Finite Temperature Description of Fermi Gases with In-medium Effective Mass

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Finite Temperature Description of Fermi Gases with In-medium Effective Mass

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Abstract

We investigate Fermi gases at finite temperature for which the in-medium effective mass may not be constant as a function of the density, the temperature, or the chemical potential. We suggest a formalism that separates the terms for which the mass is constant from the terms that explicitly treat the correction due to the in-medium effective mass. We employ the ensemble equivalence in infinite matter to treat these different terms. Our formalism is applied in nuclear matter and we show its goodness by comparing it to an exact treatment based on the numerical calculation of the Fermi integrals.

Motivation

- The study of Fermi gases at finite temperature is numerically more evolved than at zero temperature, where analytical expressions are available;
- There are several approximations for computing thermodynamic properties at finite temperature[1], but most assume a **constant in-medium mass**;
- In reality, the effective mass can vary with **density**, **temperature**, or **chemical potential**, **which is relevant** in systems such as **neutron star mergers** and **core-collapse supernovae**;
- The scope of this work is to suggest a framework where the **FFG approximations** at finite temperature could still be employed to describe the **properties of FGs**, allowing **fast calculation even at finite temperature.**

[1] Eggleton P. P. , *et. al.*, *A&A*, **23**, 325 (1973); Fukushima T., Appl. Math. Comput., **234**, 417 (2014); Gil A., *et. al.*, *CoPhC*, **283**, 108563 (2023); Johns S. M., *et. al.*, *ApJ*, **473**, 1020 (1996).

JEL Approximation

- Fundamental Thermodynamic Quantities:

$$p = \frac{\gamma}{6\pi^2}\int_0^\infty \frac{dkk^4}{(k^2+m^{*2})^{1/2}} F_D(k,\mu,T,m^*)$$

$$\epsilon = \frac{\gamma}{2\pi^2}\int_0^\infty dk\ k^2(k^2+m^{*2})^{1/2} F_D(k,\mu,T,m^*)$$

$$\sigma = -\frac{\gamma}{2\pi^2}\int_0^\infty dk\ k^2\left[F_D \ln F_D + (1-F_D)\ln(1-F_D)\right]$$

where F_D is the Fermi-Dirac distribution function: $F_D(k,\mu,T,m^*) = \dfrac{1}{e^{(\sqrt{k^2+m^{*2}}-\mu)/T}+1}$

Note that these expressions could be generalized easily to include antiparticle contributions using $\mu_{\bar{\alpha}} = -\mu_\alpha$.

- The density is also modified by the effect of finite temperature as

$$n = \frac{\gamma}{2\pi^2}\int_0^\infty dk\ k^2 F_D(k,\mu,T,m^*)\,.$$

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$$n = \frac{\gamma}{2\pi^2}\int_0^\infty dk\; k^2 F_D(k,\mu,T,m^*)\,.$$

Free Fermi Gas (FFG): In the absence of interactions among particles, the mass m^* remains constant and is equal to the bare mass of the particles.

JEL Approximation

- What is the JEL Approximation?

Johns S. M., Ellis P. J., Lattimer J. M., *ApJ*, **473**, 1020 (1996).

  - Instead of solving complex integrals, we use a mathematical shortcut;
  - JEL transforms a difficult integral into simple polynomial series;
  - This allows fast calculations with minimal loss of accuracy. Consider

$$p^* = d\bar{p}\,,\ \epsilon^* = d\bar{\epsilon}\,,\ \phi^* = d\bar{\phi}\,,\ \ \sigma^* = \frac{d}{m^*}\bar{\sigma}\,,\ n^* = \frac{d}{m^*}\bar{n}\,,\ \ d = \gamma m^{*4}/(2\pi^2)\,.$$

$$\bar{p} = \frac{fg^{5/2}(1+g)^{3/2}}{(1+f)^{J+1}(1+g)^L}\sum_{j=0}^{J}\sum_{l=0}^{L} p_{jl} f^j g^l\,,\ g = t(1+f)^{1/2} \quad \text{and} \quad t = T/m^*$$

$$\mu^* = \psi T + m^*\,, \qquad \psi = 2\sqrt{1+\frac{f}{a}} + \ln\frac{\sqrt{1+f/a}-1}{\sqrt{1+f/a}+1}\,.$$

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  - Instead of solving complex integrals, we use a mathematical shortcut;

$$\bar{n} = \frac{f[g(1+g)]^{3/2}}{(1+f)^{J+1/2}(1+g)^{L}\sqrt{1+f/a}} \sum_{j=0}^{J}\sum_{l=0}^{L} p_{jl} f^{j} g^{l} \left[1 + l + \left(\frac{1}{4} + \frac{l}{2} - J\right)\frac{f}{1+f} + \left(\frac{3}{4} - \frac{L}{2}\right)\frac{fg}{(1+f)(1+g)}\right]$$

$$p^{*} = d\bar{p},\ \epsilon^{*} = d\bar{\epsilon},\ \phi^{*} = d\bar{\phi},\ \sigma^{*} = \frac{d}{m^{*}}\bar{\sigma},\ n^{*} = \frac{d}{m^{*}}\bar{n},\ d = \gamma m^{*4}/(2\pi^{2})\,.$$

$$\bar{p} = \frac{fg^{5/2}(1+g)^{3/2}}{(1+f)^{J+1}(1+g)^{L}} \sum_{j=0}^{J}\sum_{l=0}^{L} p_{jl} f^{j} g^{l},\ g = t(1+f)^{1/2} \quad \text{and} \quad t = T/m^{*}$$

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Treatment of FG with In-Medium Effective Mass

- Mass Modification by Many-Body Correlations
  - In interacting systems, mass depends on single-particle states (k) and thermodynamic quantities such as density, temperature, and chemical potential.
- Kinetic Energy and In-Medium Effective Mass
  - The in-medium effective mass must be properly addressed, which we achieve by solving the integrals in those equations using the Gauss–Legendre method.
- Corrections to Thermodynamic Potentials
  - Our formulation separates terms assuming a constant mass (where analytical methods apply) from those explicitly treating the in-medium effective mass dependence.

Treatment of FG with In-Medium Effective Mass

- Example of In-medium Effective Masses

$$\frac{m_q^*(n,\,\delta)}{m} = [1 + 2m(C_0^\tau + \tau_3 C_1^\tau \delta)n]^{-1} \qquad q = p, n$$

convention: $\tau_3 = 1\,(-1)$ for neutrons (protons).

- Application to Symmetric Nuclear Matter

SM: $\delta = \frac{(n_n - n_p)}{n} = 0 \longrightarrow \quad \frac{m^*(n)}{m} = (1 + 2mC_0^\tau\, n)^{-1} \quad (m_p^* = m_n^* = m^*)$

- Helmholtz free energy density $f = \frac{\phi(n, T, m^*)}{n}$

$$\left.\frac{\partial X(n,\,m^*)}{\partial n}\right|_T = \left.\frac{\partial X(n,\,m^*)}{\partial n}\right|_{T,m^*} + \left.\frac{\partial X(n,\,m^*)}{\partial m^*}\right|_{T,n} \left.\frac{\partial m^*}{\partial n}\right|_T ,$$

Treatment of FG with In-Medium Effective Mass

- Pressure:

$$p \equiv n^2 \frac{\partial f}{\partial n}\Big|_T = n\frac{\partial \phi}{\partial n}\Big|_{T,m^*} + n\frac{\partial \phi}{\partial m^*}\Big|_{T,n}\frac{\partial m^*}{\partial n}\Big|_T - \phi = p^* + p_{\rm corr},$$

with

$$p^* = n\frac{\partial \phi}{\partial n}\Big|_{T,m^*} - \phi \quad \text{and} \quad p_{\rm corr} = n\frac{\partial \phi}{\partial m^*}\Big|_{T,n}\frac{\partial m^*}{\partial n}\Big|_T .$$

- Density: $n_{corr} = 0$ and $n = n^*$ (Note that since the density is the thermodynamical variable of the CE, it is not impacted by the density-dependent effective mass)

- Chemical potential: $\mu \equiv \frac{\partial \phi}{\partial n}\Big|_T = \frac{\partial \phi}{\partial n}\Big|_{T,m^*} + \frac{\partial \phi}{\partial m^*}\Big|_{T,n}\frac{\partial m^*}{\partial n}\Big|_T = \mu^* + \mu_{\rm corr},$

$$\mu_{\rm corr} = \frac{\partial \phi}{\partial m^*}\Big|_{T,n}\frac{\partial m^*}{\partial n}\Big|_T = \frac{p_{\rm corr}}{n}.$$

- Entropy: $\sigma = -\frac{\partial \phi}{\partial T}\Big|_n$, $\sigma = \sigma^*$.

Treatment of FG with In-Medium Effective Mass

- The energy density can be determined from the Euler relation

$$\epsilon = -p + \mu\, n + T\sigma \;= -(p^* + p_{\rm corr}) + (\mu^* + \mu_{\rm corr})\, n + T\sigma^* = -p^* + \mu^*\, n + T\sigma^* = \epsilon^*.$$

- Finally, the Helmholtz free energy density is

$$\phi = \epsilon - T\sigma = -p + \mu\, n = \epsilon^* - T\sigma^* = -p^* + \mu^*\, n = \phi^*$$

Treatment of FG with In-Medium Effective Mass

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$$\phi = \epsilon - T\sigma = -p + \mu\, n = \epsilon^* - T\sigma^* = -p^* + \mu^*\, n = \phi^*$$

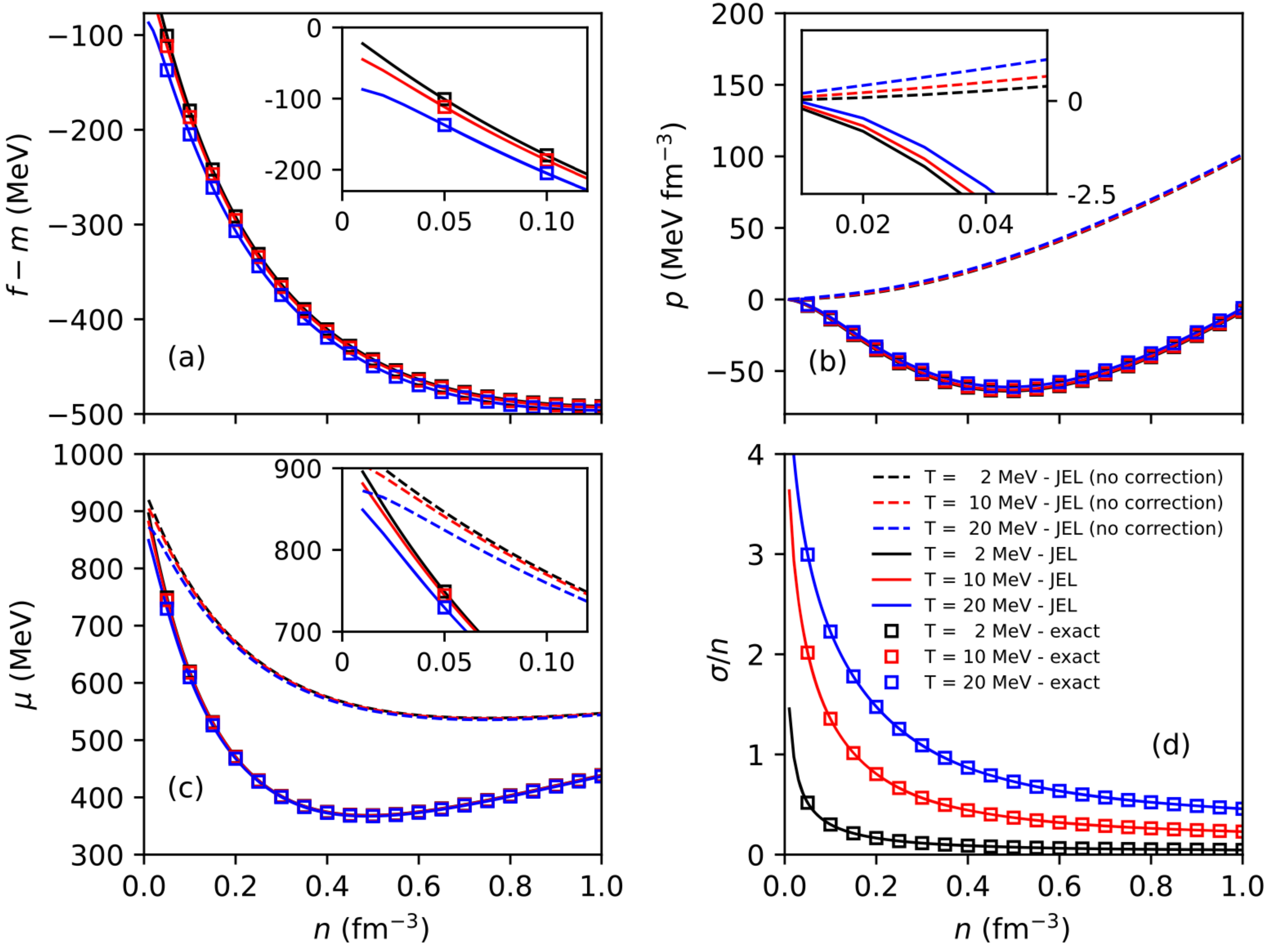
- Practical Implementation: $p_{\rm corr} = n \frac{\partial \phi}{\partial m^*}\Big|_{T,n} \frac{\partial m^*}{\partial n}\Big|_T$. $\mu^* = \psi(n)T + m^*$

  * By using the JEL approximation, it is possible to rewrite

$$\frac{\partial \phi}{\partial m^*}\Big|_{n,T} = \frac{\partial \phi^*}{\partial m^*}\Big|_{n,T} = -\frac{\partial p^*}{\partial m^*}\Big|_{n,T} + n\frac{\partial \mu^*}{\partial m^*}\Big|_{n,T} ; \quad \frac{\partial \mu^*}{\partial m^*}\Big|_{n,T} = 1, \quad \frac{\partial \phi^*}{\partial m^*}\Big|_{n,T} = \frac{1}{m^*}(\epsilon^* - 3p^*),$$

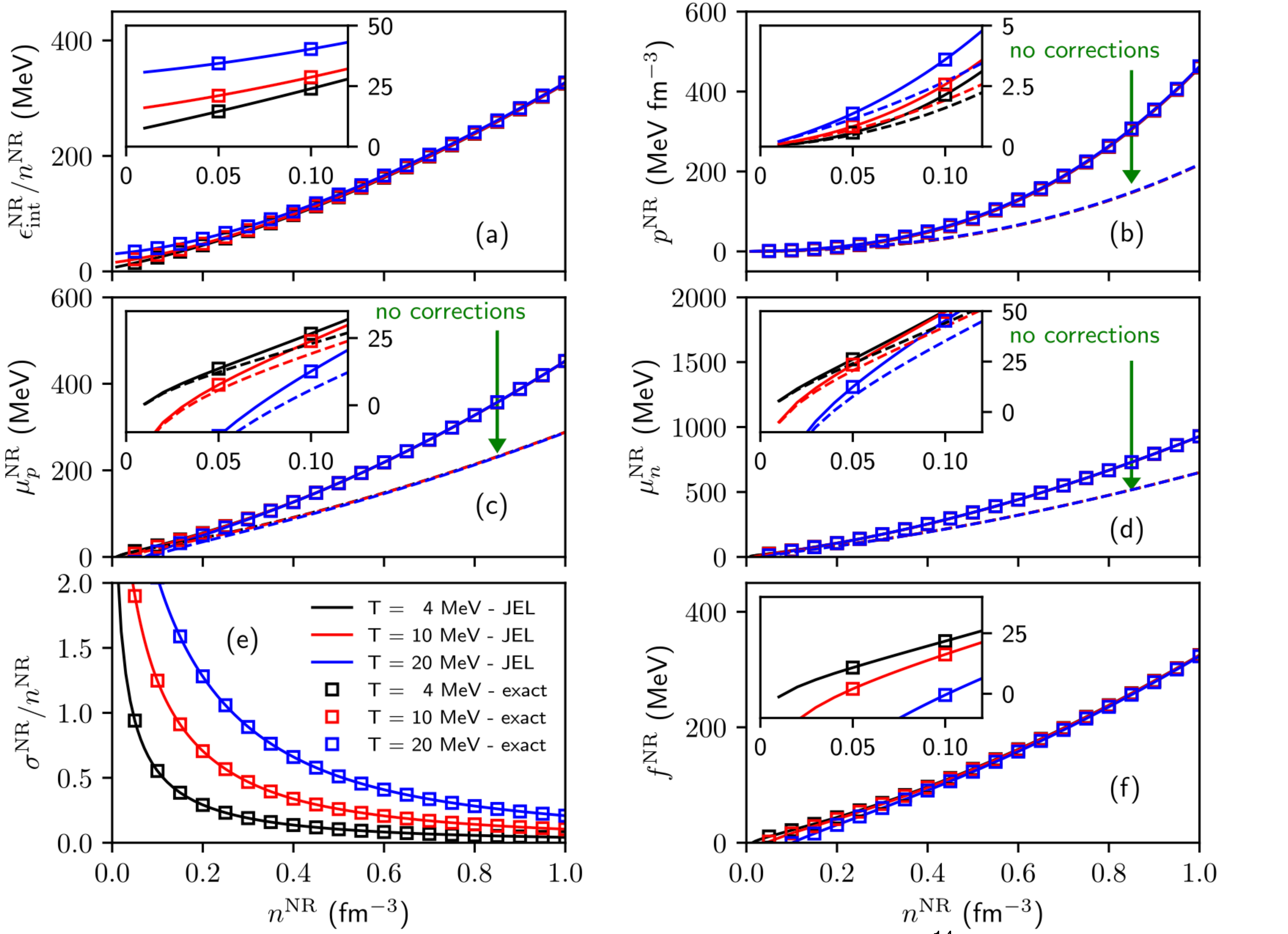
$$\Rightarrow \boxed{p_{\rm corr} = \frac{n}{m^*}(\epsilon^* - 3p^*)\frac{\partial m^*}{\partial n}\Big|_T}$$

Results



- ✓ Thermodynamic properties of the relativistic FG with density-dependent effective mass in SM, as a function of the density;
- ✓ Set of temperatures: T= 2 (black), 10 (red), and 20 MeV (blue);
- ✓ exact calculations — **squares**
- ✓ our formalism based on the JEL approximation — **solid lines**

Results



✓ Asymmetric Nuclear Matter

$$\phi = \phi(m_n^*, m_p^*, T) = \phi_n(m_n^*, T) + \phi_p(m_p^*, T)$$

$$p \equiv n^2 \frac{\partial f}{\partial n}\bigg|_{T,\delta} = \sum_{q=n,p} p_q^* + p_{\rm corr,q},$$

✓ Thermodynamic quantities of the non-relativistic FG, with density-dependent effective mass;

✓ exact calculation — squares;

✓ Our + JEL approximation — full lines;

✓for different temperatures and δ= 0.4.

Results:

PHYSICAL REVIEW C **109**, 055202 (2024)

Nucleonic models at finite temperature with in-medium effective fields

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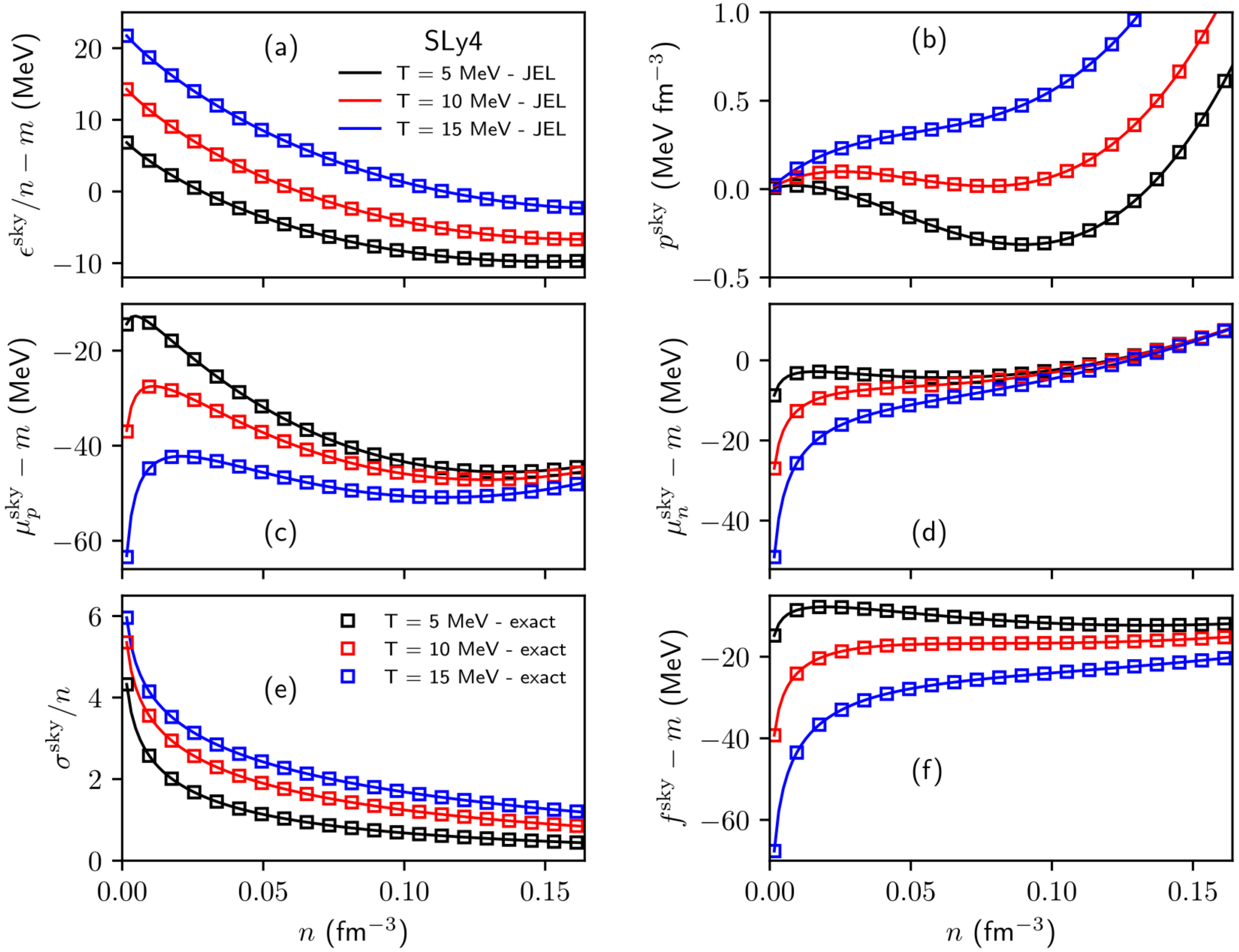
$$\phi(n, \delta, T, \{\varphi\}) = \phi^*_{\rm kin}(n, \delta, T, \{\varphi\}) + \phi_{\rm pot}(n, \delta, \{\varphi\}),$$

$$\{\varphi\} = \{m_n^*, m_p^*\}$$

$$f(n, \delta, T, \{\varphi\}) = \phi(n, \delta, T, \{\varphi\})/n$$



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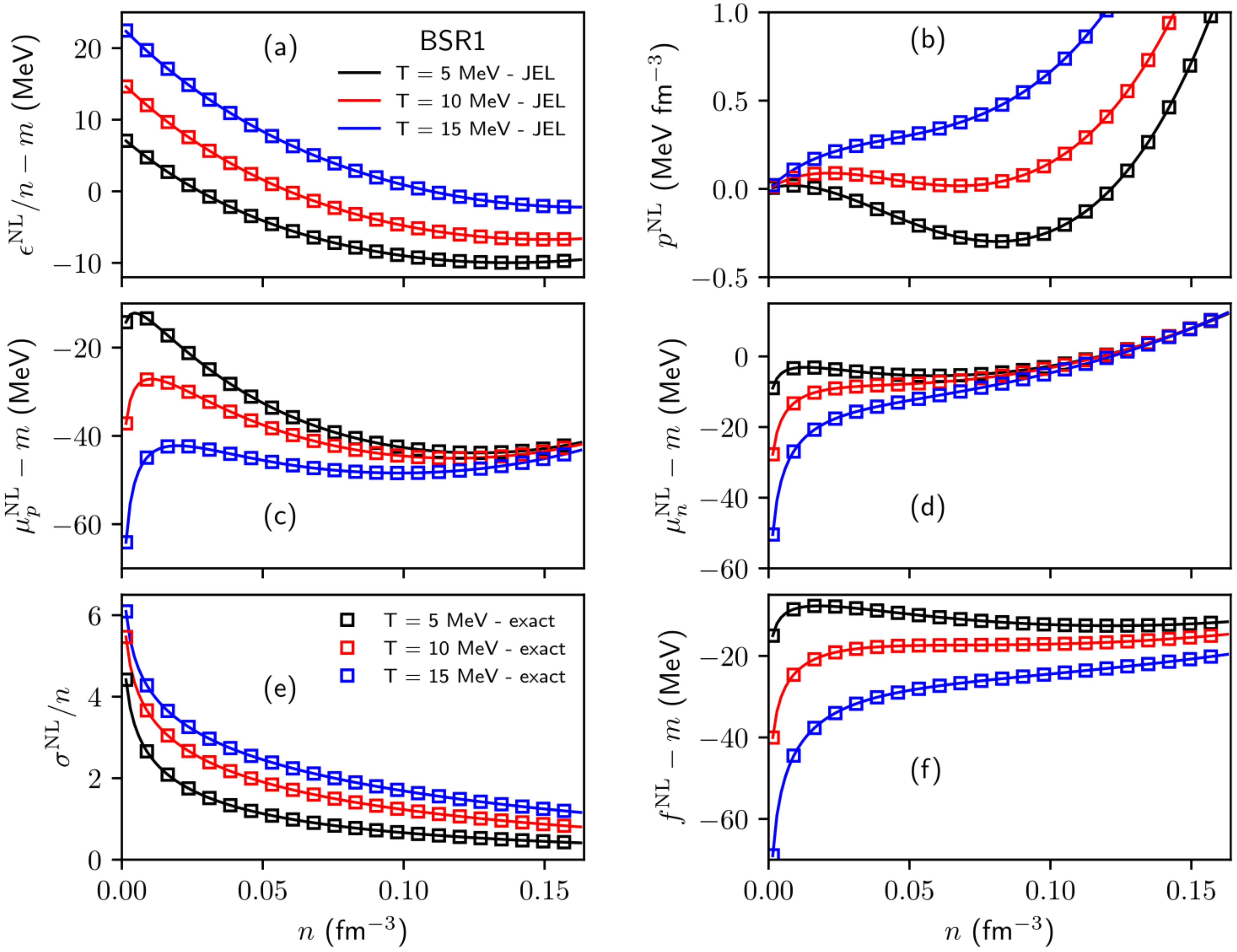
$$\phi(n, \delta, T, \{\varphi\}) = \phi^*_{\rm kin}(n, \delta, T, \{\varphi\}) + \phi_{\rm pot}(n, \delta, \{\varphi\}),$$

$$\{\varphi\} = \{\sigma_0, \delta_{(3)}, \omega_0, \rho_{0(3)}\}$$

$$f(n, \delta, T, \{\varphi\}) = \phi(n, \delta, T, \{\varphi\})/n.$$



Conclusions

- It was demonstrated how approximations for the Finite-Temperature FFG with arbitrary masses can be extended to the more general case of the FG with in-medium effective masses;
- A detailed comparison of thermodynamic quantities was performed, using an efficient approximation for the FFG (JEL approximation, order 3), showing excellent agreement with exact calculations;
- The additional dependencies of the in-medium effective mass on density, temperature, and chemical potential were explored, leading to generalized simple relations;
- Our approach is applicable in astrophysical scenarios, particularly for finite-temperature EoS relevant to neutron star mergers and core-collapse supernovae;
- For the Skyrme model we find that the JEL approximation is about 15 times faster than the numerical calculation. This factor is changed to about 30 in the case of the nonlinear relativistic model.

Thank you!