Exercise 1.
The Virasoro algebra is the universal central extension of the Witt algebra.

Let $\mathfrak{g}$ be a Lie algebra, and let A be a vector space. A 2-cocycle is a bilinear map $\omega : \mathfrak{g} \times \mathfrak{g} \to A$ which is antisymmetric, and satisfies

$$\sum^{3} \omega([X, Y], Z) = 0.$$

Given a 2-cocycle, one can form a central extension $\tilde{\mathfrak{g}} := \mathfrak{g} \oplus A$, with Lie bracket

$$\big[(X, a), (Y, b)\big]_{\tilde{\mathfrak{g}}} := \big([X, Y]_{\mathfrak{g}}, \omega(X, Y)\big)$$

which fits into a central extension $0 \to A \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$ (an extension such that $A \subset Z(\tilde{\mathfrak{g}})$). If the cocycle can written in the form

$$\omega(X, Y) = \mu([X, Y])$$

for some linear map $\mu : \mathfrak{g} \to A$ (typically not a Lie algebra homomorphism), then we say that ω is a trivial 2-cocycle, and write $\omega = d\mu$.

Theorem. *The second Lie algebra cohomology group*

$$H^2(\mathfrak{g}; A) := \frac{\{\text{2-cocycles}\}}{\{\text{trivial 2-cocycles}\}}$$

is in bijection with the set of isomorphism classes of central extensions of $\mathfrak{g}$ by A, where two central extensions $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'$ are isomorphic if there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}_A} & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathfrak{g}}} & & \\ 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}}' & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

where the two outer vertical maps are identity maps.

Proof outline. (→) We already saw how to construct a central extension from a 2-cocycle. Suppose now that $\omega_2 - \omega_1 = d\mu$. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}}_1 = \mathfrak{g} \oplus A & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ \mu & 1 \end{smallmatrix}\right)} & & \Big\| & & \\ 0 & \longrightarrow & A & \longrightarrow & \tilde{\mathfrak{g}}_2 = \mathfrak{g} \oplus A & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \end{array}$$

is an isomorphism. So the map $\{\text{2-cocycles}\} \to \{\text{central extensions}\}$ descends to a map $H^2(\mathfrak{g}; A) \to \{\text{iso classes of central extensions}\}$.

(←) Given a central extension of $\mathfrak{g}$ by A, pick a splitting

$$0 \longrightarrow A \longrightarrow \tilde{\mathfrak{g}} \underset{r}{\overset{s}{\leftrightarrows}} \mathfrak{g} \longrightarrow 0$$

(usually not a Lie algebra homomorphism) and let $\omega(X, Y) := [s(X), s(Y)] - s([X, Y])$. Given another splitting, we can write it as $s' = s + \mu$ for some $\mu : \mathfrak{g} \to A$. The corresponding cocycles satisfy $\omega' = \omega - d\mu$. So they're equal in $H^2(\mathfrak{g}; A)$. $\square$

The following lemma will be useful:

Lemma. *Let $\mathfrak{g}$ be a Lie algebra, and let $X \in \mathfrak{g}$ be such that $\mathrm{ad}(X)$ exponentiates to a 1-parameter family of automorphisms of $\mathfrak{g}$. For $\xi \in \mathfrak{g}$, let $\xi_t := \exp\big(t \cdot \mathrm{ad}(X)\big)(\xi)$, so that $\frac{d}{dt}\xi_t = [X, \xi_t]$. Then, for any 2-cocycle ω, we have*

$$[\omega] = [\omega_t] \in H^2(\mathfrak{g}),$$

where $\omega_t(\xi, \eta) := \omega(\xi_t, \eta_t)$.

Proof.

$$\begin{aligned}
\omega(\xi_T, \eta_T) - \omega(\xi, \eta) &= \int_0^T \big(\tfrac{d}{dt}\omega(\xi_t, \eta_t)\big)dt \\
\text{[}\tfrac{d}{dt}\xi_t = [X,\xi_t]\text{]} \quad &= \int_0^T \big(\omega([X, \xi_t], \eta_t) + \omega(\xi_t, [X, \eta_t])\big)dt \\
\text{[cocycle identity]} \quad &= \int_0^T \omega(X, [\xi_t, \eta_t])dt \\
\text{[}\xi \mapsto \xi_t \text{ is an automorphism]} \quad &= \int_0^T \omega(X, [\xi, \eta]_t)dt = \mu([\xi, \eta])
\end{aligned}$$

where $\mu(\xi) := \int_0^T \omega(X, \xi_t)dt$. □

Suppose that $\mathrm{ad}(X)$ exponentiates to an action of S^1 on $\mathfrak{g}$ by Lie algebra automorphisms. Then letting $\mathrm{avg}_{S^1}(\omega) := \int_{S^1} \omega_t dt$, we have

$$\big[\mathrm{avg}_{S^1}(\omega)\big] = \big[\textstyle\int_{S^1} \omega_t dt\big] = \int_{S^1}[\omega_t]dt = \int_{S^1}[\omega]dt = [\omega] \qquad \text{in} \quad H^2(\mathfrak{g}, A)$$

for any 2-cocycle ω. Given a linear map $\mu : \mathfrak{g} \to A$, let $\mu_t(\xi) := \mu(\xi_t)$, and let us define $\mathrm{avg}_{S^1}(\mu) := \int_{S^1} \mu_t dt$. If a 2-cocycle ω is trivial, i.e., if there exists μ such that $\omega = d\mu$, then there also exists an S^1-invariant μ with that same property: indeed, letting $\mu' := \mathrm{avg}_{S^1}(\mu)$ we have

$$d\mu' = d\big(\mathrm{avg}_{S^1}(\mu)\big) = \mathrm{avg}_{S^1}(d(\mu)) = \mathrm{avg}_{S^1}(\omega) = \omega.$$

From the above discussion, we deduce that

$$H^2(\mathfrak{g}; A) = \frac{\big\{\, S^1\text{-invariant 2-cocycles} \,\big\}}{\big\{ d\mu \,\big|\, \mu : \mathfrak{g} \to A,\ \ \mu \text{ is } S^1\text{-invariant} \big\}}$$

Exercise:
Let $\mathbb{W} = \mathrm{Span}\{\ell_n\}_{n \in \mathbb{Z}}$ be the Witt algebra, with Lie bracket $[\ell_m, \ell_n] = (m - n)\ell_{m+n}$. Prove that $\dim(H^2(\mathbb{W}, \mathbb{C}))$ is one-dimensional, spanned by the Virasoro cocycle.

Exercise 2.

The trivialisation $T_{\tilde{A}} : F_A(V) \to V$ associated to a lifted annulus $\tilde{A} \in \widetilde{\mathrm{Ann}}_c$.

Let $\mathfrak{g}$ be a simple Lie algebra, and let $k \in \mathbb{N}$ be a level. The corresponding chiral WZW model associates to a closed 1-manifold S the category

$$\mathcal{C}(S) = \{ \text{ level } k \text{ integrable positive energy representations of } \widetilde{L_S\mathfrak{g}} \ \}$$

where $\widetilde{L_S\mathfrak{g}} = L_S\mathfrak{g} \oplus \mathbb{C}{\cdot}K$ is the central extension of $L_S\mathfrak{g} = C^\infty(S, \mathfrak{g})$ defined by the cocycle $\omega_k(f, g) := \frac{1}{2\pi i}\int_S \langle f, dg \rangle$, and 'level k' means that K acts by k.

Let Σ be a complex cobordism, let $\mathcal{C}_{in} := \mathcal{C}(\partial_{in}\Sigma)$, and let $\mathcal{C}_{out} := \mathcal{C}(\partial_{out}\Sigma)$. When $\partial\Sigma \neq \emptyset$ (more precisely, when each connected component of Σ has non-empty boundary), we can describe the concrete functor

$$F_\Sigma : \mathcal{C}_{in} \longrightarrow \mathcal{C}_{out}.$$

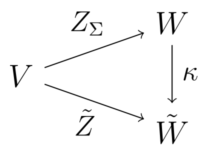
associated to the complex cobordism Σ by means of the *Segal commutation relations*:

Definition 1 Given an object $(V, \rho_V) \in \mathcal{C}_{in}$, its image $(W, \rho_W) \in \mathcal{C}_{out}$ under the functor F_Σ comes equipped with a linear map $Z_\Sigma : V \to W$ satisfying:

$$\boxed{\forall f \in \mathcal{O}(\Sigma; \mathfrak{g}) \qquad Z_\Sigma \circ \rho_V(f_{in}) = \rho_W(f_{out}) \circ Z_\Sigma} \tag{1}$$

where $f_{in/out} := f|_{\partial_{in/out}\Sigma}$.

Moreover, (W, ρ_W) and Z_Σ should be universal in the sense that for any $(\tilde{W}, \rho_{\tilde{W}}) \in \mathcal{C}_{out}$ and for any linear map $\tilde{Z} : V \to \tilde{W}$ satisfying the same relations as above, there should exist a unique morphism $\kappa : W \to \tilde{W}$ in $\mathcal{C}_{out}$ that makes the following diagram commute:



If our cobordism is an annulus $A \in \mathrm{Ann}(S)$, then the trivialization

$$T_{\tilde{A}} \ : \ F_A \longrightarrow \mathrm{id}_{\mathcal{C}(S)}$$

associated to a lift $\tilde{A} \in \widetilde{\mathrm{Ann}}_c$ is constructed as follows. For every $(V, \rho) \in \mathcal{C}(S)$, by the positive energy condition, the action $\rho : \widetilde{L_S\mathfrak{g}} \to \mathrm{End}(V)$ extends to an action, again denoted ρ, of $\widetilde{L_S\mathfrak{g}} \rtimes \widetilde{\mathrm{Diff}}_c(S)$. By definition, this means that we have actions of $\widetilde{L_S\mathfrak{g}}$ and of $\widetilde{\mathrm{Diff}}_c(S)$ on V, satisfying the following covariance relation:

$$\rho({}^\varphi f) = \rho(\varphi)\rho(f)\rho(\varphi^{-1}) \qquad \forall f \in \mathcal{C}^\infty(S, \mathfrak{g})$$

Here, $f \mapsto {}^{\varphi}f$ denotes the action of (the image of) φ in $\mathrm{Diff}(S)$ on $\mathcal{C}^\infty(S, \mathfrak{g}) \subset \widetilde{L_S\mathfrak{g}_k}$.

Since the action of $\widetilde{\mathrm{Diff}}_c(S)$ on V has positive energy, it extends to a holomorphic representation of $\widetilde{\mathrm{Ann}}_c(S)$ on V.[1] We construct the morphism $T_{\tilde{A}} : F_A(V) \to V$ making the following diagram commute

$$\begin{array}{ccc} & \xrightarrow{Z_A} & F_A(V) \\ V & & \Big\downarrow T_{\tilde{A}} \\ & \xrightarrow[\rho(\tilde{A})]{} & V \end{array}$$

by applying the universal property in Definition 1 to the object $V \in \mathcal{C}(S)$, and to the map $\rho(\tilde{A}) : V \to V$.

Exercise:
Show that the map $\rho(\tilde{A}) : V \to V$ satisfes the desired relation (1):

$$\rho(\tilde{A})\rho(f_{in}) = \rho(f_{out})\rho(\tilde{A}) \qquad \forall f \in \mathcal{O}(A; \mathfrak{g}),$$

and that we are thus in a position to invoke the universal property in Definition 1.

[1]This is stated as a theorem in [Y. Neretin. *Holomorphic continuations of representations of the group of diffeomorphisms of the circle*; translation in Math. USSR-Sb. 67 (1990), no. 1, 75–97], but the paper does not include a proof of holomorphicity.