, it's now easy to adapt the definition (5):

$$Z_{\Sigma,\varphi_1(z_1;v_i),\ldots,\varphi_n(z_n;v_n)} := \prod \left(\frac{v_i}{f_i'(0)}\right)^{\Delta_i} Z_{\Sigma^0}\big(\xi_1 \otimes \ldots \otimes \xi_n \otimes -\big). \tag{6}$$

Here, as before, $\Sigma^0 = \Sigma \backslash (f_1(\mathring{\mathbb{D}}) \sqcup \ldots \sqcup f_n(\mathbb{D}))$ for some $f_i : \mathbb{D} \to \Sigma$ satisfying $f_i(0) = z_i$.

Descendants:

There's a more general notion, called a *descendant* operator: In the definition of primary operator, just replace the tangent vectors v_i by a local coordinate $j_i : \mathbb{C} \to \Sigma : 0 \mapsto z_i$ (a finite order jet suffices) and require the equation $\varphi(z; j \circ (z \mapsto az)) = a^{\Delta} \varphi(z; j)$ to hold.

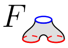
Theorem. (State-operator correspondence) *There is a natural bijection:*

$$\Big\{ \text{Operators of conformal dimension } \Delta \Big\} \longleftrightarrow \Big\{ \xi \in H_0 \,\Big|\, L_0(\xi) = \Delta\xi \Big\}. \tag{7}$$

The proof goes along the same lines.

Modularity

Let $\mathcal{C} := \mathcal{C}(S^1)$.
Recall that the bilinear functor $\boxtimes := F_{\text{Pants}} : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ is called the *fusion product*.
For $V, W \in \mathcal{C}$, we have:

F $: V \otimes W \mapsto V \boxtimes W$

and

Z $: V \otimes W \to V \boxtimes W.$

One can think of the latter as a bilinear map $V \times W \to V \boxtimes W$ (in much the same way as the tensor product $M \otimes_R N$ of two modules over a ring comes with a canonical bilinear map $M \times N \to M \otimes_R N$).

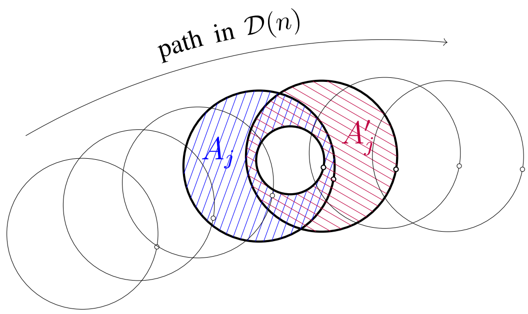
Note: The bilinear map Z_Σ genuinely depends on the complex structure on Σ. But the fusion product F_Σ is essentially independent of the complex structure.

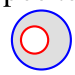
For simplicity, let us restrict to only using pairs of pants Σ that are embedded in $\mathbb{C}$, where the boundary circles are round, and parametrized by $z \mapsto az+b$ with $a, b \in \mathbb{C}$. We can easily generalise the fusion functor to define the "n-fold fusion", as follows. Let

$$\mathcal{D}(n) := \left\{ \begin{array}{c} n \text{ non-overlapping little circles in } \mathbb{D} \text{ with} \\ \partial \text{ parametrized by } z \mapsto az+b \text{ with } a, b \in \mathbb{C}\,. \end{array} \right\} \tag{8}$$

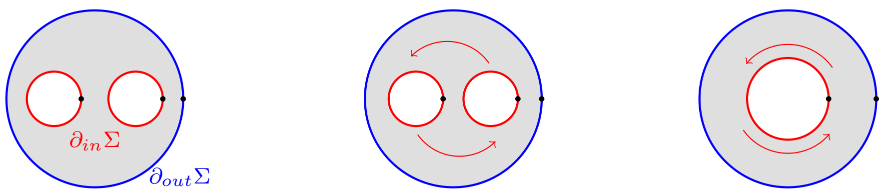
Associated to every $P \in \mathcal{D}(n)$, we have a functor $F_P : \mathcal{C} \otimes \ldots \otimes \mathcal{C} \to \mathcal{C}$.

These assemble to an action of $\mathcal{D} = \{D(n)\}_{n\in\mathbb{N}}$, the operad of framed little discs, on the category $\mathcal{C}$. Moreover, given $P_1, P_2 \in \mathcal{D}(n)$, for each homotopy class of path $\gamma : [0,1] \to \mathcal{D}(n)$ from P_1 to P_2, there is an associated invertible natural transformation $F_{P_1} \to F_{P_2}$



obtained as the composite of a zig-zag of invertible natural transformations induced by the $T_{\tilde{A}}$, where $A =$  $\in$ Univ is equipped with a canonical lift to an element $\tilde{A} \in \widetilde{\text{Univ}} \subset \widetilde{\text{Ann}}_c$. All in all, we have:

fusion product: $- \boxtimes -$ | braiding β: | twist θ:



$\Rightarrow \mathcal{C}$ is a *balanced tensor category*. (cf [video synoptic chart of tensor categories] available at https://people.math.osu.edu/penneys.2/Synoptic.mp4).

$\mathcal{C}$ is in fact *modular*.
For that, three more properties need to be checked (no extra data needed):

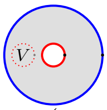
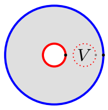
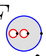
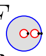
- $\mathcal{C}$ is rigid (all objects admit duals)
- it's ribbon $(\theta_X)^\vee = \theta_{X^\vee}$.

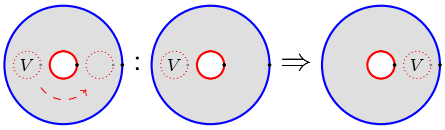
• $\mathcal{C}$ is non-degenerately braided (the S-matrix is invertible)
The last condition is equivalent to $\mathcal{Z}_2(\mathcal{C}) = \mathsf{Vec}_{\mathrm{fd}}$, where

$$\mathcal{Z}_2(\mathcal{C}) := \{V \in \mathcal{C} \,|\, \beta^+_{V,W} = \beta^-_{V,W} : V \boxtimes W \to W \boxtimes V,\ \forall W \in \mathcal{C}\}$$

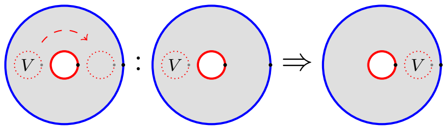
is the *Müger center* of $\mathcal{C}$, and $\beta^+_{V,W} := \beta_{V,W}$, $\beta^-_{V,W} := \beta^{-1}_{W,V}$.

Proof that the braiding is non-degenerate.

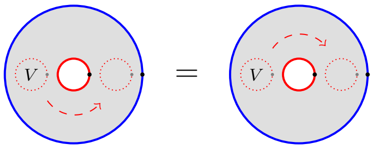
Given an object $V \in \mathcal{C}$, let us write  and  for the functors $\mathcal{C} \to \mathcal{C}$ given by $W \mapsto F$$(V \otimes W)$ and $W \mapsto F$$(W \otimes V)$, respectively. The 'underbraiding' and 'overbraiding' produce natural transformations



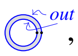
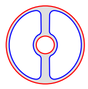
and

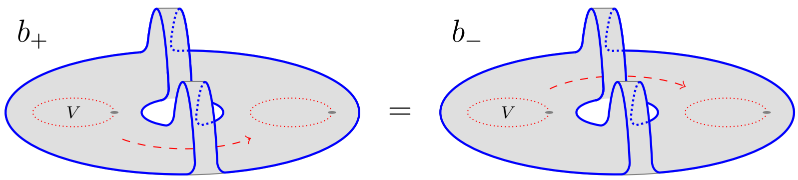


We must show that the only simple object $V \in \mathcal{C}$ that satisfies

 (9)

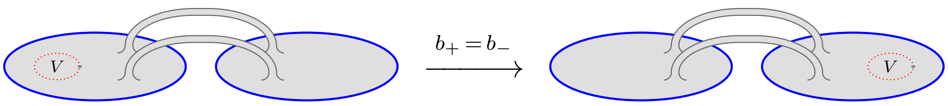
is the unit object $\mathbf{1}_{\mathcal{C}}$.

Composing the inner (red) boundary circle of (9) with , we may assume without loss of generality that it is also labelled 'out'. Further composing (9) with  we learn that



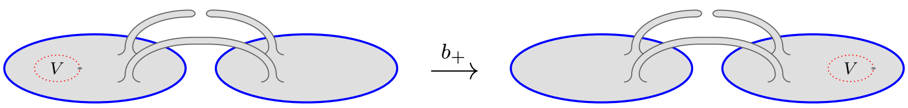
(10)

Redrawing the above cobordisms in a different way (without changing the topology), the equality (10) is an equality between two natural transformations

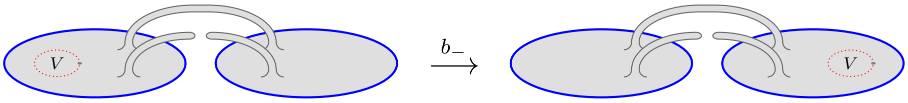


(11.a)

The natural transformation b_+ is the one where V travels through the front tube, and the natural transformation b_- is the one where it travels through the back tube. If one of the tubes is snipped, then we still have one but not the other of the two natural transformations:



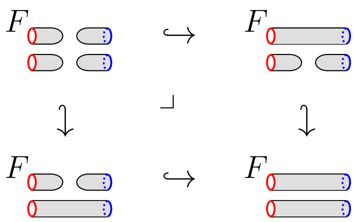
(11.b)



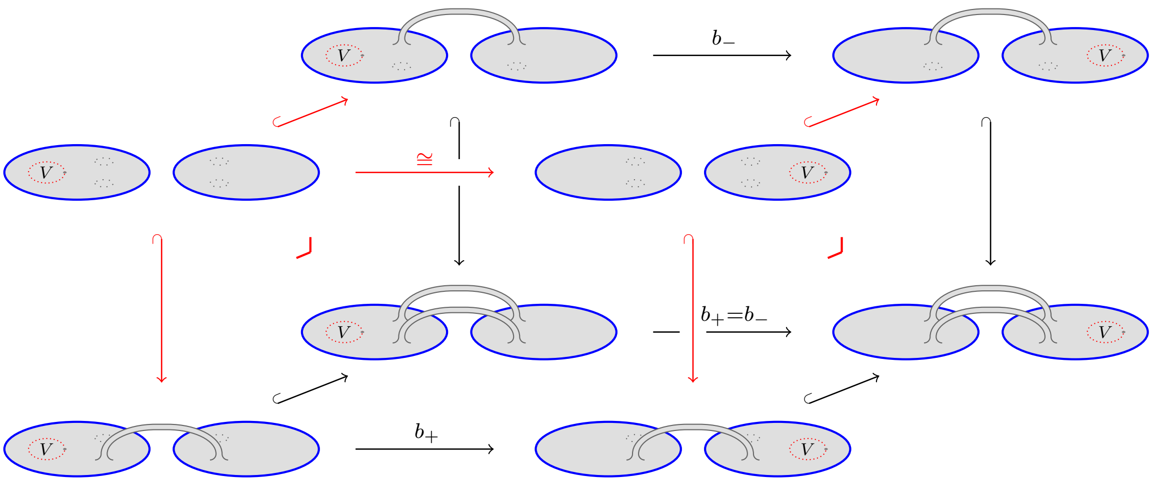
(11.c)

Claim: We have an inclusion:

It follows that we have a pullback diagram:



By the first part of the claim, the functors (11.a), (11.b), and (11.c) fit into a commutative diagram (the black arrows in the diagram below). And by the second part of the claim, we can complete this commutative by taking pullbacks (the red arrows):



The maps labelled 'b_+', 'b_-', and '$b_+ = b_-$' are all isomorphisms, therefore so is their pushout. That pushout, which is an isomorphism, is a map $V \otimes \mathbf{1}_\mathcal{C} \to \mathbf{1}_\mathcal{C} \otimes V$ in $\mathcal{C} \otimes \mathcal{C}$. So we must have $V \cong \mathbf{1}_\mathcal{C}$. □