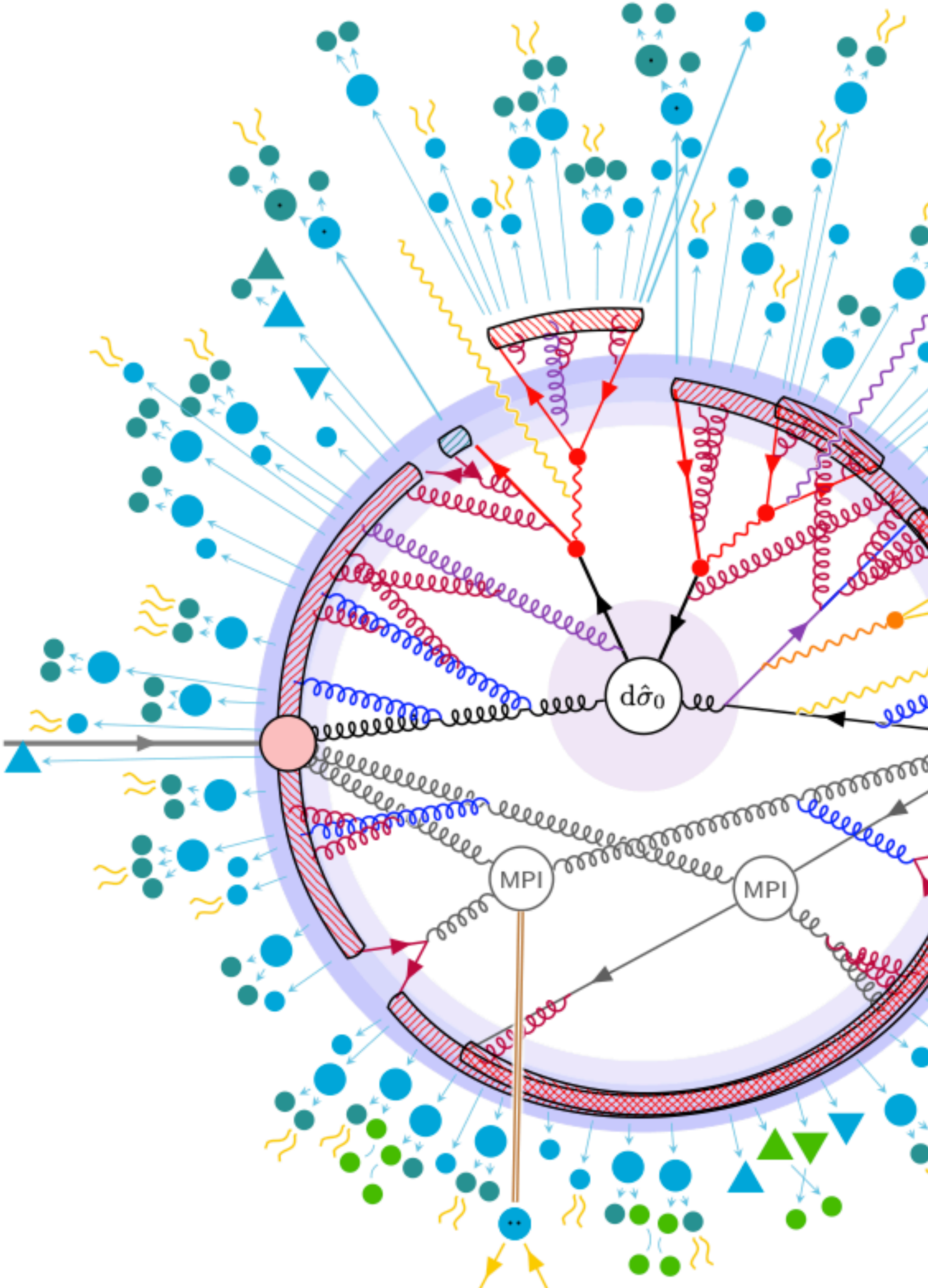

Introduction to QCD

Lecture 2

HASCO Summer School 2024
Göttingen

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(University of Wuppertal)

BERGISCHE
UNIVERSITÄT
WUPPERTAL

Overview

Lecture 1 (yesterday):

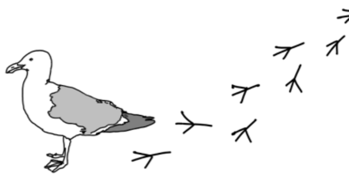
- Hadrons, partons, and all that
- Colour charges
- QCD Lagrangian and Feynman rules
- QCD at test
- QCD-improved parton model

Lecture 2 (today):

- Fixed-order calculations and jets
- QCD radiation
- Running coupling and confinement
- QCD in event generators

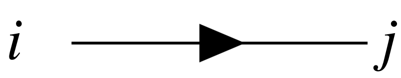
Recap: birdtracks I

Colour factors can most easily be calculated using birdtrack diagrams

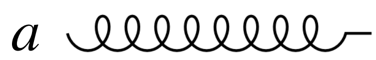


[Keppeler 1707.07280]

- Kronecker deltas are represented by colour lines
  (+ implicit summation over colour indices)

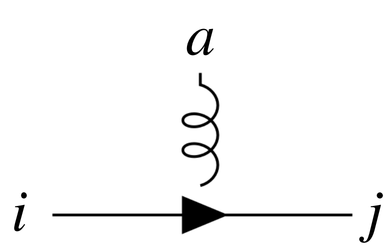


$= \delta^i_j$

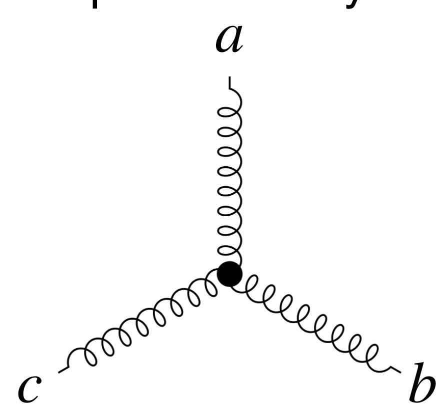


$= \delta^{ab}$

- generators and structure constants are represented by vertices

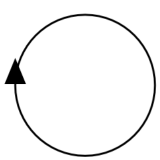
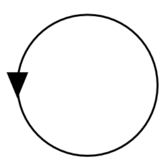


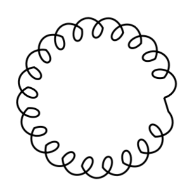
$= (T^a)^i_j$

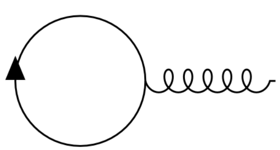


$= \mathrm{i} f^{abc}$

- there are N_C quark colours and $N_\mathrm{C}^2 - 1$ gluon colours


$=$

$= N_\mathrm{C}$


$= N_\mathrm{C}^2 - 1$

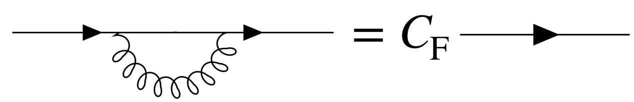
- generators are traceless  $= 0$

Recap: birdtracks II

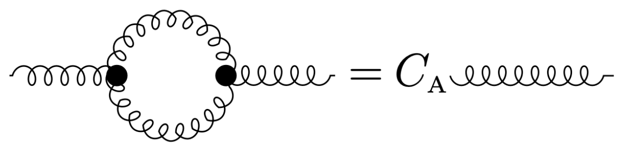
Rewrite $SU(N_{\mathrm{C}})$ identities as birdtracks

- **Casimir invariants**

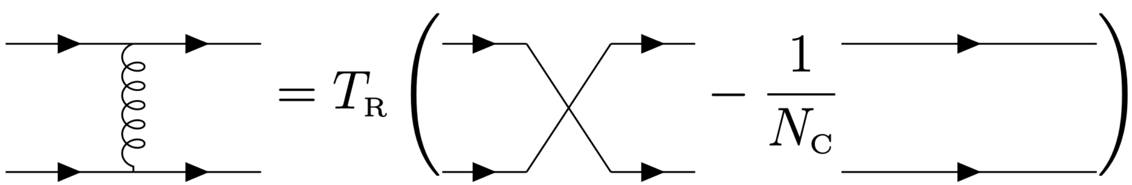
Fundamental Casimir:



Adjoint Casimir:



- **Fierz identity**:



Recap: perturbative QCD

Idea:

at high scales $\alpha_s \approx 0.1 \Rightarrow$ series expansion in powers of the strong coupling α_s

$$d\sigma \sim C_0 + \alpha_s C_1 + \underbrace{\alpha_s^2 C_2}_{\text{small}} + \underbrace{\alpha_s^3 C_3}_{\text{smaller}} + \underbrace{\quad\dots\quad}_{\text{negligible?}}$$

→ improve prediction by successively correcting leading-order approximation

(leading order, next-to-leading order, next-to-next-to-leading order, …)

Example:

$$\frac{1}{7} = \frac{1}{10}\left(1 - \frac{3}{10}\right)^{-1} \approx \frac{1}{10}(\underset{\text{LO}}{1} + \underset{\text{NLO}}{0.3} + \underset{\text{NNLO}}{0.09} + \underset{\text{N3LO}}{0.027} + \dots)$$

Exact: 0.142857143

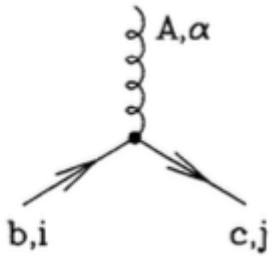
	LO	NLO	NNLO	N3LO
Sum:	0.1	0.13	0.139	0.1417
Error:	30%	9%	3%	1%

Need: set of universal rules to calculate cross sections order by order

Recap: Feynman rules of QCD — vertices

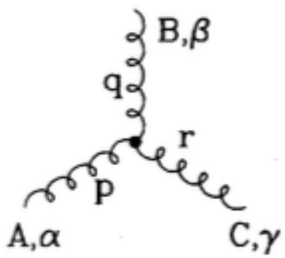
Three types of vertices in QCD

- quark-gluon vertex (∼ fermion-photon vertex in QED)



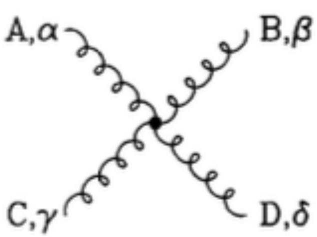
$$-ig\ (t^A)_{cb}\ (\gamma^\alpha)_{ji}$$

- pure gluon vertices (result of non-abelian structure of $\mathrm{SU}(3)$)



$$-g\ f^{ABC}[(p-q)^\gamma g^{\alpha\beta}+(q-r)^\alpha g^{\beta\gamma}+(r-p)^\beta g^{\gamma\alpha}]$$

(all momenta incoming, $p+q+r = 0$)

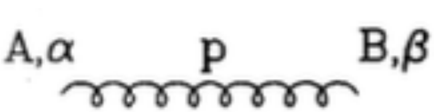


$$-ig^2\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^2\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^2\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$

Recap: Feynman rules of QCD — propagators

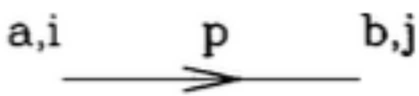
Propagators $\hat{=}$ Green's functions of inhomogeneous equations of motion

- gluon propagator (vector propagator)



$$= \delta^{AB} \frac{-g^{\alpha\beta} + (1-\lambda)\frac{p^\alpha p^\beta}{p^2 + \mathrm{i}\varepsilon}}{p^2 + \mathrm{i}\varepsilon} \overset{\lambda=1}{=} \delta^{AB} \frac{-g^{\alpha\beta}}{p^2 + \mathrm{i}\varepsilon}$$

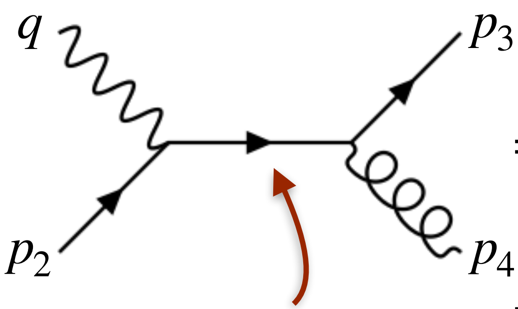
- quark propagator (spinor propagator)



$$= \delta^a_b \frac{\mathrm{i}\not{p} + m}{p^2 - m^2 + \mathrm{i}\varepsilon}$$

Recap: first-order corrections in DIS

Consider the s-channel contribution to $\gamma^* \mathrm{q} \to \mathrm{qg}$



$$= \bar{u}(p_3)(-\mathrm{i}g_s)T^a\gamma^\mu \frac{\mathrm{i}(\not{p}_3 + \not{p}_4)}{(p_3+p_4)^2}(-\mathrm{i}e)\gamma^\nu u(p_2)\varepsilon_\nu(q)\varepsilon^*_\mu(p_4)$$

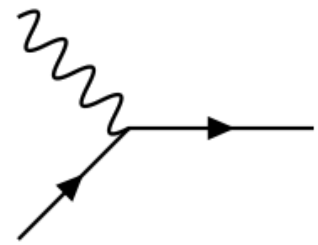
In the collinear limit $3 \parallel 4$, the intermediate quark goes on-shell $p_{34}^2 \to 0$

$$= \bar{u}(p_3)(-\mathrm{i}g_s)T^a\gamma^\mu \frac{\mathrm{i}\not{p}_{34}}{p_{34}^2}(-\mathrm{i}e)\gamma^\nu u(p_2)\varepsilon_\nu(q)\varepsilon^*_\mu(p_4)$$

$$\to \bar{u}(p_3)(-\mathrm{i}g_s)T^a\gamma^\mu \frac{\mathrm{i}\sum_\lambda u_\lambda(p_{34})\bar{u}_\lambda(p_{34})}{p_{34}^2}(-\mathrm{i}e)\gamma^\nu u(p_2)\varepsilon_\nu(q)\varepsilon^*_\mu(p_4)$$

collinear emissions factorise!

$$= g_s T^a \frac{1}{p_{34}^2}\sum_\lambda \left[\bar{u}(p_3)\not{\varepsilon}^*(p_4)u_\lambda(p_{34})\right] \times$$



Recap: scaling violations

In the collinear limit, the squared amplitude becomes

$$|\mathcal{M}_{\gamma^* \mathrm{q}\to\mathrm{qg}}|^2 \sim g_\mathrm{s}^2 \frac{1}{p_{34}^2} P_{qg}(z) |\mathcal{M}_{\gamma^* \mathrm{q}\to\mathrm{q}}|^2, \;\; P_{qg}(z) = C_\mathrm{F}\frac{1+z^2}{1-z}$$

The effect of the gluon emission on the cross section is given by

$$\sigma_{\gamma^* \mathrm{q}\to\mathrm{qg}} \sim \sigma_{\gamma^* \mathrm{q}\to\mathrm{q}} g_\mathrm{s}^2 \frac{1}{8\pi^2}\int_{\mu^2}^{Q^2} \frac{\mathrm{d}s_{34}}{s_{34}} P_{qg}(z) = \sigma_{\gamma^* \mathrm{q}\to\mathrm{q}} \frac{\alpha_\mathrm{s}}{2\pi} P_{qg}(z) \log\frac{Q^2}{\mu^2}$$

reference scale

We can now write the structure functions as

$$2F_1(x,Q^2) = \sum_i Q_i^2 \int_x^1 \frac{\mathrm{d}y}{y} f_i(y) \left(\delta\left(1-\frac{x}{y}\right) + \frac{\alpha_\mathrm{s}}{2\pi} P_{qg}\left(\frac{x}{y}\right) \log\frac{Q^2}{\mu^2} \right)$$

parton distribution function

no scaling

logarithmic scaling

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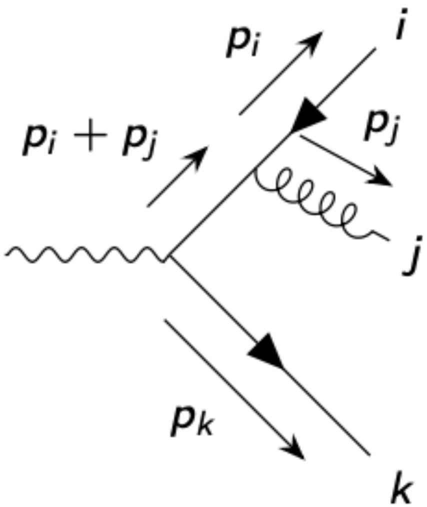
Consider real correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



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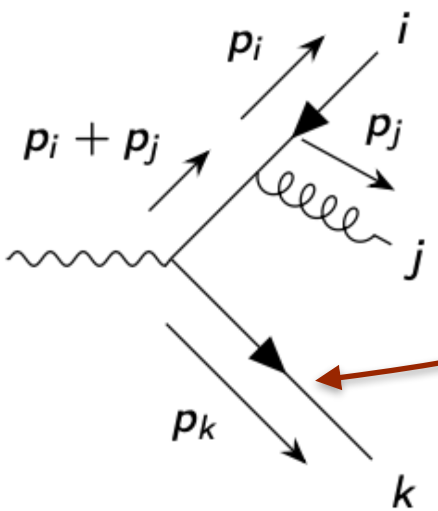
Real corrections

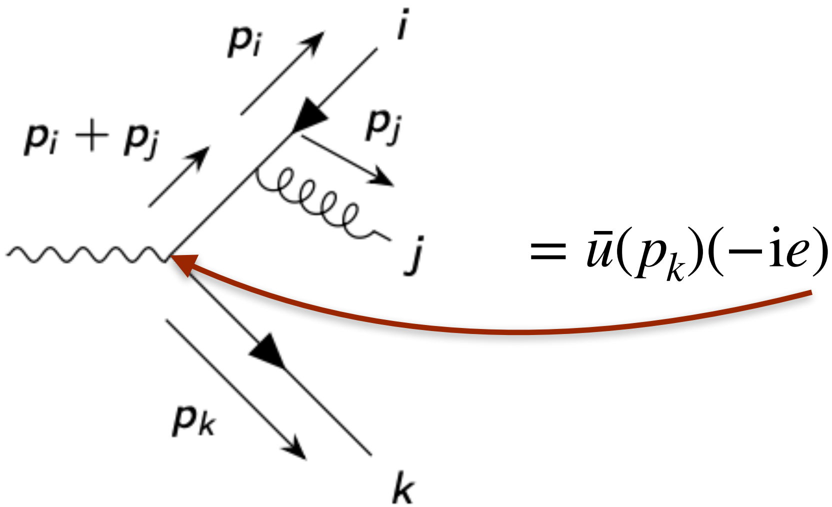
Consider real correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



$= \bar{u}(p_k)$

Real corrections

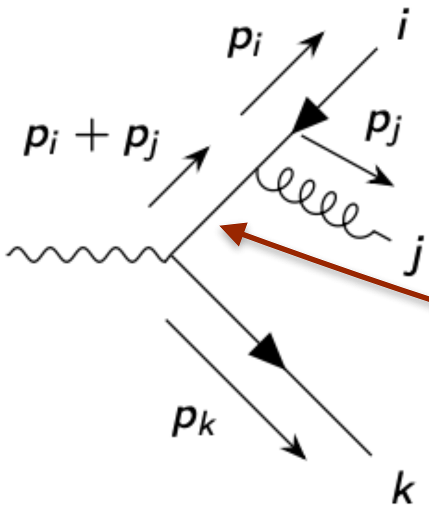
Consider real correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



$$= \bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)$$

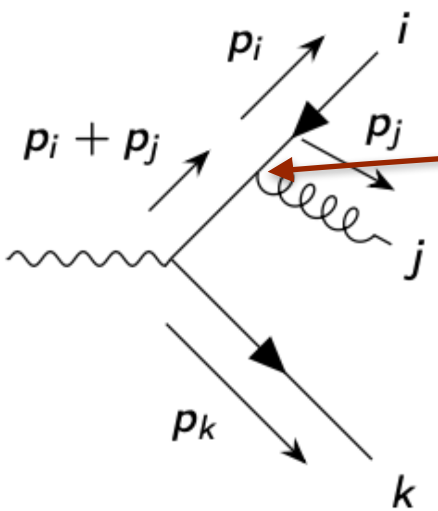
Real corrections

Consider real correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



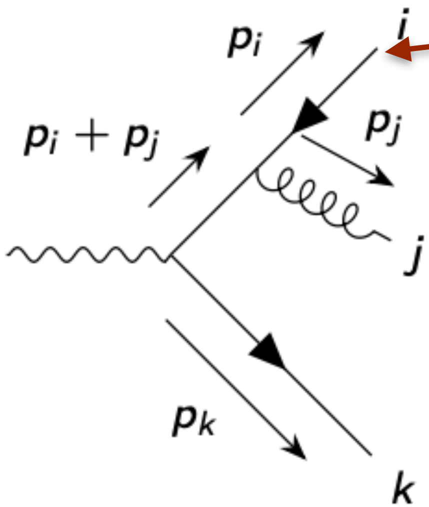
$$= \bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)\frac{\mathrm{i}(\not{p}_i + \not{p}_j)}{(p_i + p_j)^2}$$

Consider real correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



$$= \bar{u}(p_k)(-ie)\not{\epsilon}_\mu(q)\frac{i(\not{p}_i + \not{p}_j)}{(p_i + p_j)^2}(-ig_s)T^a\gamma^\mu$$

Consider real correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



$$= \bar{u}(p_k)(-ie)\not{\varepsilon}_\mu(q)\frac{i(\not{p}_i + \not{p}_j)}{(p_i + p_j)^2}(-ig_s)T^a\gamma^\mu v(p_i)$$

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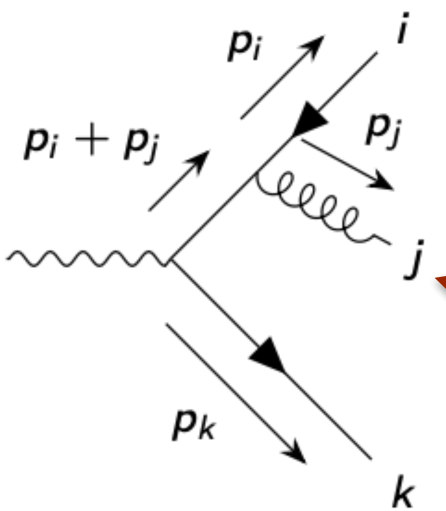

$$= \bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)\frac{\mathrm{i}(\not{p}_i + \not{p}_j)}{(p_i + p_j)^2}(-\mathrm{i}g_s)T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

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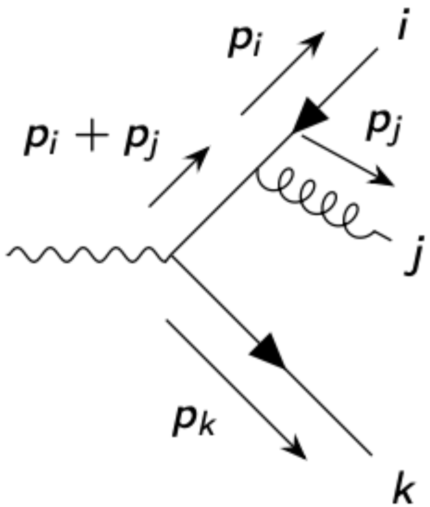
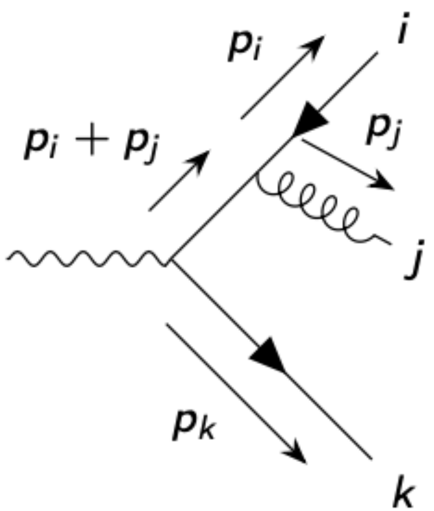


$$= \bar{u}(p_k)(-ie)\not{\epsilon}_\mu(q)\frac{i(\not{p}_i + \not{p}_j)}{(p_i + p_j)^2}(-ig_s)T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

$$= \bar{u}(p_k)(-ie)\not{\epsilon}_\mu(q)\frac{i(\not{p}_i + \not{p}_j)}{2p_ip_j}(-ig_s)T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

massless partons $p_i^2 = 0, p_j^2 = 0$

Consider real correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



$$= \bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)\frac{\mathrm{i}(\not{p}_i + \not{p}_j)}{(p_i + p_j)^2}(-\mathrm{i}g_s)T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

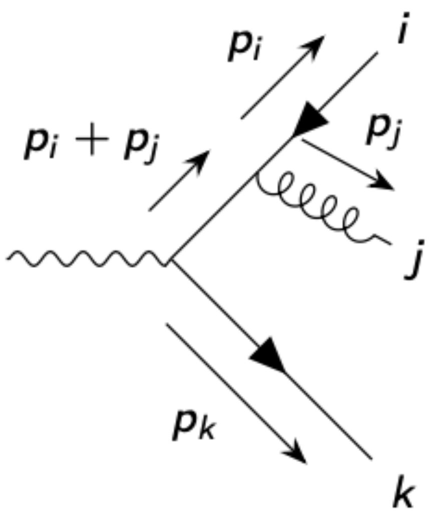
$$= \bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)\frac{\mathrm{i}(\not{p}_i + \not{p}_j)}{2p_ip_j}(-\mathrm{i}g_s)T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

In the soft limit $p_j \to 0$, the leading term is

$$\to \bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)\frac{\mathrm{i}\gamma^\nu p_{i\,\nu}}{2p_ip_j}(-\mathrm{i}g_s)T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

Real corrections

Consider real correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



$$= \bar{u}(p_k)(-ie)\not{\epsilon}_\mu(q)\frac{i(\not{p}_i + \not{p}_j)}{(p_i + p_j)^2}(-ig_s)T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

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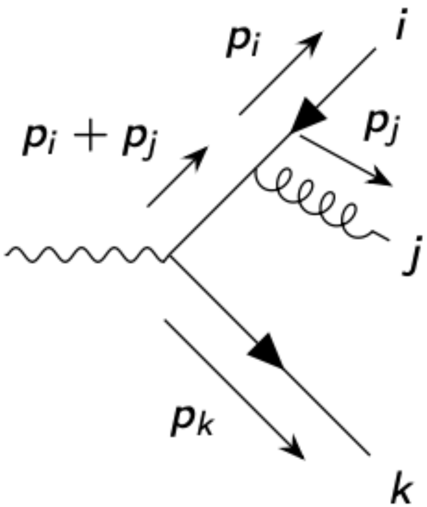
$$\to \bar{u}(p_k)(-ie)\not{\epsilon}_\mu(q)\frac{i\gamma^\nu p_{i\,\nu}}{2p_ip_j}(-ig_s)T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

$$\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu = 2g^{\mu\nu}$$
$$\gamma^\mu p_{i\,\mu} v(p_i) = 0$$

$$= \bar{u}(p_k)(-ie)\not{\epsilon}_\mu(q)\frac{ip_i\varepsilon^*(p_j)}{p_ip_j}(-ig_s)T^a v(p_i)$$

Real corrections

Consider real correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



$$= \bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)\frac{\mathrm{i}(\not{p}_i + \not{p}_j)}{(p_i + p_j)^2}(-\mathrm{i}g_\mathrm{s})T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

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In the soft limit $p_j \to 0$, the leading term is

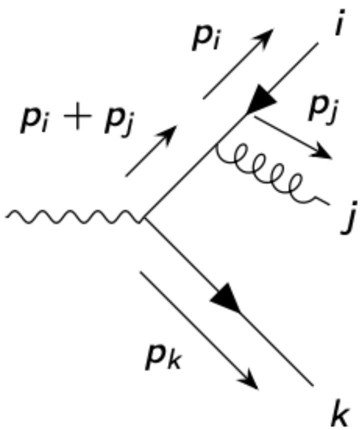
$$\to \bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)\frac{\mathrm{i}\gamma^\nu p_{i\,\nu}}{2p_ip_j}(-\mathrm{i}g_\mathrm{s})T^a\gamma^\mu v(p_i)\varepsilon^*_\mu(p_j)$$

$$= \bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)\frac{\mathrm{i}p_i\varepsilon^*(p_j)}{p_ip_j}(-\mathrm{i}g_\mathrm{s})T^a v(p_i)$$

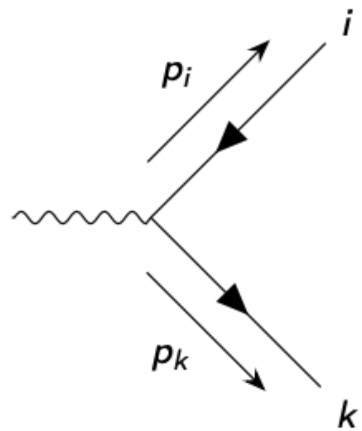
soft gluon emissions factorise!

$$= g_\mathrm{s}T^a\frac{p_i\varepsilon^*(p_j)}{p_ip_j}\,\bar{u}(p_k)(-\mathrm{i}e)\not{\epsilon}_\mu(q)v(p_i)$$

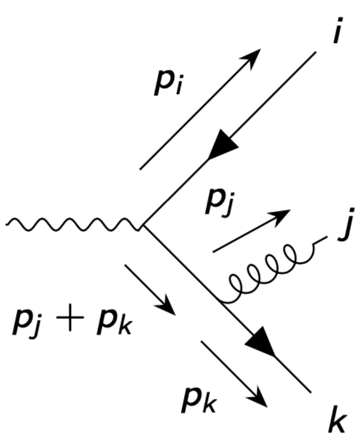
In the soft limit $p_j \to 0$, the leading term in the amplitude $\gamma^* \to \mathrm{q\bar{q}g}$ is



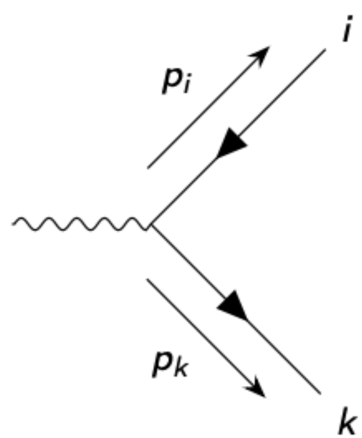
$$= g_s T^a \frac{p_i \varepsilon^*(p_j)}{p_i p_j} \times$$



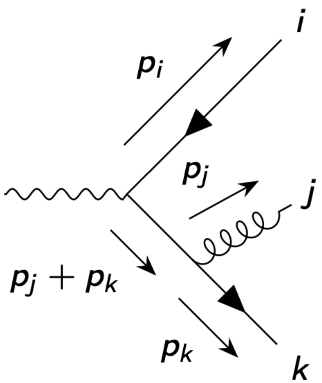
and similarly



$$= g_s T^a \frac{p_k \varepsilon^*(p_j)}{p_j p_k} \times$$



In the soft limit $p_j \to 0$, the leading term in the squared amplitude $\gamma^* \to q\bar{q}g$ is



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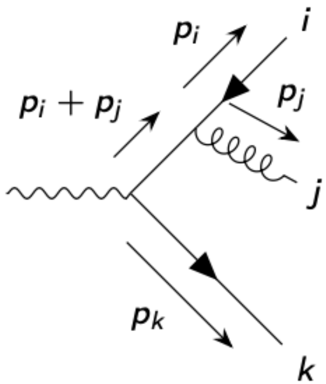
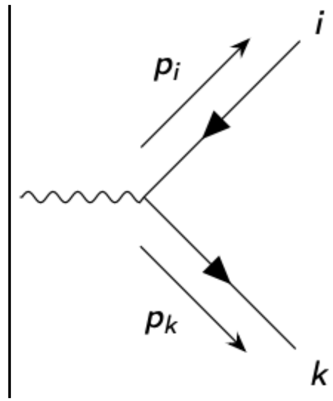


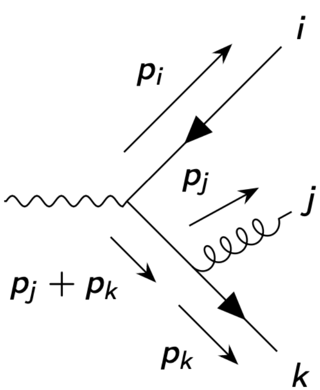
$\Bigg)^* = g_s^2 T^a T^b \dfrac{p_k \varepsilon^*(p_j)}{p_j p_k} \dfrac{p_i \varepsilon(p_j)}{p_i p_j} \times \Bigg|$



$\Bigg|^2$

Sum over polarisations $\sum \varepsilon_\mu(p_j)\varepsilon_\nu^*(p_j) = -\, g_{\mu\nu}$ and colours $T^a T^b = C_F N_C$

$2 \times$



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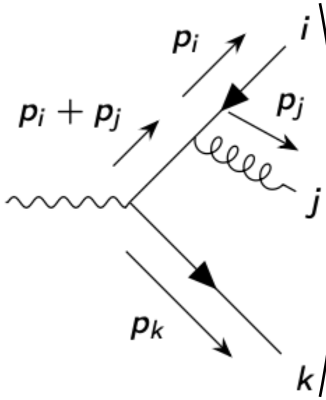
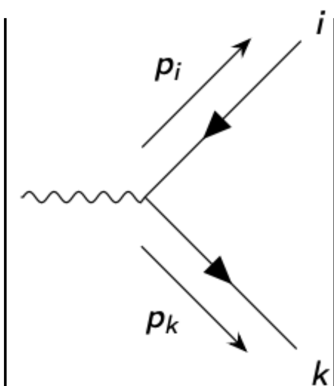


$\Bigg)^* = C_F g_s^2 \dfrac{2 p_i p_k}{(p_i p_j)(p_j p_k)} \times \Bigg|$



$\Bigg|^2$

universal factor: soft eikonal

Soft eikonal

Squared matrix elements factorise in the soft limit

$$|\mathcal{M}_{n+1}|^2 \propto \frac{2p_ip_k}{(p_ip_j)(p_jp_k)}|\mathcal{M}_n|^2$$

The same is true for the phase space $d\Phi_{n+1} = d\Phi_{+1} d\Phi_n$

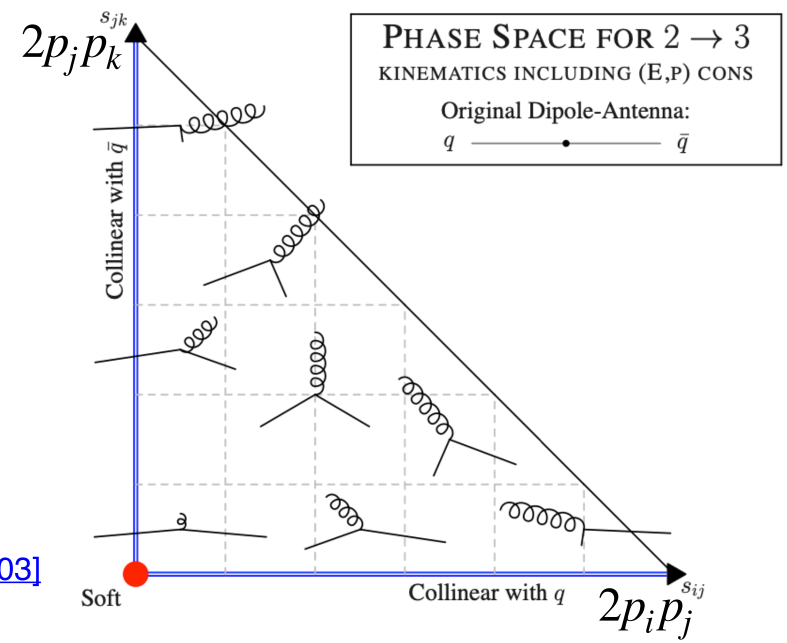
In a specific reference frame, use $p_ip_j = E_iE_j(1-\cos\theta_{ij})$ to write eikonal as:

$$\frac{2p_ip_k}{(p_ip_j)(p_jp_k)} = \frac{1}{E_j^2(1-\cos\theta_{ij})(1-\cos\theta_{jk})}$$

This **diverges** in the

- soft limit $E_j \to 0$
- collinear limits $\theta_{ij} \to 0$ or $\theta_{jk} \to 0$

Divergences and factorisation in the soft and collinear limits are universal features of QCD amplitudes!



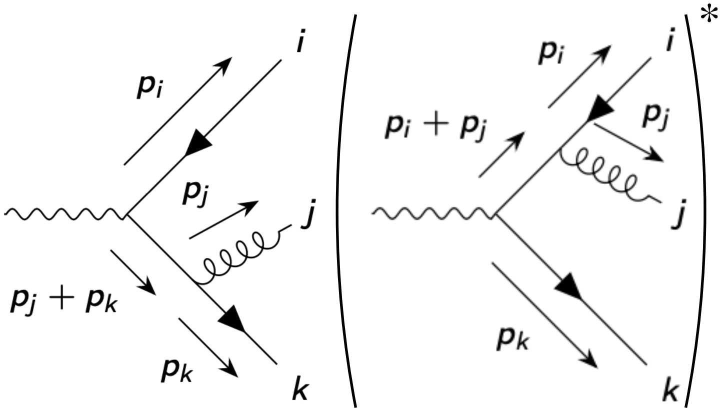
[Giele, Kosower, Skands PRD 84 (2011) 054003]

Infrared-finite cross sections

Problem:
how can we calculate physical results if QCD amplitudes are infrared divergent?

Solution:
by taking the full $\mathcal{O}(\alpha_s)$ correction into account!

So far, we considered the real correction…



$\propto g_s$ $\qquad\qquad$ $\propto g_s$

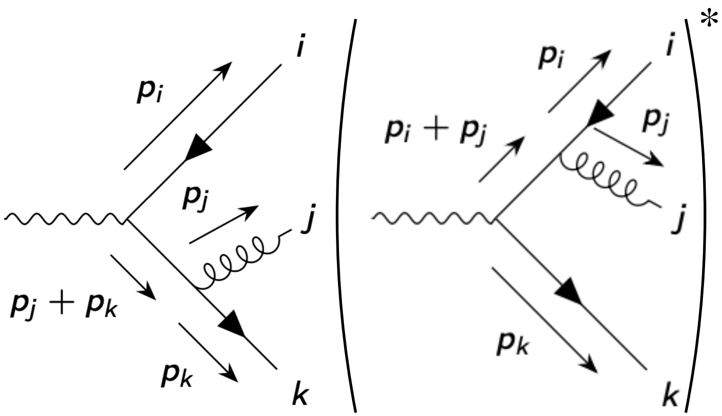
$\rightarrow \mathcal{O}(\alpha_s)$

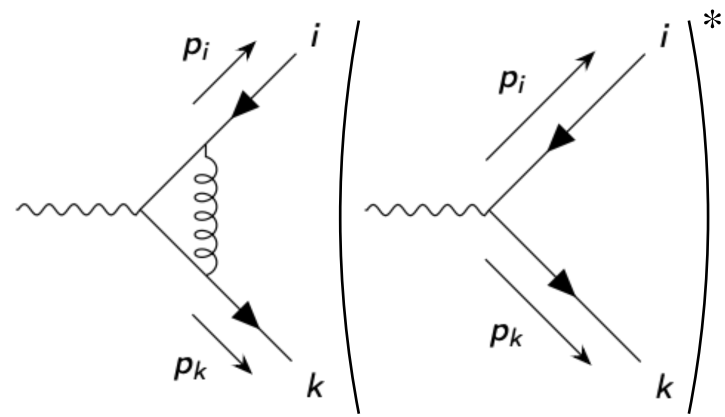
Infrared-finite cross sections

Problem:
how can we calculate physical results if QCD amplitudes are infrared divergent?

Solution:
by taking the full $\mathcal{O}(\alpha_s)$ correction into account!

So far, we considered the real correction… but the virtual correction contributes too!





$\propto g_s \qquad \propto g_s \qquad\qquad \propto g_s^2 \qquad \propto g_s^0$

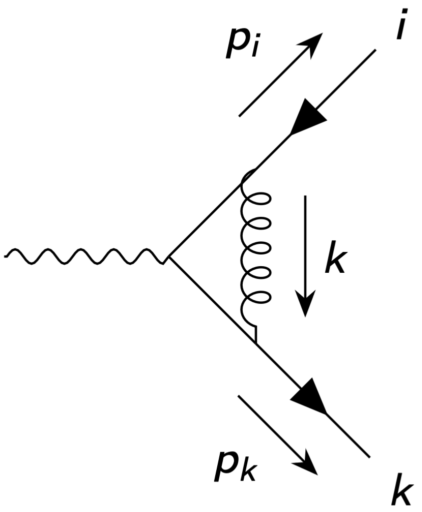
$\rightarrow \mathcal{O}(\alpha_s) \qquad\qquad\qquad \rightarrow \mathcal{O}(\alpha_s)$

The inclusive cross section $\sigma^{\mathrm{NLO}} = \sigma^{\mathrm{R}} + \sigma^{\mathrm{V}}$ is **infrared finite**!

[Bloch, Nordsieck PR 52 (1937) 54] [Kinoshita JMP 3 (1962) 650] [Lee, Nauenberg PR 133 (1964) B1549]

Virtual corrections

Consider virtual correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



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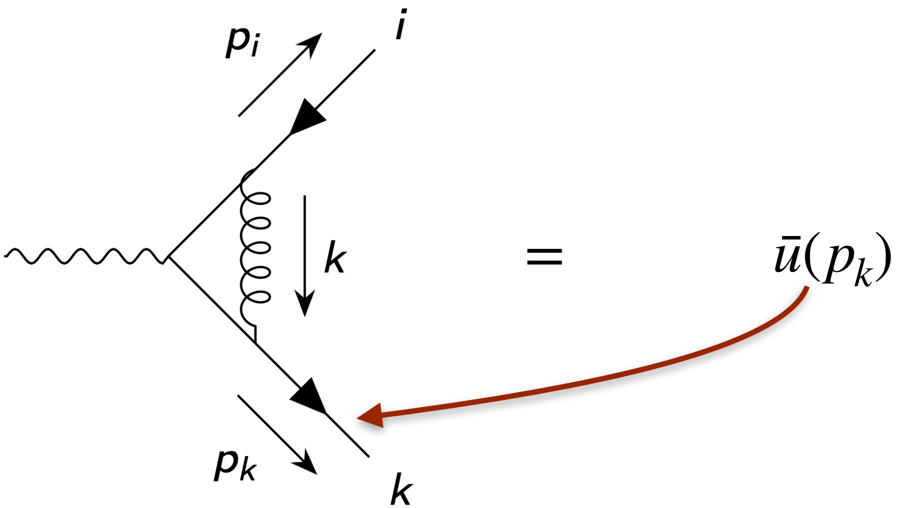

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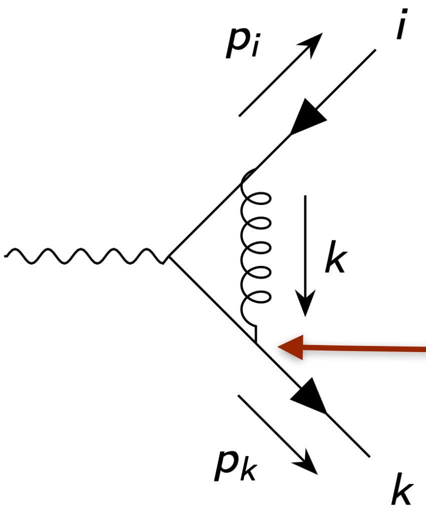

$$= \quad \bar{u}(p_k)(-\mathrm{i}g_s)T^a\gamma^\mu$$

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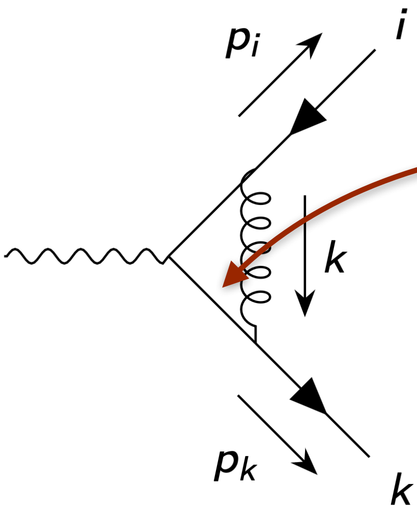

$$= \quad \bar{u}(p_k)(-\mathrm{i}g_s)T^a\gamma^\mu \frac{\mathrm{i}(\not{p}_k - \not{k})}{(p_k - k)^2}$$

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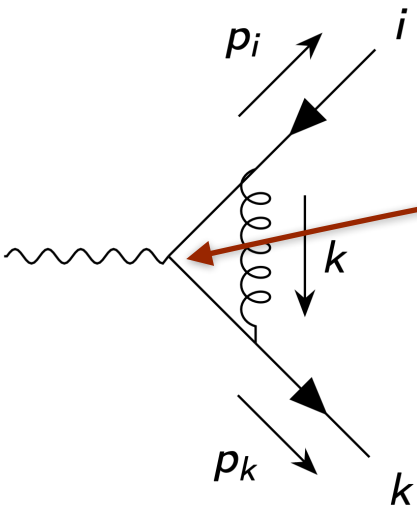

$$= \quad \bar{u}(p_k)(-\mathrm{i}g_s)T^a\gamma^\mu \frac{\mathrm{i}(\not{p}_k - \not{k})}{(p_k - k)^2}(-\mathrm{i}e)\not{\varepsilon}_\mu(q)$$

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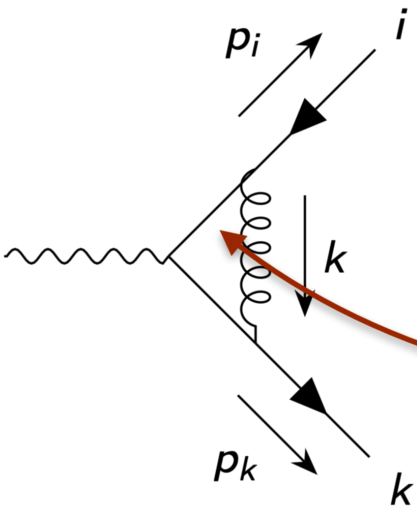

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$$\times \frac{\mathrm{i}(\not{p}_i + \not{k})}{(p_i + k)^2}$$

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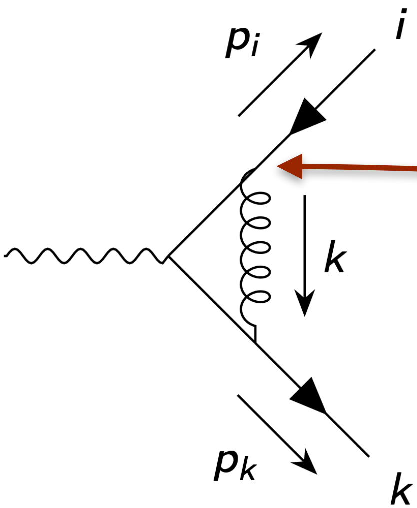
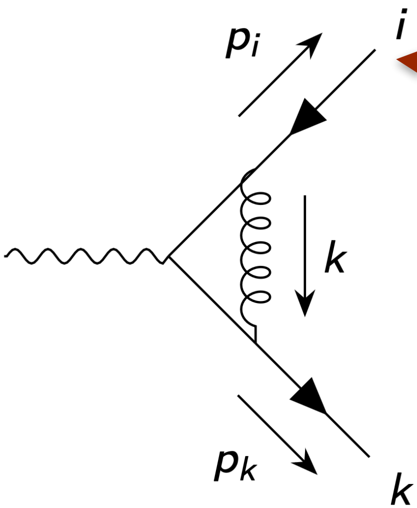


$$= \bar{u}(p_k)(-\mathrm{i}g_s)T^a\gamma^\mu \frac{\mathrm{i}(\not{p}_k - \not{k})}{(p_k - k)^2}(-\mathrm{i}e)\not{\varepsilon}_\mu(q) \times \frac{\mathrm{i}(\not{p}_i + \not{k})}{(p_i + k)^2}(-\mathrm{i}g_s)T^a\gamma^\nu$$

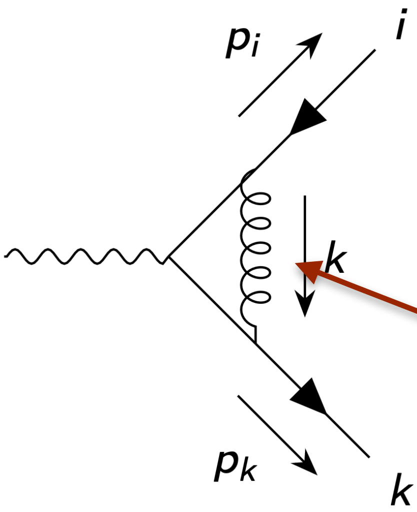
Consider virtual correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



$$= \bar{u}(p_k)(-\mathrm{i}g_s)T^a\gamma^\mu \frac{\mathrm{i}(\not{p}_k - \not{k})}{(p_k - k)^2}(-\mathrm{i}e)\not{\epsilon}_\mu(q) \times \frac{\mathrm{i}(\not{p}_i + \not{k})}{(p_i + k)^2}(-\mathrm{i}g_s)T^a\gamma^\nu v(p_i)$$

Virtual corrections

Consider virtual correction to $e^+e^- \to \gamma^*/Z \to q\bar{q}$



$$= \bar{u}(p_k)(-\mathrm{i}g_s)T^a\gamma^\mu \frac{\mathrm{i}(\not{p}_k - \not{k})}{(p_k-k)^2}(-\mathrm{i}e)\not{\varepsilon}_\mu(q)$$

$$\times \frac{\mathrm{i}(\not{p}_i + \not{k})}{(p_i+k)^2}(-\mathrm{i}g_s)T^a\gamma^\nu v(p_i)\frac{-\mathrm{i}g_{\mu\nu}}{k^2}$$

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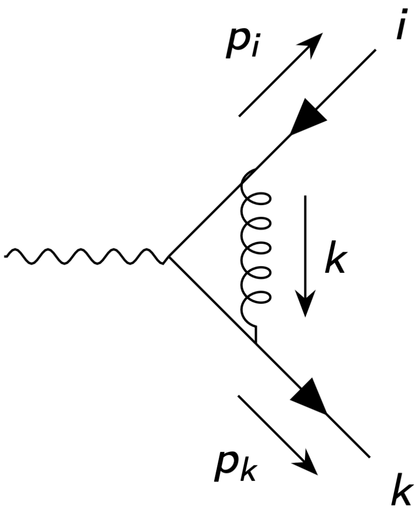

We have to integrate over the free loop momentum in $D = 4 - 2\varepsilon$ dimensions:

$$= \int \frac{dk^D}{(2\pi)^D} \bar{u}(p_k)(-ig_s)T^a\gamma^\mu \frac{i(\not{p}_k - \not{k})}{(p_k - k)^2}(-ie)\not{\varepsilon}_\mu(q)$$
$$\times \frac{i(\not{p}_i + \not{k})}{(p_i + k)^2}(-ig_s)T^a\gamma^\nu v(p_i)\frac{-ig_{\mu\nu}}{k^2}$$

Virtual corrections

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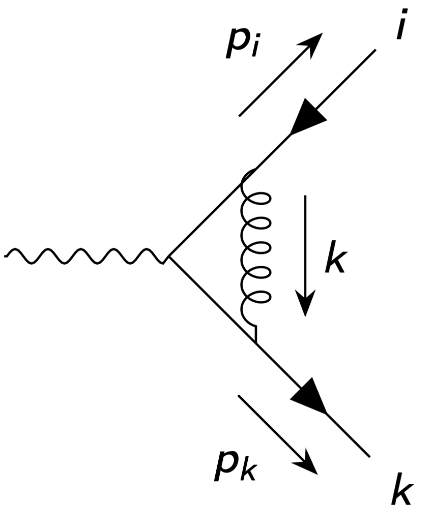

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The integration is cumbersome but straightforward and yields an integral of type (although beyond the scope of these lectures…)

$$\mu^{4-D}\int \frac{dk^D}{(2\pi)^D}\frac{1}{k^2(p_i + k)^2(p_k - k)^2} \sim \frac{e^{\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)}\left(\frac{\mu^2}{2p_ip_k}\right)^\varepsilon\left(-\frac{1}{\varepsilon^2} + \mathcal{O}\left(\varepsilon^{-1}\right)\right)$$

The limit $\varepsilon \to 0$ is not well-defined!

$\gamma_E = 0.55721...$ is the Euler-Mascheroni constant

The virtual correction yields an integral of type

$$\mu^{4-D}\int\frac{\mathrm{d}k^D}{(2\pi)^D}\frac{1}{k^2(p_i+k)^2(p_k-k)^2}\sim\frac{\mathrm{e}^{\varepsilon\gamma_\mathrm{E}}}{\Gamma(1-\varepsilon)}\left(\frac{\mu^2}{2p_ip_k}\right)^{\varepsilon}\left(-\frac{1}{\varepsilon^2}+\mathcal{O}\left(\varepsilon^{-1}\right)\right)$$

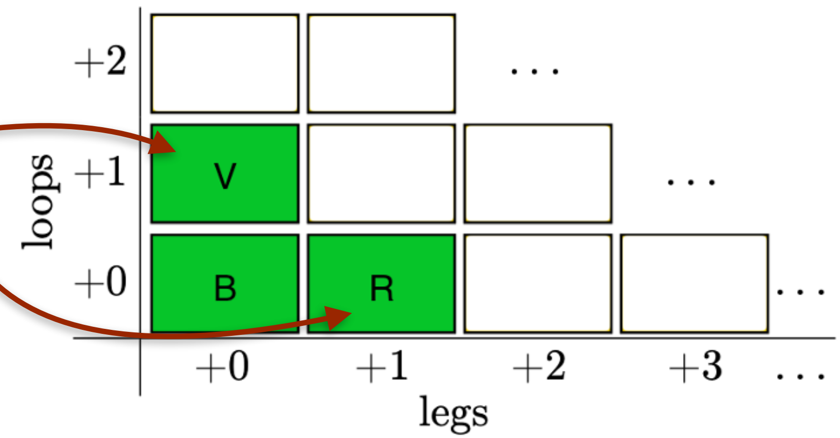
The integral of the eikonal over the phase space of the emission yields:

$$\mu^{4-D}\int\frac{\mathrm{d}p_j^D}{(2\pi)^D}\frac{p_ip_k}{(p_ip_j)(p_jp_k)}\sim\frac{\mathrm{e}^{\varepsilon\gamma_\mathrm{E}}}{\Gamma(1-\varepsilon)}\left(\frac{\mu^2}{2p_ip_j}\right)^{\varepsilon}\left(\frac{1}{\varepsilon^2}+\mathcal{O}\left(\varepsilon^{-1}\right)\right)$$

The sum of all virtual and all integrated real corrections cancels all poles in ε
$\Rightarrow$ the limit $\varepsilon\to 0$ is well-defined!

The cancellation of poles is simple for inclusive cross sections $\sigma^{\mathrm{NLO}}=\sigma^{\mathrm{R}}+\sigma^{\mathrm{V}}$, but what if we want to calculate a differential cross section $\mathrm{d}\sigma^{\mathrm{NLO}}/\mathrm{d}O$ for some observable O?

Problem:
Higher-order corrections **not separately finite**, but total cross section should be!
- explicit poles in virtual corrections
- implicit poles (singularities) in real corrections



Idea:
Introduce universal and simple counter term that subtracts singular behaviour:

$$\mathrm{d}\sigma^{\mathrm{NLO}} = \mathrm{d}\sigma^{\mathrm{V}} + \mathrm{d}\sigma^{\mathrm{T}} + \int_{+1} \left[\mathrm{d}\sigma^{\mathrm{R}} - \mathrm{d}\sigma^{\mathrm{S}}\right]$$

with $\mathrm{d}\sigma^{\mathrm{T}} - \int_{+1} \mathrm{d}\sigma^{\mathrm{S}} = 0$

Toy model:
NLO correction to “two-jet” observable $O_2(x_1, x_2)$

$$\left.\frac{d\sigma^{\text{NLO}}}{dx_1 dx_2}\right|_{D=4-2\varepsilon} = \underbrace{\left[\frac{M_2^0(x_1,x_2)}{\varepsilon} + M_2^{1,\text{finite}}(x_1,x_2)\right] O_2(x_1,x_2)}_{d\sigma^{\text{V}}} + \int_0^1 \underbrace{M_3^0(x_1,x_2,x_3)\, O_3(x_1,x_2,x_3)}_{d\sigma^{\text{R}}}\, x_3^{-\varepsilon} dx_3$$

and assume the following IR behaviour of the matrix element and observable:

$$\lim_{x_3\to 0} M_3^0(x_1,x_2,x_3) = \frac{1}{x_3} M_2^0(x_1,x_2)\,, \quad \lim_{x_3\to 0} O_3(x_1,x_2,x_3) = O_2(x_1,x_2)$$

The single-unresolved limit $x_3 \to 0$ can be subtracted from $d\sigma^{\text{R}}$ by

$$d\sigma^{\text{S}}(x_1,x_2,x_3) = \frac{1}{x_3} M_2^0(x_1,x_2)\, O_2(x_1,x_2)$$

so that

$$\lim_{x_3\to 0}\left[d\sigma^{\text{R}}(x_1,x_2,x_3) - d\sigma^{\text{S}}(x_1,x_2,x_3)\right] = 0$$

The real subtraction term can easily be integrated in D dimensions

$$\int_0^1 d\sigma^{S}(x_1, x_2, x_3)\, x_3^{-\varepsilon} dx_3 = \int_0^1 \frac{1}{x_3} M_2^0(x_1, x_2)\, O_2(x_1, x_2)\, x_3^{-\varepsilon} dx_3 = -\frac{M_2^0(x_1, x_2)}{\varepsilon}\, O_2(x_1, x_2)$$

so that

$$\lim_{\varepsilon \to 0} \left[d\sigma^{V}(x_1, x_2) + d\sigma^{T}(x_1, x_2) \right] = M_2^{1,\text{finite}}(x_1, x_2)\, O(x_1, x_2)$$

Then $d\sigma^{V} + d\sigma^{T}$ and $d\sigma^{R} - d\sigma^{S}$ are separately infrared finite and

$$\frac{d\sigma^{\text{NLO}}}{dx_1 dx_2} = M_2^{1,\text{finite}} O_2(x_1, x_2) + \int_0^1 \left[M_3^0(x_1, x_2, x_3)\, O_3(x_1, x_2, x_3) - \frac{1}{x_3} M_2^0(x_1, x_2)\, O_2(x_1, x_2) \right] dx_3$$

can be evaluated with $\varepsilon = 0$

The toy subtraction formalism relied on the fact that

$$\lim_{x_i \to 0} O_{n+1}(x_1, \ldots, x_i, \ldots, x_{n+1}) = O_n(x_1, \ldots, x_n)$$

This is a manifestation of **infrared and collinear safety** (IRC safety).

In perturbative QCD, observables are only calculable if they are insensitive to arbitrarily soft and collinear radiation.
Otherwise, the cancellation of real and virtual singularities is spoiled.

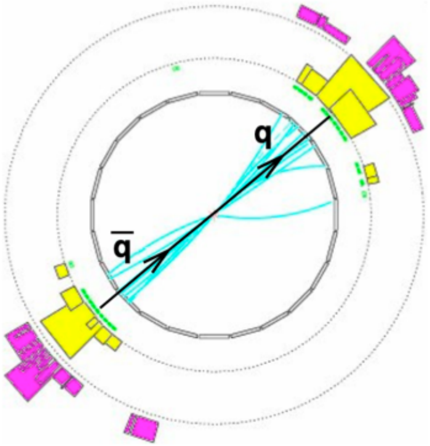
Example of IRC-unsafe observables: particle multiplicities ("how many quarks?")

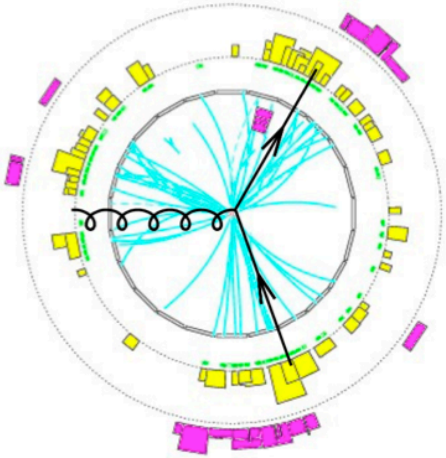
IRC-safe observables are often defined in terms of **jets**.

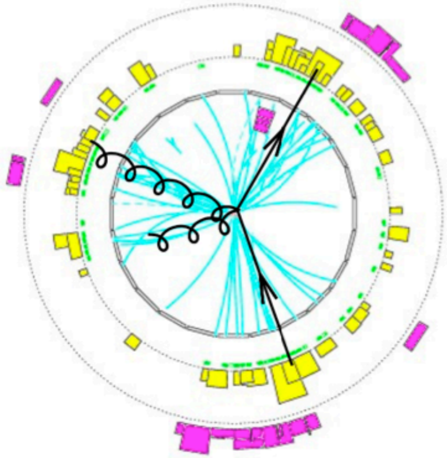
Instead of looking at individual partons, look at collimated sprays of partons.

- initial partons radiate further partons (“Bremsstrahlung”)
- hard radiation generates new jets
- soft/collinear radiation generates jet substructure

Jet counting not always obvious!







Sequential jet algorithms help to quantify what we mean by a jet

Algorithmic procedure:

1. compute distance measure d_{ij} for all pairs of final-state particles i, j and beam distance d_{iB} for all final-state particles i
2. find minimum of all d_{ij} and d_{iB}
   A. if one of the d_{ij} is the smallest, combine i, j into a pseudo-particle
   B. if one of the d_{iB} is the smallest, i as a jet and removed from the algorithm
3. start over from step 1 until all objects are clustered

Distance measures are subject to choice, e.g. generalised k_{T}-algorithm:

$$d_{ij} = \min(p_{\mathrm{T}i}^{2p}, p_{\mathrm{T}j}^{2p}) \frac{\Delta R_{ij}^2}{R^2}, \quad d_{iB} = p_{\mathrm{T}i}^{2p}, \quad \Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

→ jets defined by “cone size” R (input parameter)

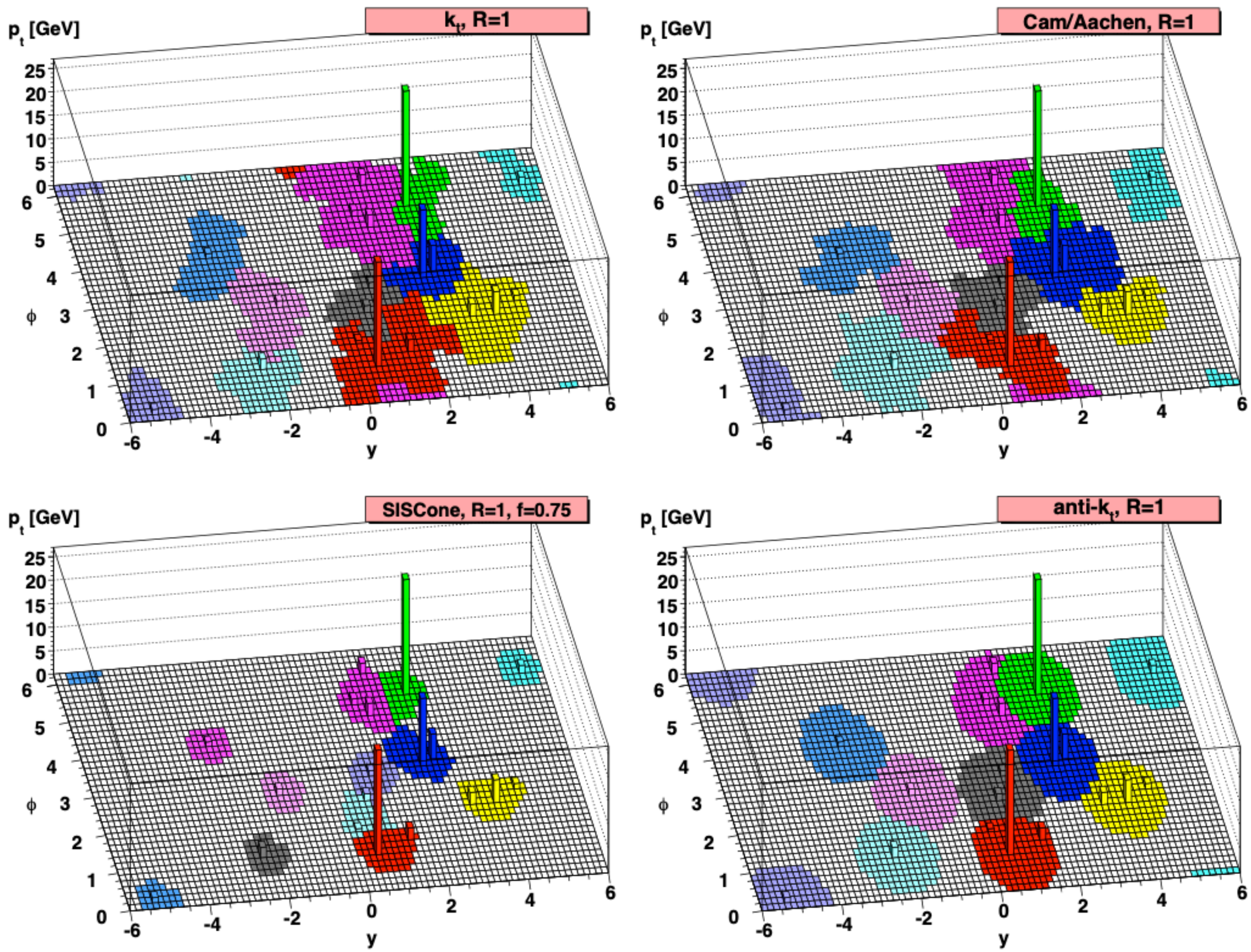
→ different choices of p yield different geometric properties of the jets

Jet algorithms — illustration

$$d_{ij} = \min(p_{Ti}^{2p}, p_{Tj}^{2p})\frac{\Delta R_{ij}^2}{R^2}\,, \quad d_{iB} = p_{Ti}^{2p}, \quad \Delta R_{ij}^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

$p = 1$

$p = 0$



$p = -1$

[Salam EPJC 67 (2010) 637]

Overview

Lecture 1 (yesterday):

- Hadrons, partons, and all that
- Colour charges
- QCD Lagrangian and Feynman rules
- QCD at test
- QCD-improved parton model

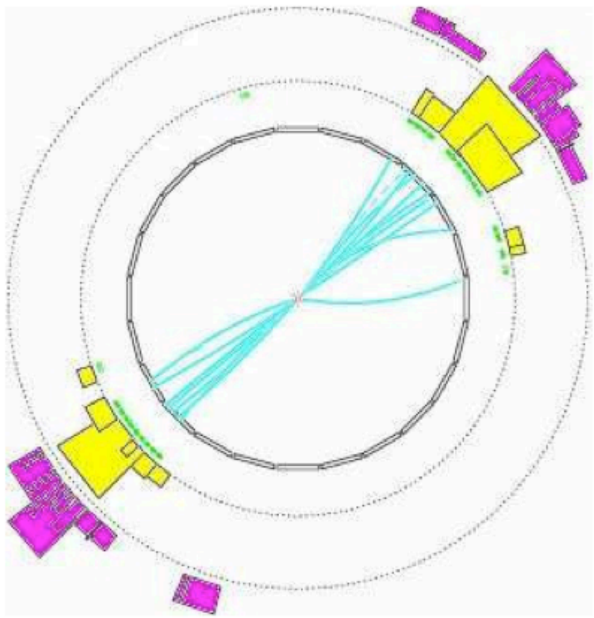
Lecture 2 (today):

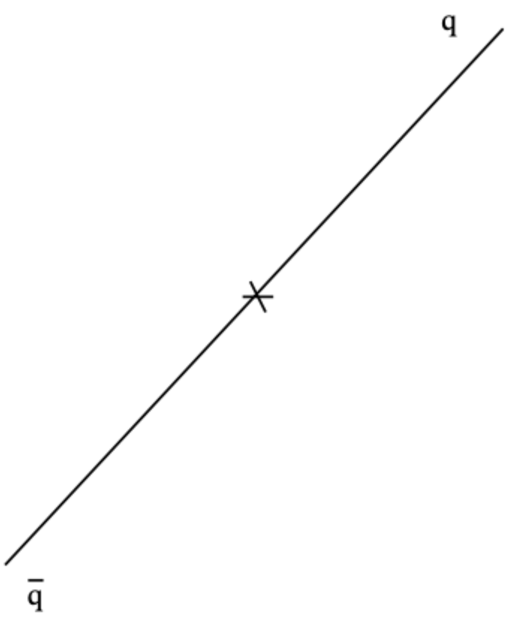
- Fixed-order calculations and jets
- QCD radiation
- Running coupling and confinement
- QCD in event generators

QCD radiation

So far, jets consist of a handful of partons only (1 at LO, 2 at NLO, …).

But additional radiation driven by divergent propagators!





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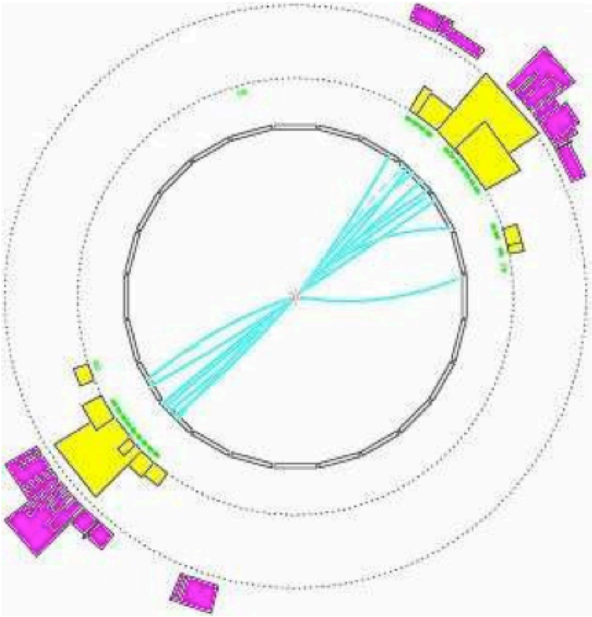

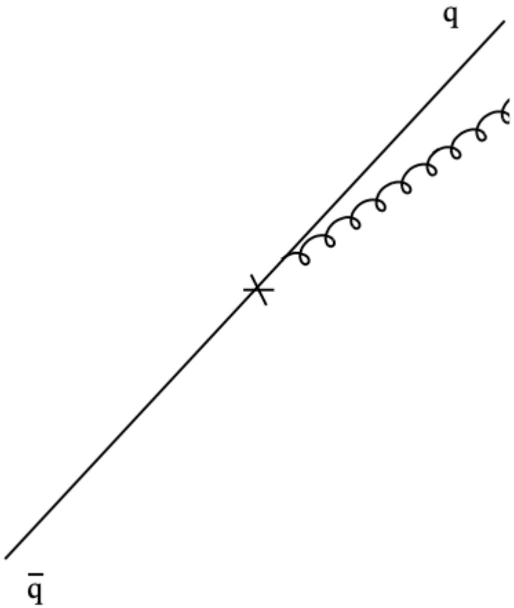

QCD radiation

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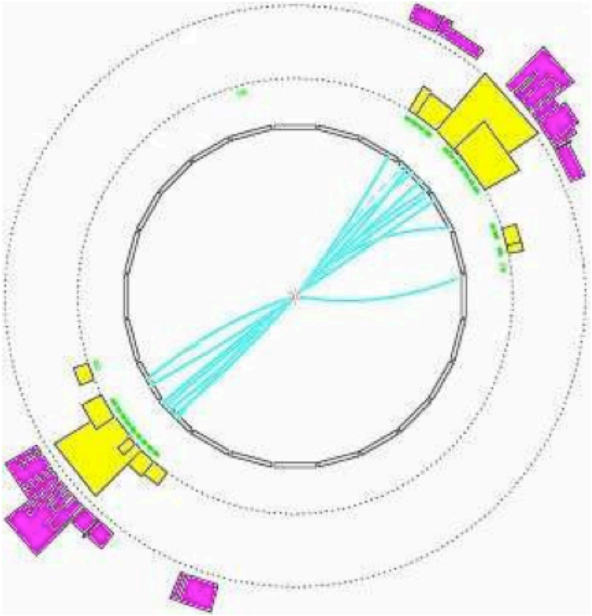

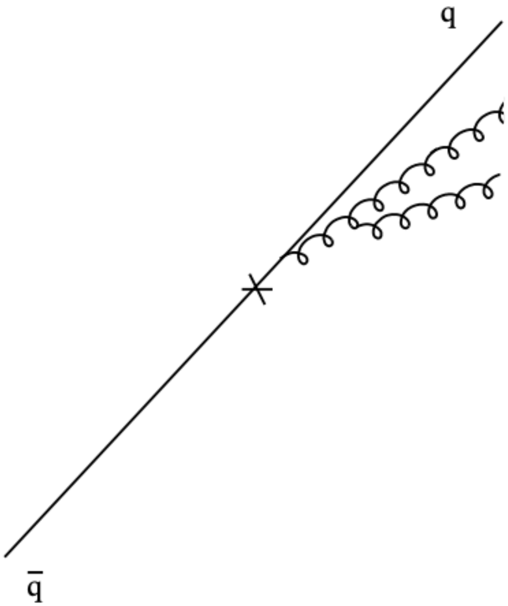

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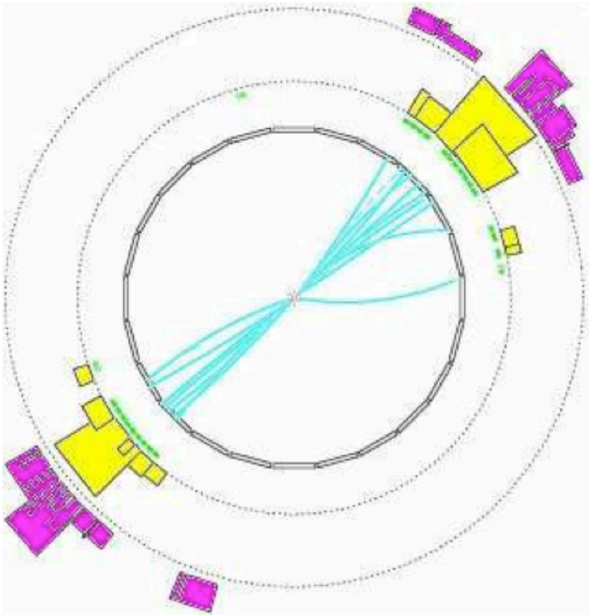
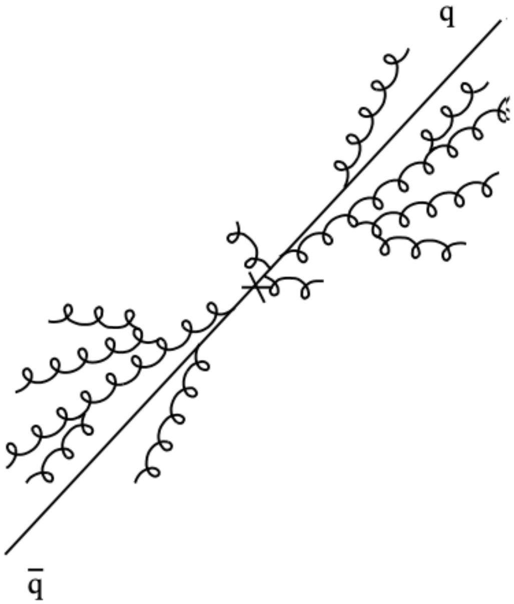
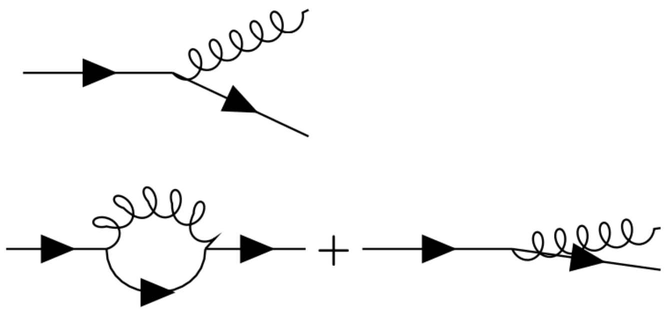
But additional radiation driven by divergent propagators!





Parton branching can occur in two ways:



observed

unobserved

Assume that parton evolution conserves probability (unitarity).

The probability for n emissions is given by Poisson statistics

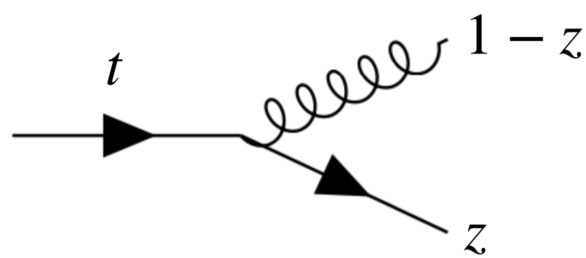
$$P(n, \lambda) = \frac{\lambda^n e^{-\lambda}}{n!}$$

In parton showers this is called Sudakov factor

with "decay probability" λ.

Express decay probability in terms of splitting functions in the collinear limit:

$$\lambda \to \lambda(Q^2, t) = \int_t^{Q^2} \frac{\mathrm{d}t'}{t'} \int \mathrm{d}z \, \frac{\alpha_s}{2\pi} P(z)$$



By construction, the parton shower is unitary:

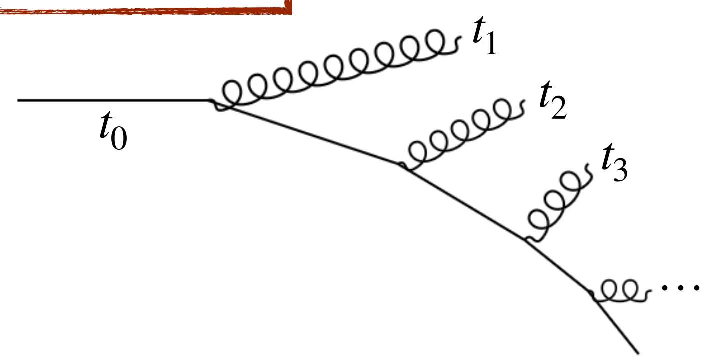
$$\Delta(t_0, t) = \mathrm{e}^{-\lambda(t_0, t)}$$

$$\mathrm{d}\sigma = \Delta(t_0, t_h)\, \mathrm{d}\sigma + \int_{t_h}^{t_0} \frac{\mathrm{d}t}{t} \int \mathrm{d}z \, \Delta(t_0, t)\, P(z)\, \mathrm{d}\sigma$$

no branching between t_0 and t_h

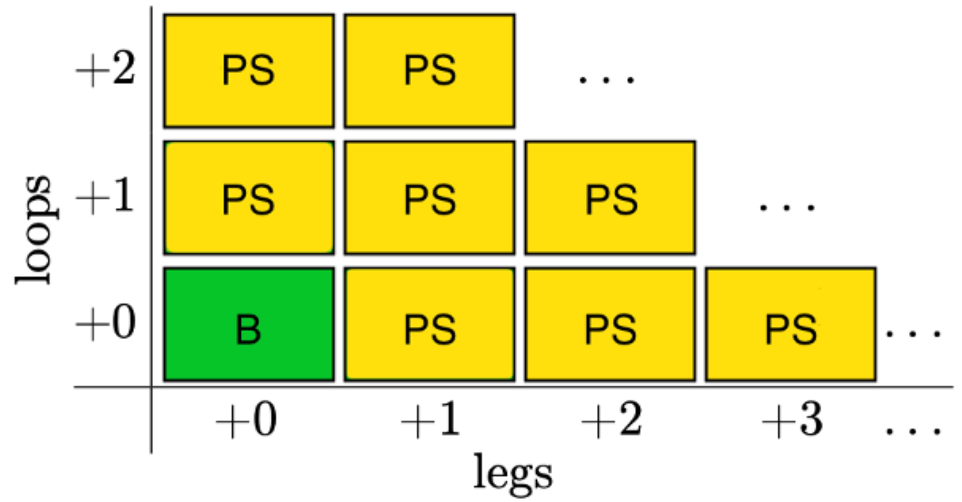
branching at scale t

Algorithmically: starting from t_0 solve $\Delta(t_0, t)$ for next branching scale t until hadronisation scale t_h is reached.



Starting from a hard scale t_0, a parton shower models additional radiation under the assumption that it is **soft** and/or **collinear** and **ordered** $(t_0 > t_1 > t_2 > \ldots > t_h)$

Additional loops and legs are only modelled approximately.



Overview

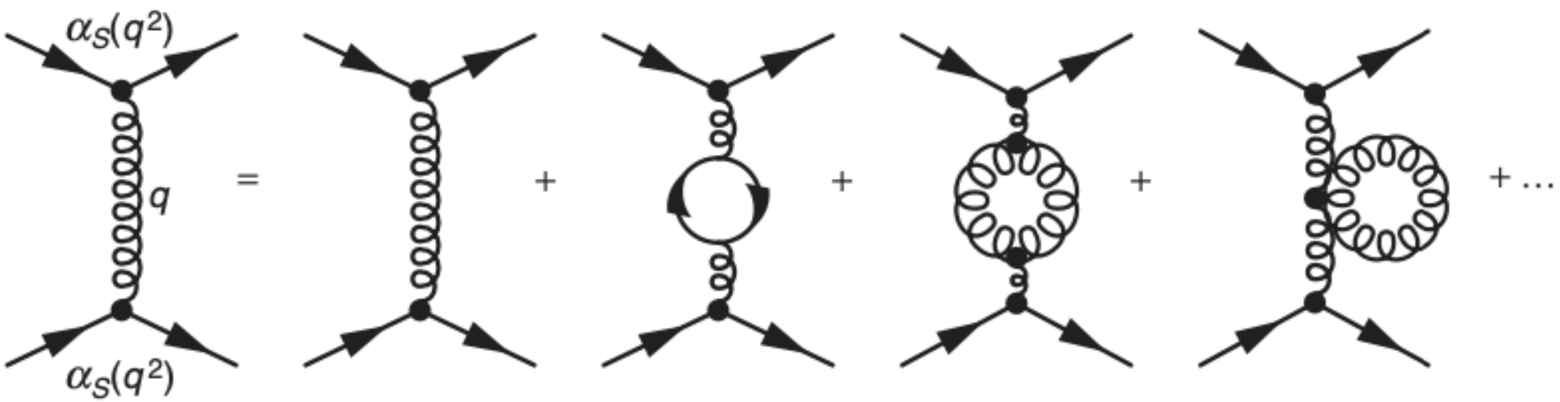
Lecture 1 (yesterday):

- Hadrons, partons, and all that
- Colour charges
- QCD Lagrangian and Feynman rules
- QCD at test
- QCD-improved parton model

Lecture 2 (today):

- Fixed-order calculations and jets
- QCD radiation
- Running coupling and confinement
- QCD in event generators

The strength of QCD interactions is governed by α_s, but each interaction receives infinitely many unobservable corrections, e.g. in $qq \to qq$:



The corresponding loop integrals require ultraviolet renormalisation:

- UV divergences are cancelled at unphysical scale μ
- Universal higher-order terms are absorbed into the definition of α_s

As a result, α_s becomes scale dependent with logarithmic scale dependency:

$$\frac{\mathrm{d}\alpha_s(\mu^2)}{\mathrm{d}\log\mu^2} = \beta(\alpha_s), \quad \beta(\alpha_s) = -\,\alpha_s^2(\beta_0 + \beta_1\alpha_s + \beta_2\alpha_s^2 + \ldots)$$

Note: the dependence on μ vanishes at all orders in perturbation theory.

The running coupling — leading-order result

At first order in perturbation theory

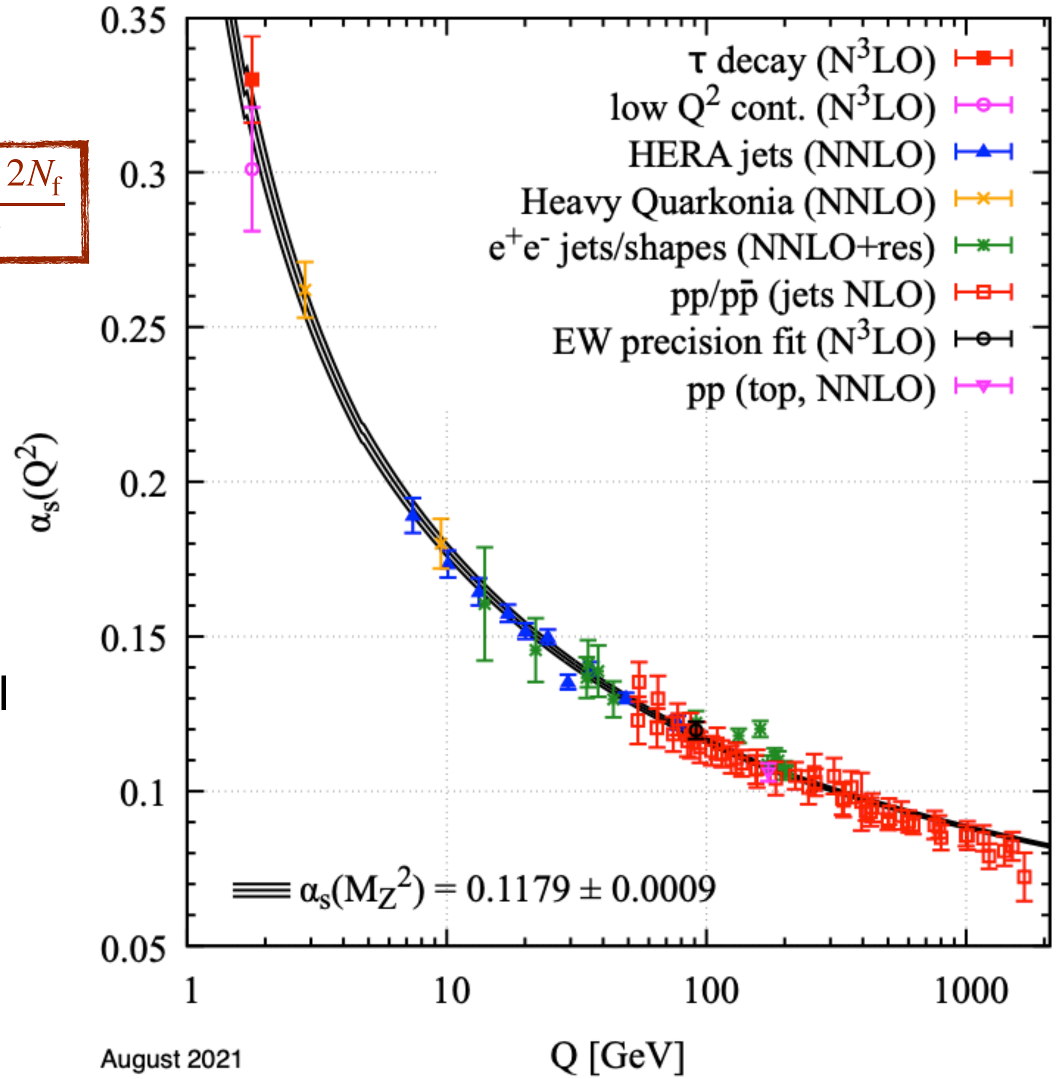
$$\frac{d\alpha_s(\mu^2)}{d\log\mu^2} = -\beta_0\alpha_s^2(\mu^2) \qquad \beta_0 = \frac{11C_A - 2N_f}{12\pi}$$

so that

$$\alpha_s(\mu^2) = \frac{\alpha_s(\mu_0^2)}{1 + \alpha_s(\mu_0^2)\beta_0 \log\frac{\mu^2}{\mu_0^2}}$$

Alternatively in terms of fundamental scale at which α_s diverges:

$$\alpha_s(\mu^2) = \frac{1}{\beta_0 \log\frac{\mu^2}{\Lambda_{QCD}^2}}$$



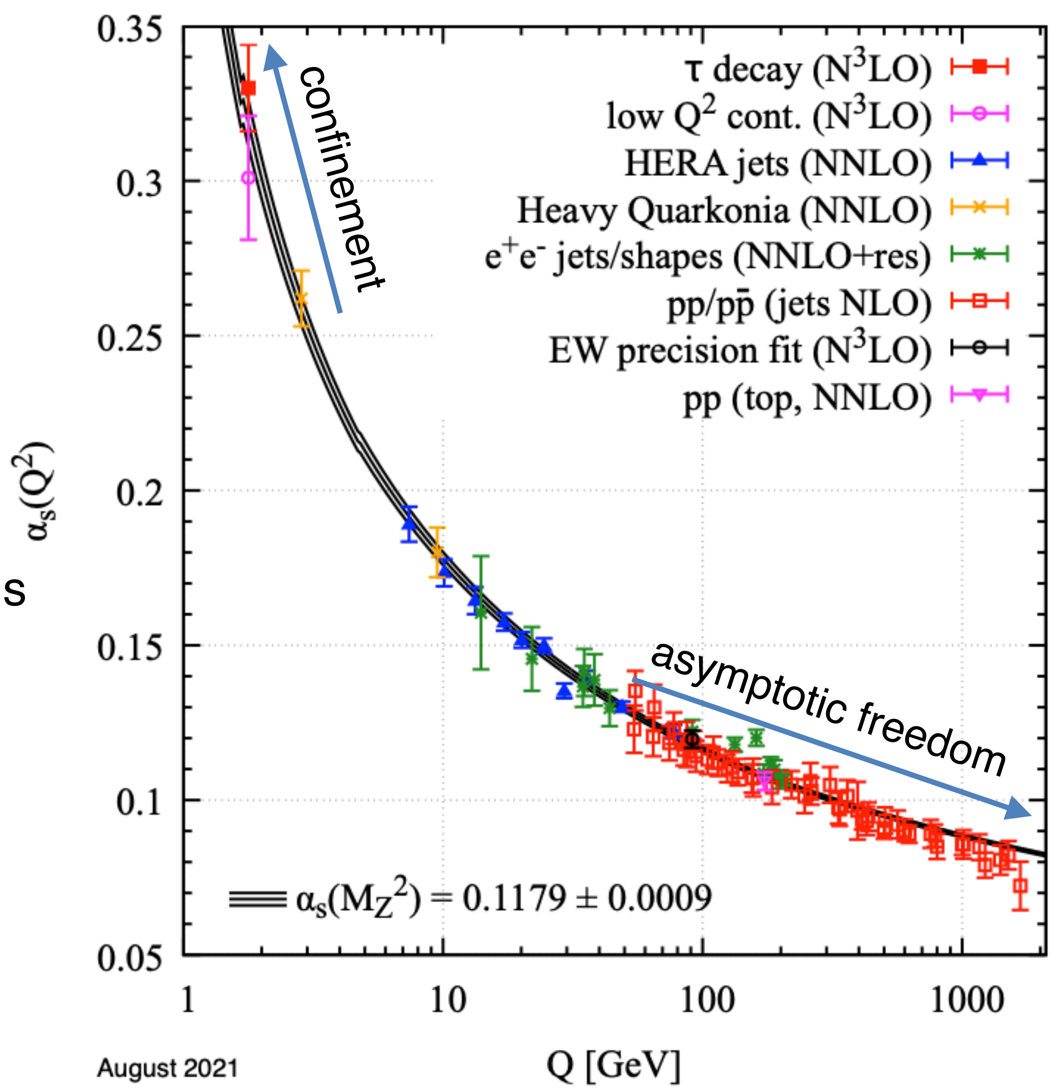
[PDG]

At Z-boson mass $\alpha_s(m_Z) \approx 0.118$

Asymptotic freedom:
Nobel Prize 2004: Gross, Polizer, Wilczek
At high energy scales $\alpha_s \to 0$, quarks and gluons are quasi-free

Confinement:
At Λ_{QCD} $\alpha_s \to \infty$ (Landau pole), quarks and gluons bound in hadrons

- perturbation theory valid if $\mu \gg \Lambda_{QCD}$
- typical jet transverse momenta: $p_T \sim 50\ \mathrm{GeV} - 5\ \mathrm{TeV}$



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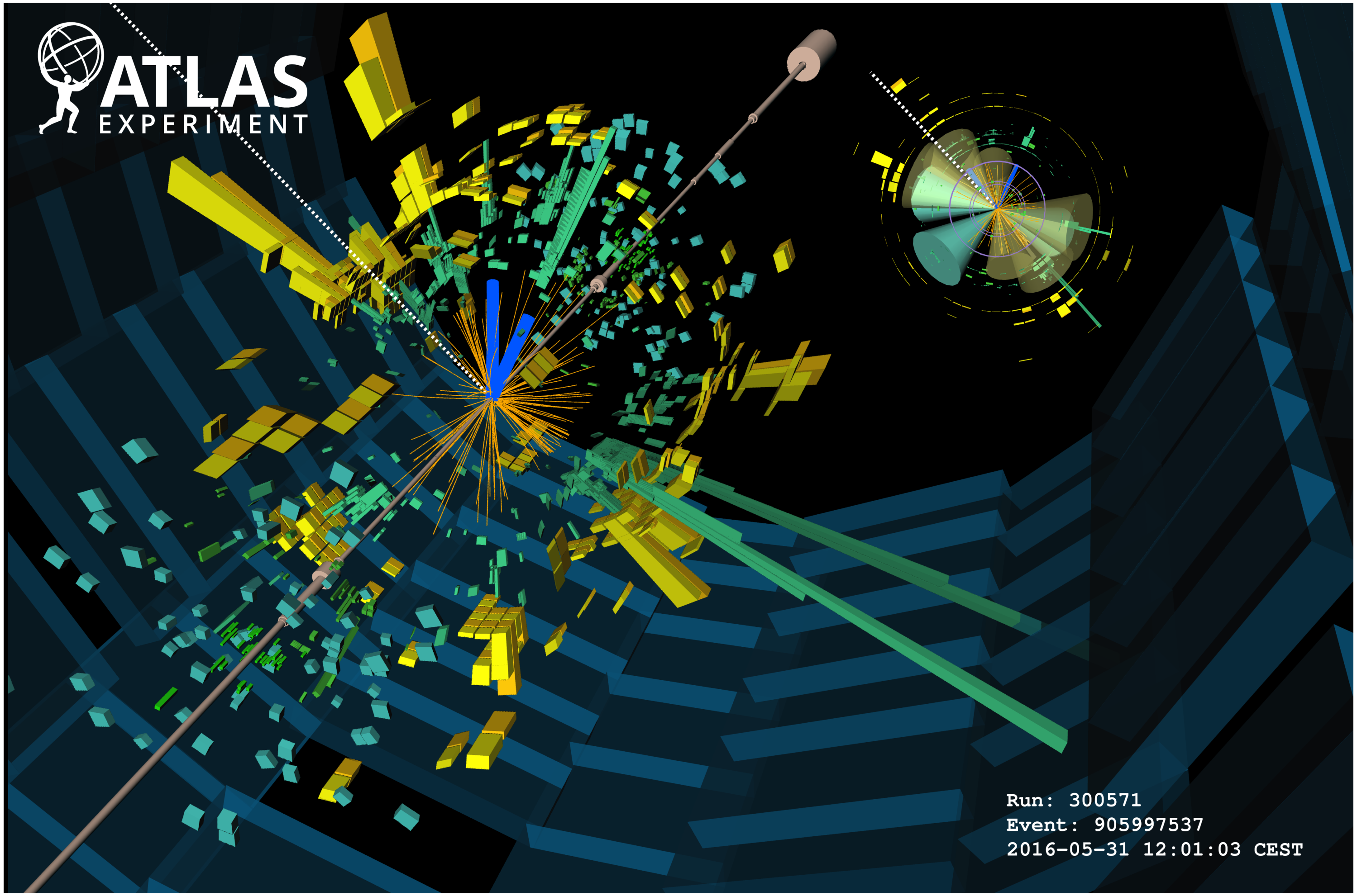
How hard can it be?

Approximate all contributing amplitudes for this…

…to all orders …including non-perturbative effects

…then integrate it over a ~300 dimensional phase space



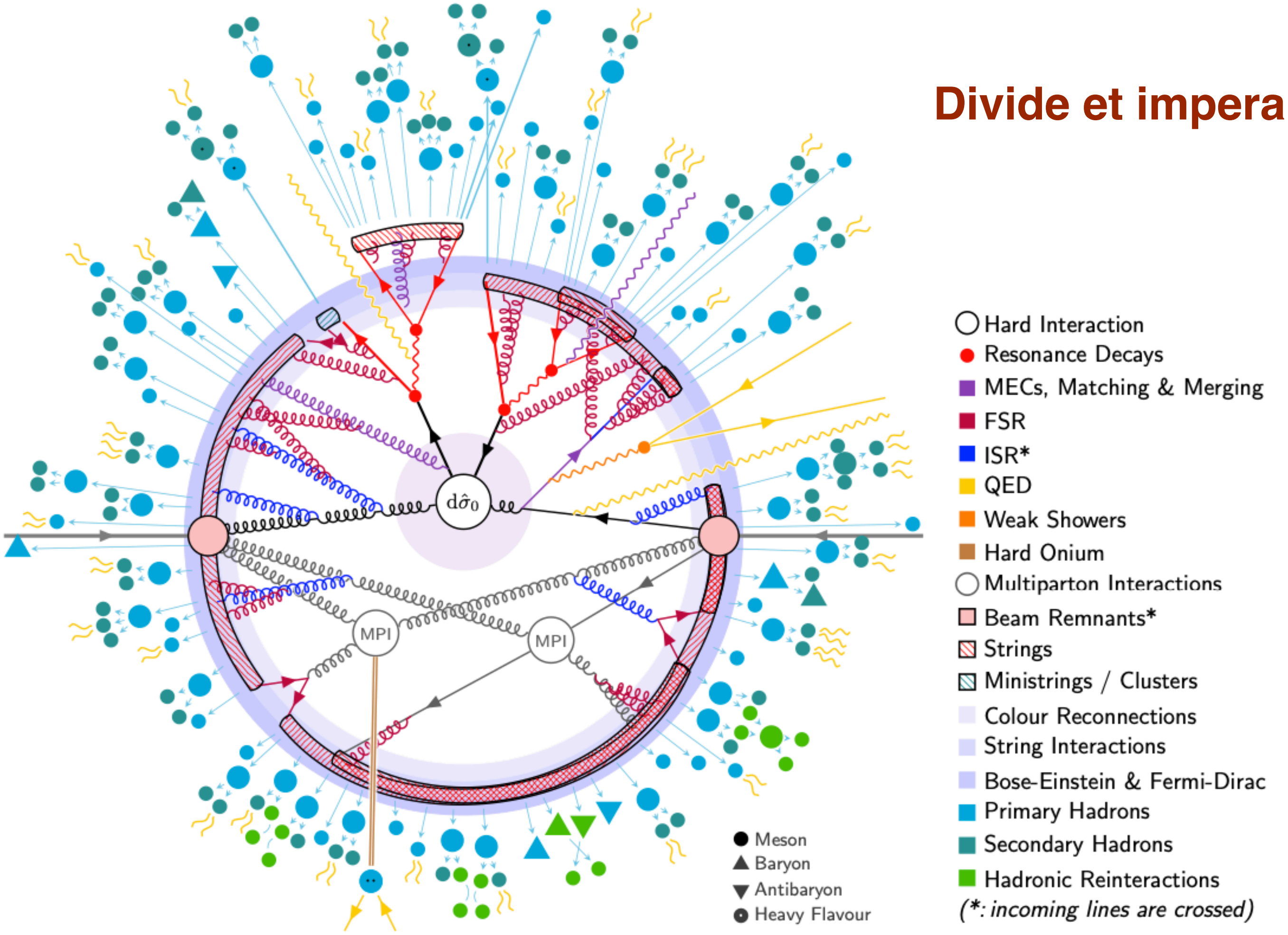
slide adapted from P. Skands

Approximate all contributing amplitudes for this…

…to all orders …including non-perturbative effects

…then integrate it over a ~300 dimensional phase space

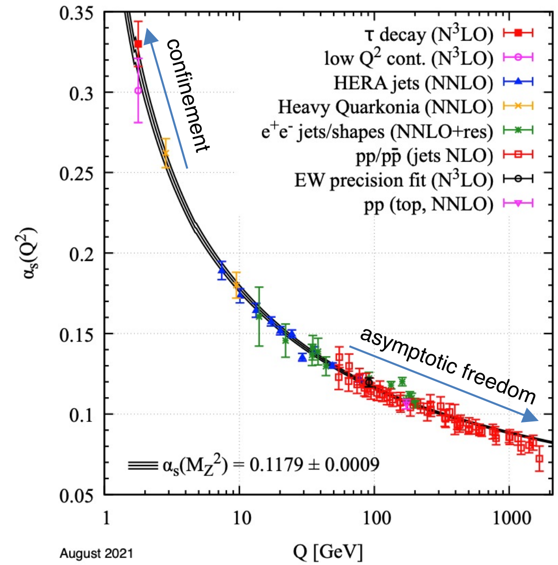
Divide et impera!



Step 1: fixed-order perturbation theory

A priori, confinement prohibits a perturbative calculation.

At high scales, quarks and gluons behave *almost* free
⇒ factorise **short-distance** (perturbative) from **long-distance** (non-perturbative) physics.



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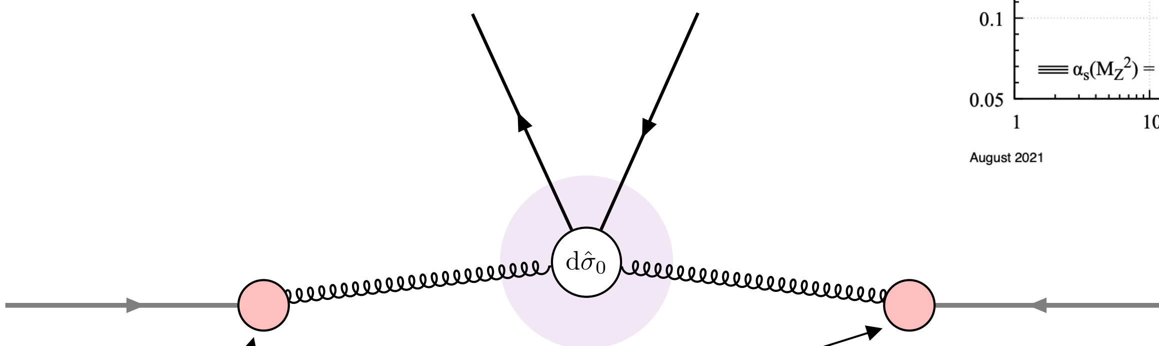


probability to find gluon in proton A

probability to find gluon in proton B

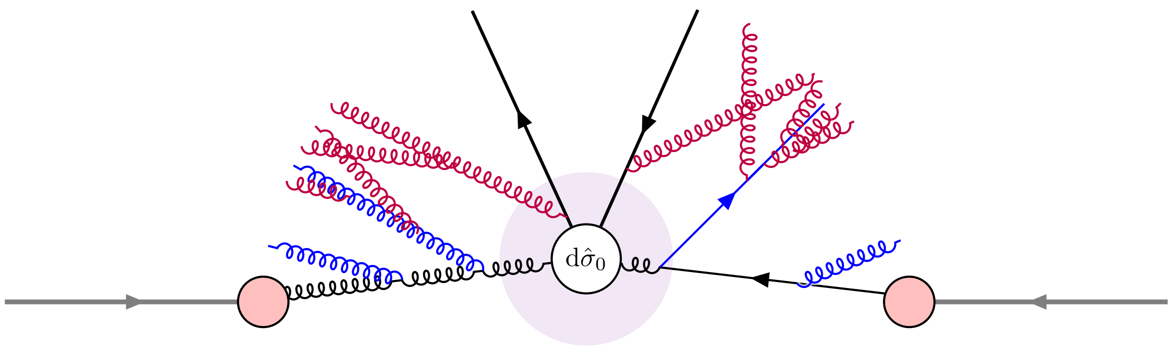
$$\mathrm{d}\sigma_{pp\to X} = \sum_{a,b} \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2\, f_a(x_a, \alpha_s(\mu_R), \mu_F)\, f_b(x_b, \alpha_s(\mu_R), \mu_F)\, \mathrm{d}\hat{\sigma}_0(x_1, x_2, \alpha_s(\mu_R), \mu_R, \mu_F) + \mathcal{O}\left((\Lambda/Q)^p\right)$$

Parton distribution functions (PDFs)
non-perturbative

hard scattering
perturbative

power corrections
non-perturbative

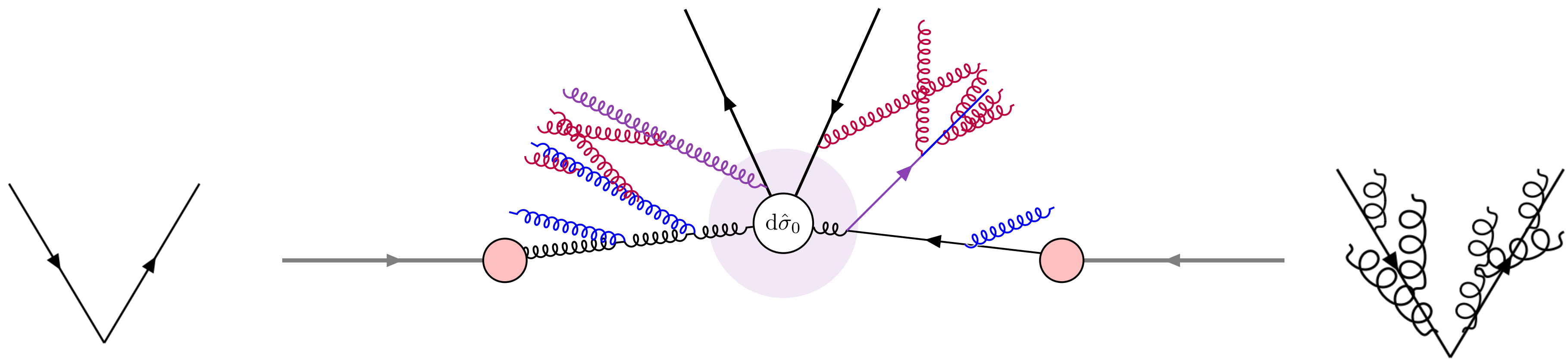
Scattered partons carry **QCD** and/or **electric charges.**
This induces radiative corrections via QCD/electromagnetic bremsstrahlung.



Parton showers dress a fixed-order calculation with **additional radiation**, describing the evolution from the **parton level** (quarks, gluons, …) at large scales to the **particle level** (hadrons) at small scales.

Step 3: combine fixed-order and all orders

Large complementarity between fixed-order calculations and parton showers, so ideally combine them!



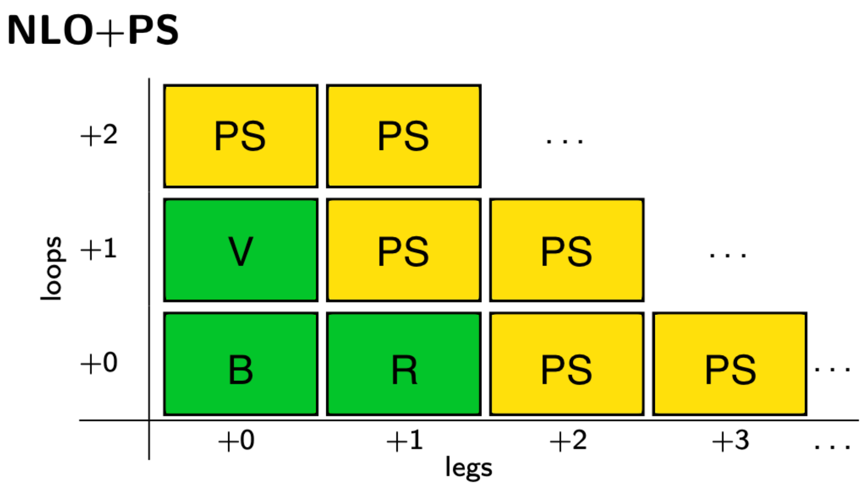
Fixed-order calculations → hard jets

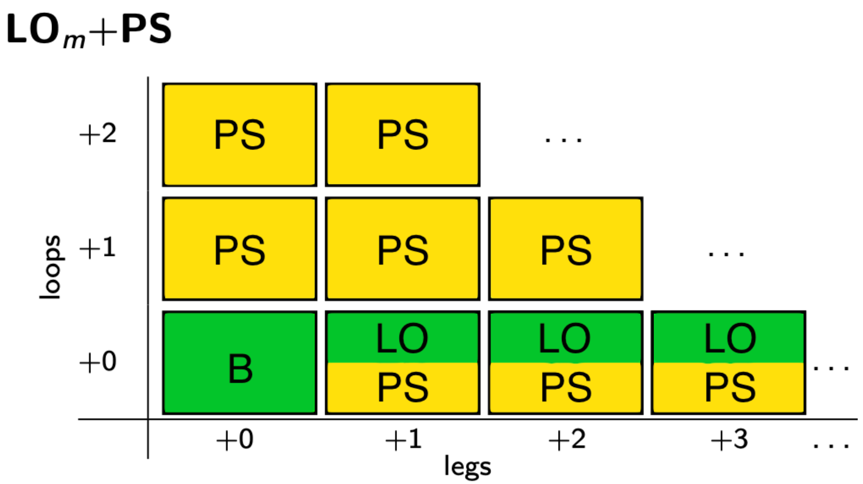
- reliable at high scales, without large scale hierarchies
- accurate predictions for limited number of legs (+ loops)
- determines perturbative accuracy (LO, NLO, NNLO, …)

Parton showers → jet substructure

- reliable in unresolved regions, with large scale hierarchies
- approximate predictions for many particles
- determines logarithmic accuracy (LL, NLL, NNLL, …)

Matching and merging





Matching

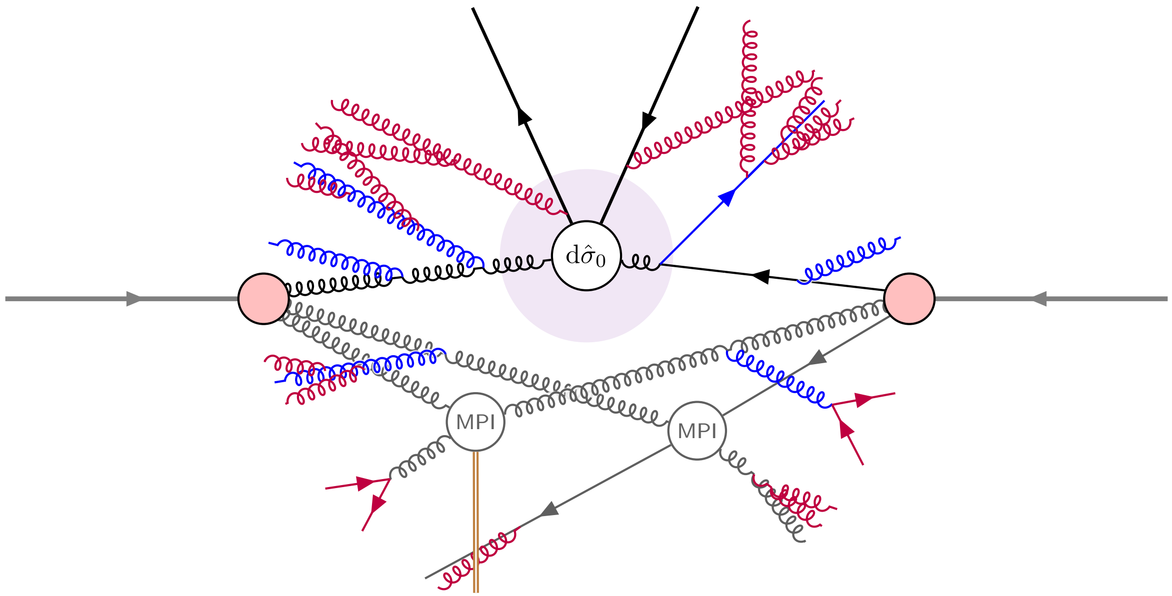
combine a fixed-order (typically NLO) calculation with a parton shower, **avoiding double-counting** of emissions

Merging

combine **multiple** (N)LO event samples into a **single** one, accounting for **shower radiation** and **avoiding double-counting**

What happens after one parton has been extracted from each proton?

Remnants can undergo multiple scatters → multi-parton interactions (MPI)



How exactly the underlying event is modelled depends…

After first interaction, scale of MPI is sampled according to

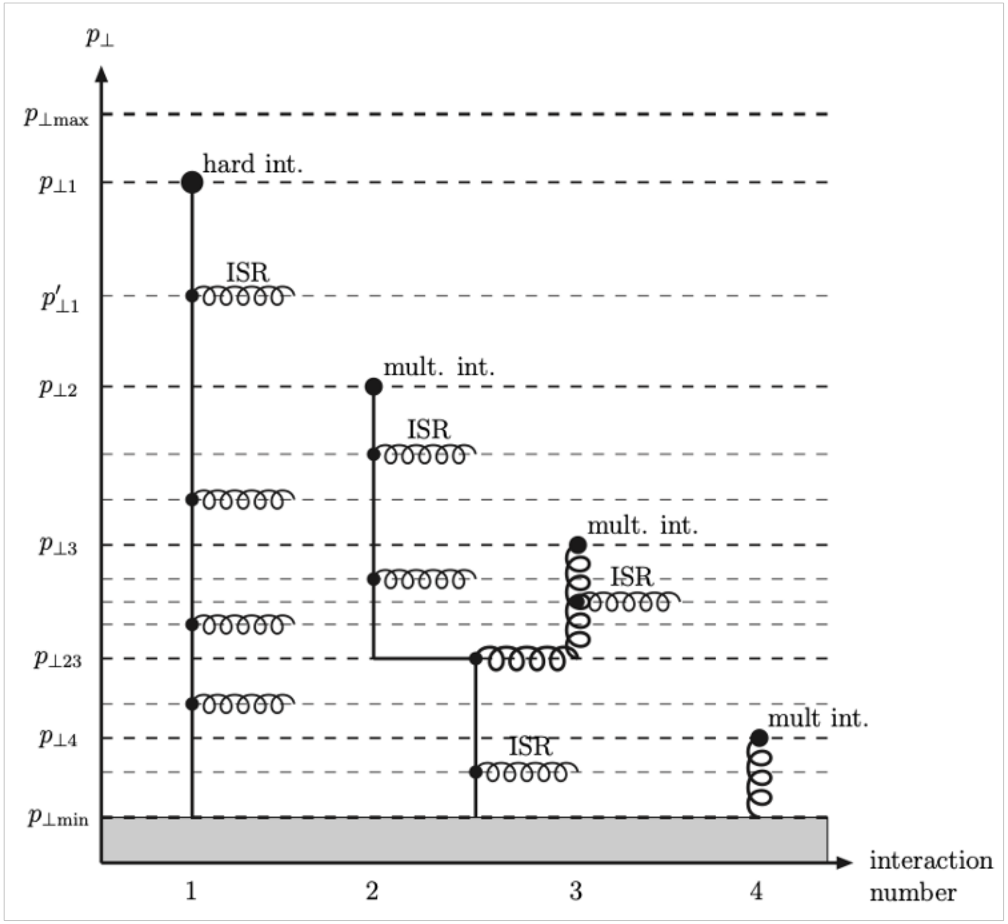
$$\frac{\mathrm{d}\mathcal{P}_{\mathrm{MPI}}}{\mathrm{d}p_{\mathrm{T}}^2} = \frac{1}{\sigma_{\mathrm{nd}}}\frac{\mathrm{d}\sigma(p_{\mathrm{T}}^2)}{\mathrm{d}p_{\mathrm{T}}^2}\exp\left(-\int_{Q^2}^{Q_0^2}\mathrm{d}p_{\mathrm{T}}^2\,\frac{1}{\sigma_{\mathrm{nd}}}\frac{\mathrm{d}\sigma(p_{\mathrm{T}}'^2)}{\mathrm{d}p_{\mathrm{T}}'^2}\right)$$

Compare to parton shower:

$$\frac{\mathrm{d}\mathcal{P}_{\mathrm{Shower}}}{\mathrm{d}p_{\mathrm{T}}^2} = \frac{\mathrm{d}p_{\mathrm{T}}^2}{p_{\mathrm{T}}^2}\int_0^1\frac{\alpha_s}{2\pi}P(z)\exp\left(-\int_{Q^2}^{Q_0^2}\frac{\mathrm{d}p_{\mathrm{T}}^2}{p_{\mathrm{T}}'^2}\int_0^1\frac{\alpha_s}{2\pi}P(z)\right)$$

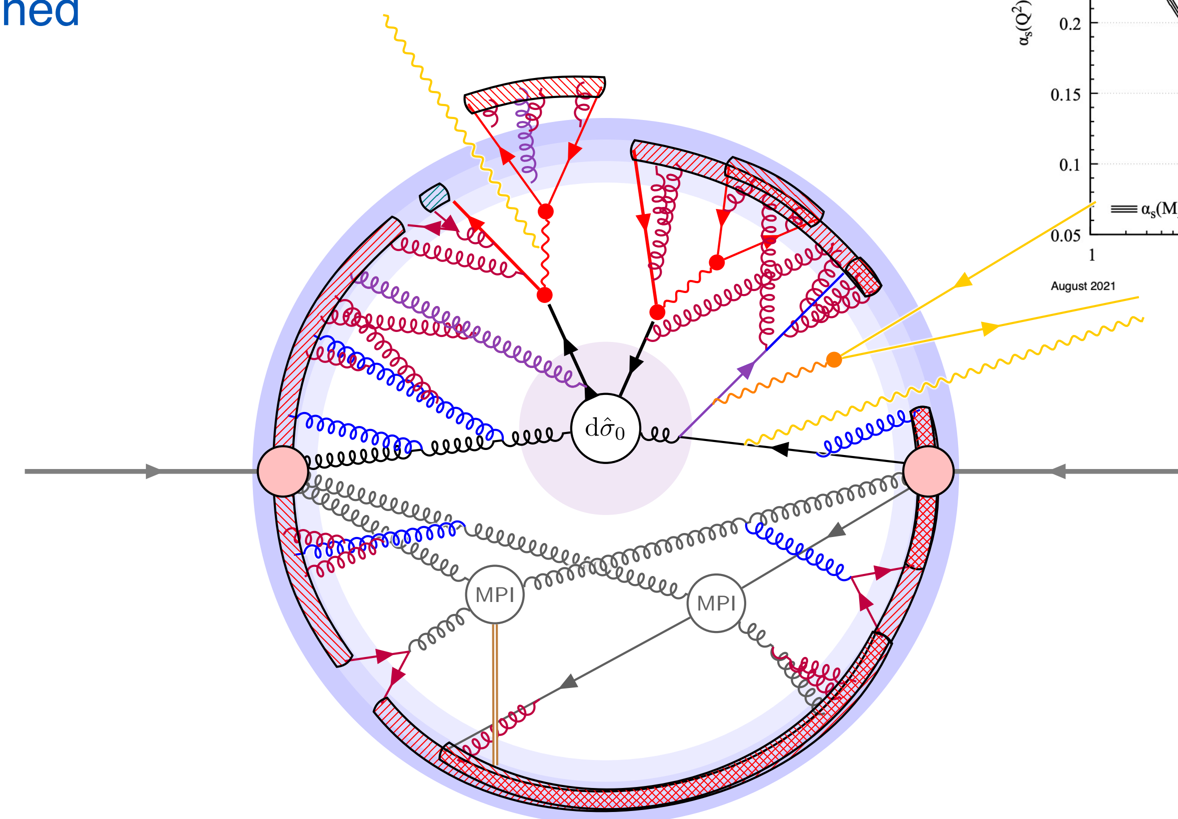
Motivates to interleave MPI and parton-shower evolution

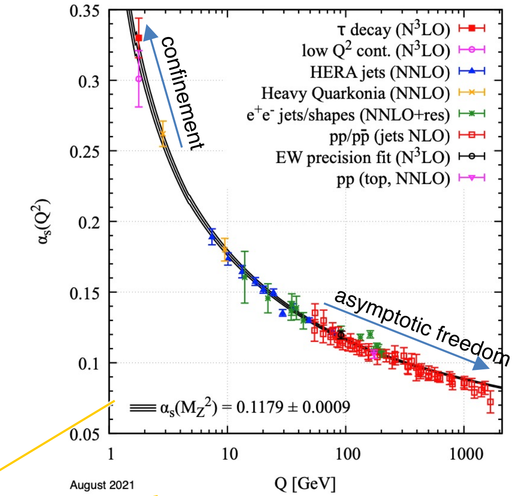
$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}p_{\mathrm{T}}^2} = \left(\frac{\mathrm{d}\mathcal{P}_{\mathrm{MPI}}}{\mathrm{d}p_{\mathrm{T}}^2} + \frac{\mathrm{d}\mathcal{P}_{\mathrm{Shower}}}{\mathrm{d}p_{\mathrm{T}}^2}\right) \times \exp\left(-\int_{Q^2}^{Q_0^2}\left[\frac{\mathrm{d}\mathcal{P}_{\mathrm{MPI}}}{\mathrm{d}p_{\mathrm{T}}'^2} + \frac{\mathrm{d}\mathcal{P}_{\mathrm{Shower}}}{\mathrm{d}p_{\mathrm{T}}'^2}\right]\right)$$



[Sjöstrand, Skands EPJC 39 (2005) 129-154]

At energies ~ 1 GeV, the parton shower is terminated, partons become confined

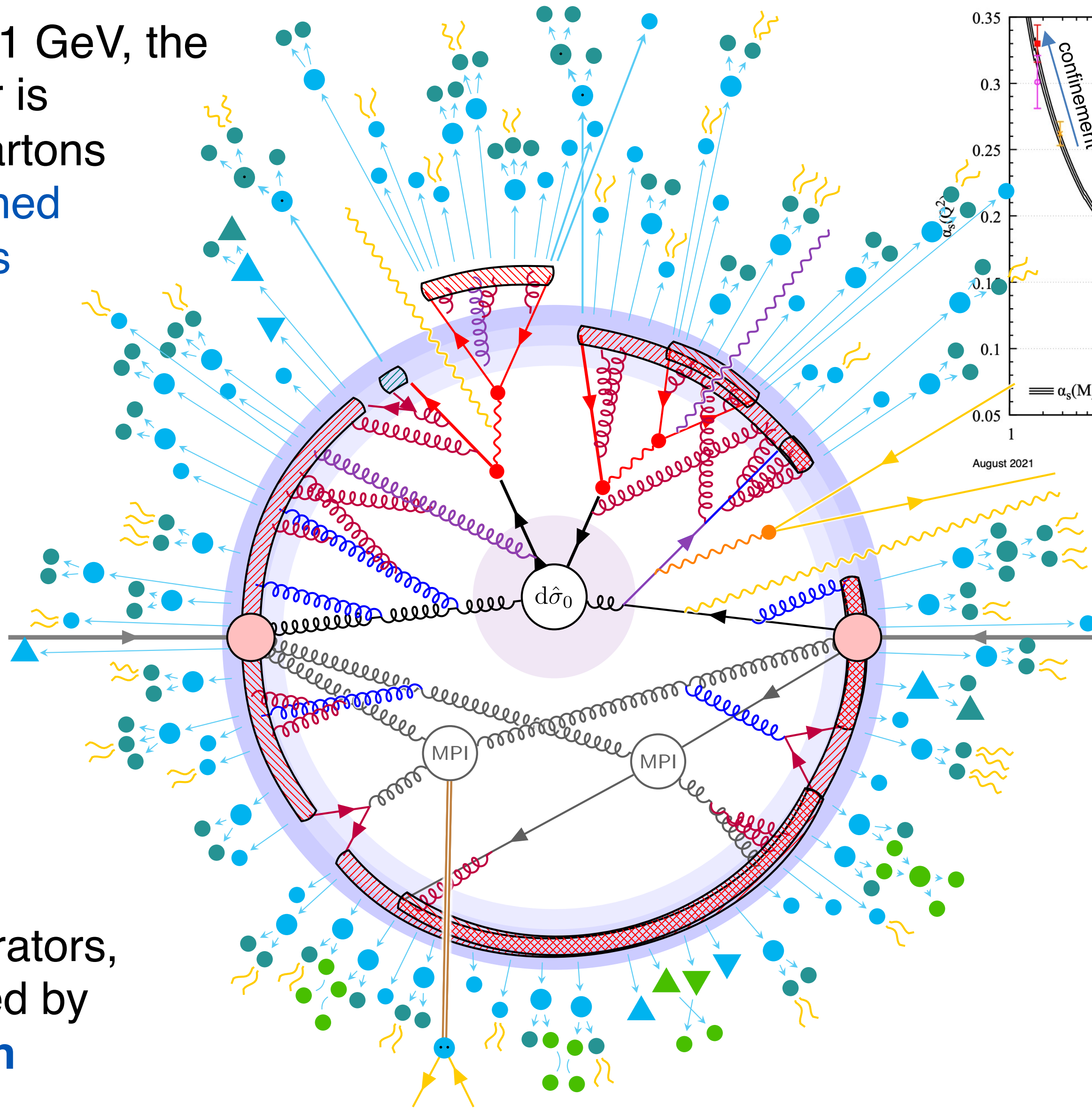


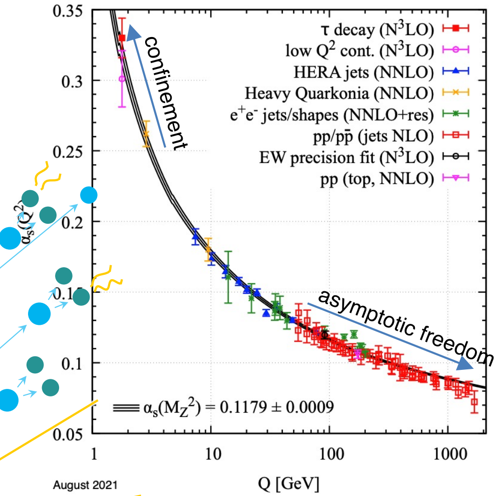


[PDG]

Step 5: confinement (and fragmentation)

At energies ~ 1 GeV, the parton shower is terminated, partons become confined inside hadrons





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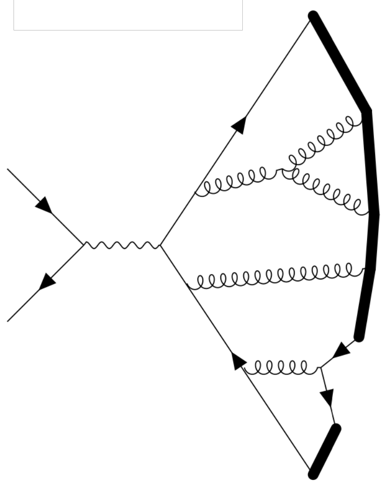
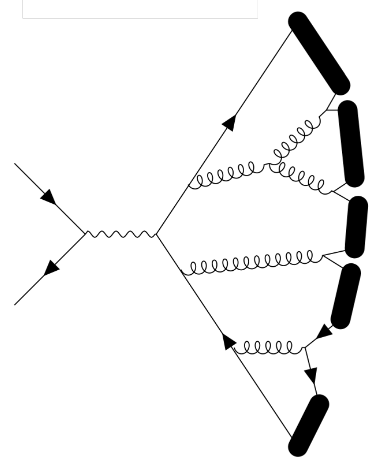
In event generators, this is modelled by **hadronisation**

(Lund) string model

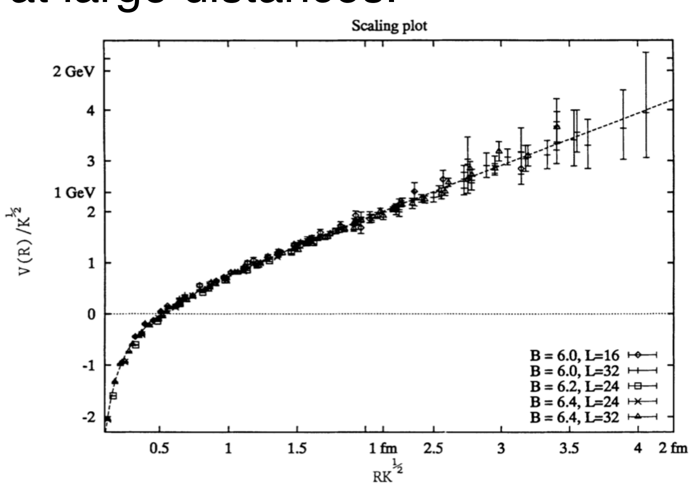


cluster model

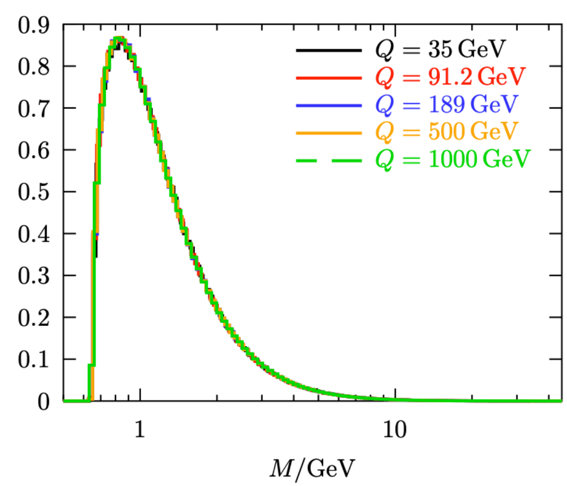


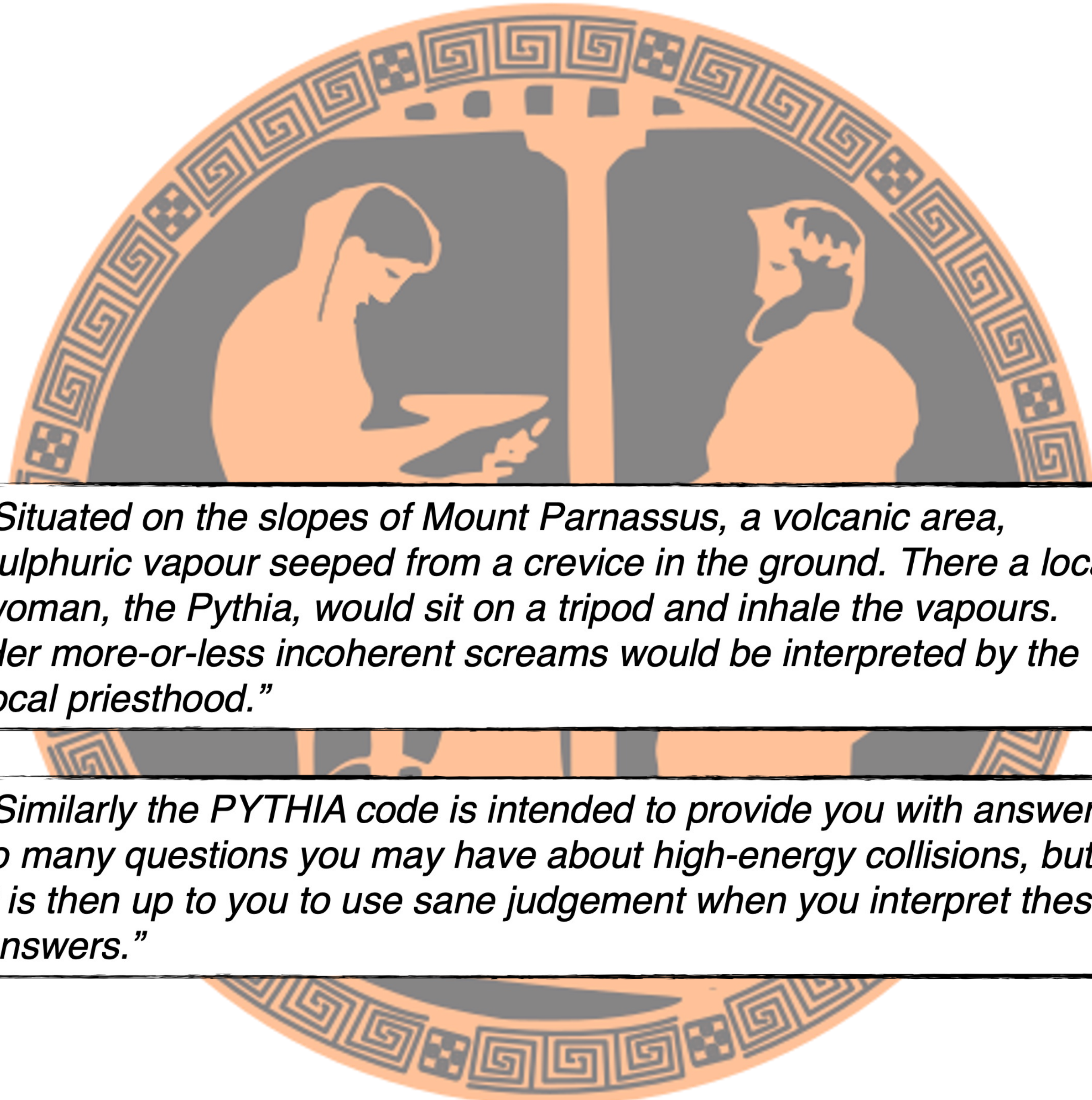
Both are heuristic models!

Based on linear strong potential at large distances.



Based on preconfinement and local parton-hadron duality.





"Situated on the slopes of Mount Parnassus, a volcanic area, sulphuric vapour seeped from a crevice in the ground. There a local woman, the Pythia, would sit on a tripod and inhale the vapours. Her more-or-less incoherent screams would be interpreted by the local priesthood."

"Similarly the PYTHIA code is intended to provide you with answers to many questions you may have about high-energy collisions, but it is then up to you to use sane judgement when you interpret these answers."