The structure of the Proton

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Outline

Lecture I: How do we know the nucleon has a structure

$e^- \mu^-$ elastic scattering

$e^- p^+$ elastic and inelastic scattering

Parton model e Bjorken scaling

The proton structure

Lecture II: Factorization

The operator product expansion

The Collins way

The formal definition of PDFs

One loop-corrections

PDFs from first principles?

Lecture IIII: PDFs and the path towards a 3-D picture

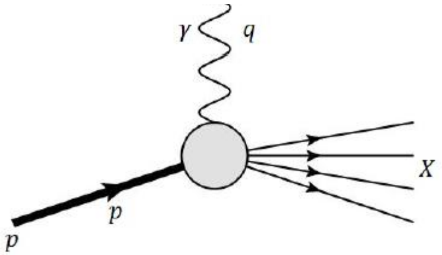
PDFs and quasi-PDFs

GPDs

TMD-PDFs

Recent results

The Operator product expansion



$\langle X|J^\nu(0)|P\rangle$

The hadronic tensor is then

$$W^{\mu\nu}(P,X) = \frac{1}{4\pi}\sum_S \sum_X \langle P,S|J^\mu(0)|X\rangle\langle X|J^\nu(0)|P,S\rangle(2\pi)^4\delta^4(p_X - P - q)$$

$$= \frac{1}{4\pi}\sum_S \sum_X \int d^4z\, e^{-i(p_X-P-q)}\langle P,S|J^\mu(0)|X\rangle\langle X|J^\nu(0)|P,S\rangle$$

$$= \frac{1}{4\pi}\sum_S \int d^4z\, e^{iq\cdot z}\langle P,S|J^\mu(z)J^\nu(0)|P,S\rangle$$

$$= \frac{1}{4\pi}\sum_S \int d^4z\, e^{iq\cdot z}\langle P,S|[J^\mu(z),J^\nu(0)]|P,S\rangle$$

From momentum conservation

Hadronic tensor can be seen as the Fourier transform of the commutator of currents

But why are we doing that?

Where the main contribution to this FT comes from?

$$\sum_S \int d^4z\, e^{iq\cdot z} \langle P,S|[J^\mu(z), J^\nu(0)]|P,S\rangle$$

Riemann-Lebesgue theoreman: only the contribution from finite $q.z$ contributes

LAB frame: $P = (M, 0,0,0), q = \left(\nu, 0,0, -\sqrt{\nu^2 + Q^2}\right)$

Within the Bjorken limit: $q^+ = \frac{1}{\sqrt{2}}(q^0 + q^3) \sim -Q^2/2\nu$ $\quad q^- = \frac{1}{\sqrt{2}}(q^0 - q^3) \sim \nu$

$\Longrightarrow$ $q.z = q^+z^- + q^-z^+$ is finite only if $z^- \sim \nu/Q^2$ and $z^+ \sim 1/\nu$

From causality, $[J^\mu(z), J^\nu(0)]$ is zero for $z^2 < 0$

Bjorken limit, $\nu, Q^2 \to \infty$, integral is dominated by the $z^2 = 2z^+z^- - z_\perp^2 \sim 0$ region

DIS is light-cone dominated!

Continuing with the OPE

Let $j^\mu(z) =: \overline{\psi}(z)\gamma^\mu\psi(0):$ be the electromagnetic current. Then

$$T[J^\mu(z)J^\nu(0)] =: j^\mu(z)j^\nu(0): \quad + \quad (one\ propagator\ term) \quad + \quad (two\ propagators\ term)$$

Not singular

$\frac{i}{2\pi^2}\frac{\gamma_\mu z^\mu}{z^2 - i\epsilon}$ singular

$\frac{i}{2\pi^2}\frac{\gamma_\mu z^\mu}{z^2 - i\epsilon}$ singular $\frac{i}{2\pi^2}\frac{\gamma_\mu z^\mu}{z^2 - i\epsilon}$ singular

In general

$$T[j^\mu(z)j^\nu(0)] \to \sum_{n=0}^{\infty} i^{n-1} C_i^n(z^2 - i\epsilon) z^{\mu_1} z^{\mu_2} \cdots O^{i,n}_{\mu_1\mu_2\cdots\mu_n}(0)\,, \qquad z^2 \to 0$$

W. Zimmermann, Annals Phys. 77, 570 (1973)

R. A. Brandt and G. Preparata, Nucl. Phys. B 27, 541 (1972)

$C^n(z^2 - i\epsilon)$, singular Wilson coefficients calculated in perturbation theory

$O^i_{\mu_1\mu_2\cdots\mu_n}(0)$, set of i local operators

What are these local operators?

Let:

- d_j be the dimension of the current
- d_O be the dimension of the operator
- n be the dimendion of the product $z^{\mu_1} z^{\mu_2} \cdots z^{\mu_n}$

Thus, $[C] = 2d_j + n - d_O \equiv 2d_j - \tau$ Dimension of the singular coefficient

where $\tau = d_O - n$ is defined as the twist of the operator

The smaller the twist, the more singular is the
Wilson coefficient associated with the operator O

Simplest choice: $\bar{\psi}\psi$ has twist **3**,

But $\bar{\psi}(0)\gamma_{\mu_1}\psi(0)$ has twist **2**

$\bar{\psi}(0)\gamma_{\mu_1} D_{\mu_2} D_{\mu_3} \cdots D_{\mu_n}\psi(0)$ also has twist **2** !

How can this be used?

$$\frac{2F_2}{\nu M}$$

Hadronic tensor

Known as Mellin moments

rd Compton scattering
tude

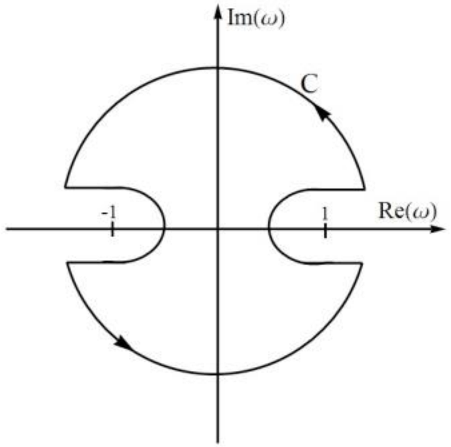
$\omega = \frac{1}{x}$

$$M_L^{(n)}(Q^2) = \int_0^1 dx x^{n-2} F_L(x, Q^2) = \sum_i A_{L,n}^{(i)} C_{L,n}^{(i)}(Q^2)$$

$$M_2^{(n)}(Q^2) = \int_0^1 dx x^{n-2} F_2(x, Q^2) = \sum_i A_{2,n}^{(i)} C_{2,n}^{(i)}(Q^2)$$

$\boldsymbol{A_n}$ matrix elements of local operators

$\boldsymbol{C_n}$ perturbatively computed Wilson coefficients



$$\langle P | O_{\mu_1 \mu_2 \cdots \mu_n}(0) | P \rangle = \left\langle P \left| \overline{\psi}(0) \gamma_{\mu_1} D_{\mu_2} D_3 \cdots D_{\mu_n} \psi(0) \right| P \right\rangle = \boldsymbol{A_n} P^{\mu_1} P^{\mu_2} \ldots P^{\mu_n}$$

ME of the operator

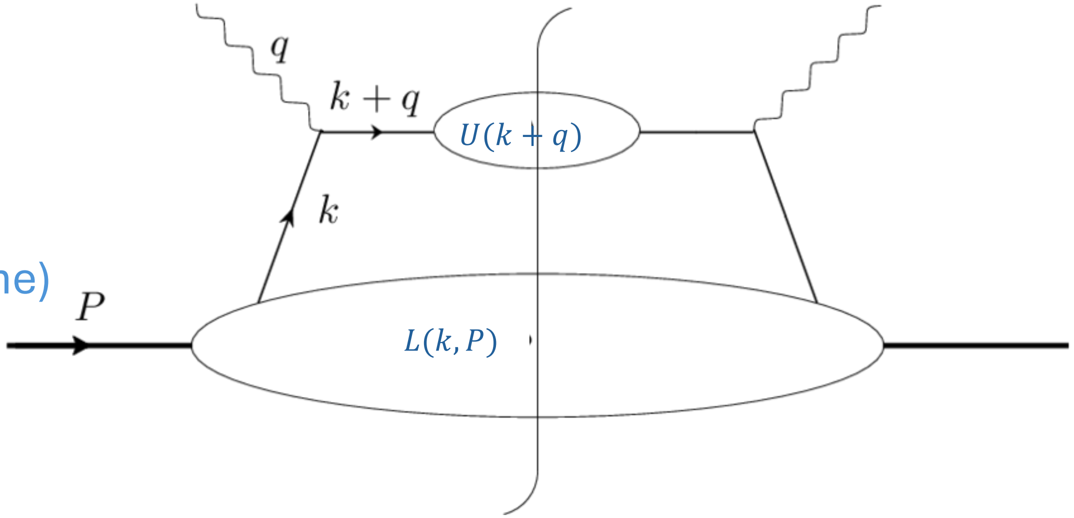
$$\frac{1}{2\pi} \oint_c \omega^{-n} T^{\mu\nu} \, d\omega = 2 \int_0^1 dx \, x^{n-2} W^{\mu\nu}$$

Relation between the Compton amplitude and the hadronic tensor

Collinear factorization and PDFs: Collins way

Hadronic tensor

$$W^{\mu\nu} = \sum_j \frac{e_j^2}{4\pi} \int \frac{d^4k}{(2\pi)^4} Tr\left[\gamma^\mu U_j(k+q)\gamma^\nu L(k,P)\right]$$



External particles without transverse momentum (Breit frame)

$$P^\mu = \left(P^+, \frac{M^2}{2P^+}, \vec{0}_T\right) \qquad q^\mu = \left(-xP^+, \frac{Q^2}{2xP^+}, \vec{0}_T\right)$$

Parton momentum

$$k^\mu = (\xi P^+, k^-, \vec{k}_T)$$

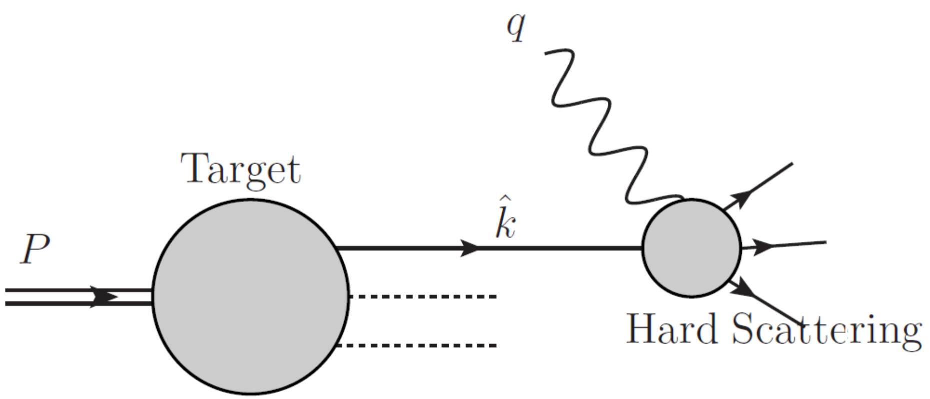
$$k^\mu + q^\mu = \left((\xi - x)P^+, \frac{Q^2}{2xP^+} + k^-, \vec{k}_T\right)$$

$$\xi = \frac{k^+}{P^+}$$ Internal variable

$$x = -\frac{q^+}{P^+} \to x_B \; as \; Q^2 \to \infty$$ External variable

$$x_B = \frac{Q^2}{2P \cdot q} = \frac{Q^2}{2P^+q^- + 2P^-q^+} = \frac{x}{1 - \frac{x^2M^2}{Q^2}}$$

Leading power in m/Q: neglect the small components of momentum, $k^-, \vec{k}_T$ and $(\xi - x)P^+$ with respect to Q



Collinear factorization makes the following approximation for the U part:

$$k^\mu \to \hat{k}^\mu \simeq (\xi P^+, 0, \vec{0}_T)$$

With a Lorentz transformation that takes to zero the transverse momentum of the scattered parton

$$k^\mu + q^\mu \to \hat{k}^\mu + \hat{q}^\mu \simeq \Big(\underbrace{k^+ + q^+ - \frac{k_T^2}{2q^-}}_{\equiv l^+}, q^-, \vec{0}_T\Big)$$

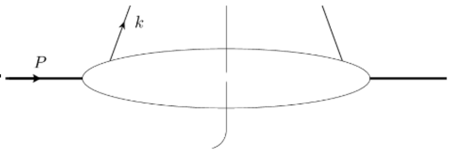
The hadronic tensor can, then, be factorized into two distinct parts

$$W^{\mu\nu} \simeq \sum_j \frac{e_j^2}{4\pi} Tr\left[\gamma^\mu \int \frac{dl^+}{2\pi} U_j(l^+, q^-, \vec{0}_T) \,\vdots\, \gamma^\nu \underbrace{\int \frac{dk^-}{2\pi} \frac{d^2k_T}{(2\pi)^2} L\left((xP^+, k^-, \vec{k}_T), P\right)}_{q_j(x)}\right] + \mathcal{O}\left(\frac{m^2}{Q^2}\right)$$

Which can be written as the FT of the ME of a LC correlation

The formal expression for the quark distribution

$$q_j(x) = Tr\,\frac{\gamma^+}{2}\int\frac{dk^- d^2k_\perp}{(2\pi)^4}$$



γ^+ for unpolarized

$\gamma^+\gamma_5$ for helicity

Which can be written as

$$q(x) = \frac{1}{4\pi}\int dz^- e^{-ip^+z^-}\langle P|\overline{\psi}(z)\,\gamma^+\,\psi(0)|P\rangle \equiv \int dz^- e^{-ixP^+z^-} f(z^-)$$

Gauge invariance introduces a Wilson line $\longrightarrow e^{ig\int_0^{z^-} A^+(z')dz'}$

With moments given by

$$a_n = \int_{-1}^{+1} dx x^{n-1} q(x) = \int_{-\infty}^{+\infty} dx x^{n-1} q(x) \;= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty}\frac{dz^-}{4\pi}\frac{1}{(-iP^+)^{n-1}}\left(D^{+(n-1)}e^{-ixP^+z^-}\right)f(z^-)$$

$$\Longrightarrow \quad a_n = \frac{i^{n-1}}{2(P^+)^n}\langle P|\overline{\psi}(0)\gamma^+ D^{+(n-1)}\psi(0)|P\rangle$$

The moments of the quark distributions are given by the ME of local operators, as we saw before!

Specifically,

From the definition of quark distributions: $$a_n = \frac{i^{n-1}}{2(P^+)^n} \langle P|\overline{\psi}(0)\gamma^+ D^{+(n-1)}\psi(0)|P\rangle$$

From the operator product expansion: $$\left\langle P \middle| O_{\mu_1\mu_2\cdots\mu_n}(0) \middle| P \right\rangle = \left\langle P \middle| \overline{\psi}(0)\gamma_{\mu_1} D_{\mu_2} D_3 \cdots D_{\mu_n}\psi(0) \middle| P \right\rangle = \boldsymbol{A_n} P^{\mu_1} P^{\mu_2} \dots P^{\mu_n}$$

Which are equivalent for $\mu_1 = \mu_2 = \cdots \mu_n = +$

The $n = 2$ case is of particular interest: $$a_2 = \frac{i}{2(P^+)^2} \langle P|\overline{\psi}(0)\gamma^+ D^+\psi(0)|P\rangle = \int_{-1}^{1} dx x q(x) = \langle x\rangle_q$$

Which is the momentum fraction carried by the quark q

Can be computed within lattice QCD

Computing $\langle x \rangle$ using lattice

$$\bar{T}^q_{\mu\nu} = -\frac{(i)^{\kappa_{\mu\nu}}}{4}\bar{q}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu - \delta_{\mu\nu}\frac{1}{2}\gamma_\rho \overleftrightarrow{D}_\rho\right)q$$

Quark EMT

$$\bar{T}^g_{\mu\nu} = (i)^{\kappa_{\mu\nu}}\left(F_{\mu\rho}F_{\nu\rho} + F\right.$$

$\kappa_{\mu\nu} = \delta_{\mu 4}\delta_{\nu 4}$

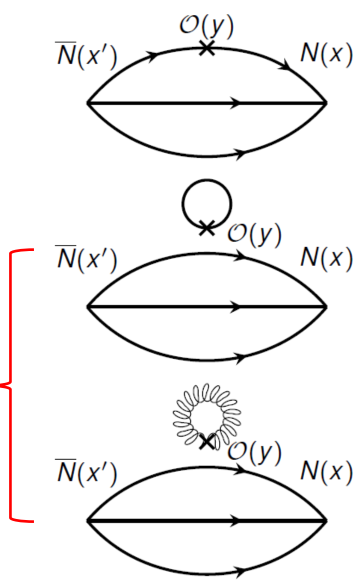
Average momentum $\langle x \rangle$

$$\langle \pi(\mathbf{p})|\bar{T}^X_{\mu\nu}|\pi(\mathbf{p})\rangle = 2\langle x\rangle^X\left(p_\mu p_\nu - \delta_{\mu\nu}\frac{\ }{4}\right)$$

	π	K	p (B-ensemble)
$\langle x\rangle_{u,R}$	0.249(28)	0.269(09)	0.354(30)
$\langle x\rangle_{d,R}$	0.249(28)	0.059(09)	0.188(19)
$\langle x\rangle_{s,R}$	0.036(15)	0.339(11)	0.052(12)
$\langle x\rangle_{c,R}$	0.013(16)	0.028(21)	0.019(09)
$\langle x\rangle_{g,R}$	0.402(53)	0.422(67)	0.427(92)
$\langle x\rangle_{q,R}$	0.575(79)	0.683(50)	0.618(60)
$\langle x\rangle_{u+d-2s,R}$	0.438(18)	−0.362(08)	− − −
$\langle x\rangle_{u+d+s-3c,R}$	0.521(51)	0.494(36)	− − −
$\langle x\rangle_{g,R} + \langle x\rangle_{q,R}$	0.984(89)	1.13(11)	1.04(11)

Ensemble	a [fm]
cB211.072.64 (B)	0.0796(1
cC211.060.80 (C)	0.0682(1
cD211.054.96 (D)	0.0569(1

Connected



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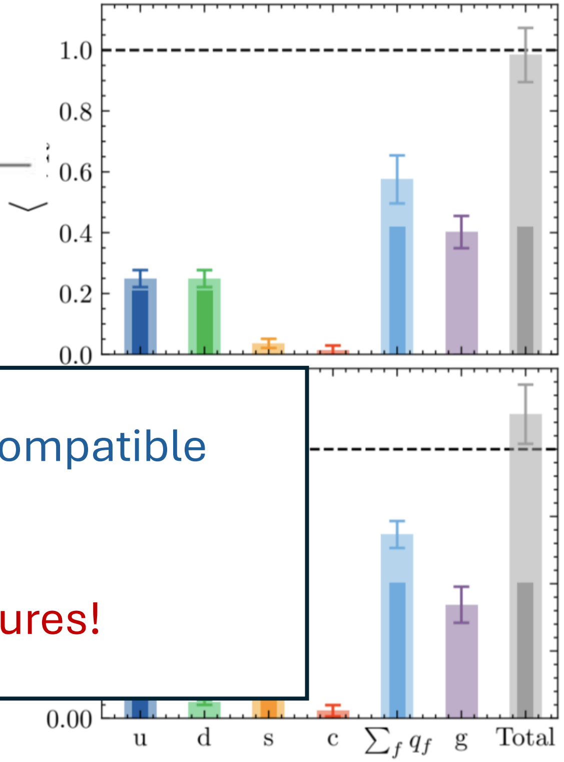


2405.08529

To appear in PRL

The total quark and gluon contributions are compatible among the proton, the pion and the kaon

Despite their very different quark structures!

Consequences to the mass decompostion of hadrons

Mass decomposition by Ji

Ji, PRL74, 1071 (1995)

$$M = M_q + M_g + M_m + M_a$$

$$M_q = \frac{3}{4} M \left(\sum_f \langle x \rangle_{f,R} - \frac{b}{1+\gamma_m} \right)$$

Quark contribution
γ_m is the mass and
dimension

$$M_g = \frac{3}{4} M \langle x \rangle_{g,R}$$

Gluon contribution

$$M_m = \frac{1}{4} M \left(\frac{(4+\gamma_m) b}{1+\gamma_m} \right)$$

Quark mass contribution
b can be obtained from

$$2M^2 b = (1+\gamma_m)\langle (m\overline{\psi}\psi)_R \rangle$$

$$M_a = \frac{1}{4} M (1-b)$$

Trace anomaly contribution

$$2M^2(1-b) = \frac{\beta}{2g}\langle (F^2)^R \rangle$$

From our results, we have

. $M_g^{\pi,K,p} \approx 0.3 M^{\pi,K,p}$!

But there are in fact multiple ways to decompose the mass

It depends on how the anomaly contribution is distributed

Decomposition is not unique

Metez, Pasquini, Rodini, PRD102,114042 (2020)

$$M = M'_q + M'_g + M'_m + M'_a$$

$$M'_q = \frac{3}{4} M \sum_f \langle x \rangle_f^R$$

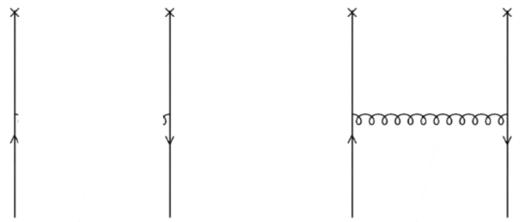
$$M'_g = \frac{3}{4} M \langle x \rangle_g^R$$

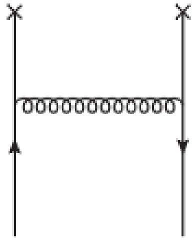
$$M'_m = \frac{1}{4} M \left(\frac{(1+y) b}{1+\gamma_m} + \frac{x(1-b)2g}{\beta} \right)$$

$$M'_a = \frac{1}{4} M \left(1 - \frac{(1+y) b}{1+\gamma_m} - \frac{x(1-b)2g}{\beta} \right)$$

One-loop corrections

$$q(x) = \frac{1}{4\pi}\int d^4z e^{-ip\cdot z}\langle P|\overline{\psi}(z)\,\gamma^+\,\psi(0)|P\rangle$$





$$= -ig^2 C_F \int \frac{dk^+dk^-d^2k_\perp}{(2\pi)^4}\frac{\bar{u}(p)\gamma^\mu\, k\cdot\gamma\,\gamma^+ k\cdot\gamma\,\gamma_\mu u(p)}{(k^2)^2(p-k)^2}\,\delta\left(x - \frac{k^+}{p^+}\right)$$

$$p = (\xi P^+, 0{,}0{,}0);\;\; \xi = \frac{p^+}{P^+}$$

$k^2 + i\epsilon = 2xp^+\Big(k^-$ …

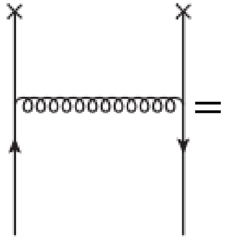
$(p-k)^2 + i\epsilon = -2p^+(1-x)\Big(k^- + \frac{\ldots}{2p^+(1-x)} - i\epsilon\Big)$

DR used for IR and UV divergences

Only the physical region $0 < x < 1$ contributes

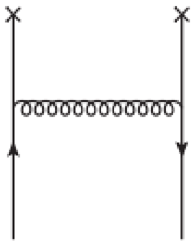
… pole in the … her in the … omplex plane

For $x > 1$ or $x < 0$, the poles are either on the lower half or on the upper half of the complex plane



$$= 2\alpha_s C_F(1-x)\int \frac{d^2k_\perp}{(2\pi)^2}\frac{\overline{u}(p)\gamma^+u(p)}{k_\perp^2} = \frac{\alpha_s}{2\pi}4p^+(1-x)\left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} + \ln\left(\frac{\mu^2}{\mu_F^2}\right)\right)$$

Infinite momentum frame (IMF)

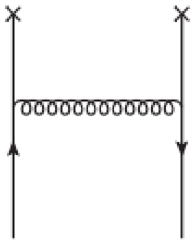


$$= -ig^2 C_F \int \frac{dk^0 dk^3 d^2 k_\perp}{(2\pi)^4} \frac{\bar{u}(p)\gamma^\mu k\cdot\gamma\gamma^3 k\cdot\gamma\gamma_\mu u(p)}{(k^2)^2 (p-k)^2} \delta\left(x - \frac{k^3}{p^3}\right)$$

$$k^2 + i\epsilon = \left(k^0 - \sqrt{k_\perp^2 + x^2(p^3)^2} + i\epsilon\right)\left(k^0 + \sqrt{k_\perp^2 + x^2(p^3)^2} - i\epsilon\right)$$

$$(p-k)^2 + i\epsilon = \left(k^0 - p^3 - \sqrt{k_\perp^2 + (1-x)^2(p^3)^2} + i\epsilon\right)\left(k^0 - p^3 + \sqrt{k_\perp^2 + (1-x)^2(p^3)^2} - i\epsilon\right)$$

Integrating in k^0 and taking the $p^3 \to \infty$ limit:



$$= 2\alpha_s C_F (1-x) \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{\bar{u}(p)\gamma^3 u(p)}{k_\perp^2} \quad = \frac{\alpha_s}{2\pi} 4p_3 (1-x) \left(\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} + \ln\left(\frac{\mu^2}{\mu_F^2}\right)\right)$$

With $0 < x < 1$

LC and IMF have the same IR and UV behaviour and are equivalente

Unfortunately, they can not be computed within LQCD

A reflection on the interplay between large and small momentum

Definition of the quark distribution

$$q(x) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} dz^- e^{-ixP^+z^-}\langle P|\overline{\psi}(z^-)\gamma^+\psi(0)|P\rangle \qquad P = (P^0, 0,0, P^3)$$

Projecting the “good” and “bad” components:

$$\Lambda_\pm = \frac{1}{2}\gamma^\mp\gamma^\pm \Longrightarrow \psi_\pm = \Lambda_\pm\psi \qquad v^+ = (v^+, v^-, \vec{v}_T), v^\pm = \frac{v^0 \pm v^3}{\sqrt{2}}$$

$$q(x) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} dz^- e^{-ixP^+z^-}\langle P|\psi_+^\dagger(z^-)\psi_+(0)|P\rangle$$

quark number density
for the good component

Had we chosen something else for γ^μ

$$\overline{\psi}\gamma^3\psi = \psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-$$

$$\overline{\psi}\gamma^0\psi = \psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-$$

And there would be a contamination from the bad component, spoiling the probability interpretation

But it raises the question: can a distribution off the LC, say in the third direction be used if we know how to subtract these contaminations?

$$q(x) = \frac{1}{4\pi}\int dz e^{-ixzP^3}\langle P|\overline{\psi}(z)\gamma^3\psi(0)|P\rangle$$

Two main messages of today

We can factorize at the position or at the momentum space

As the LC and IMF PDFs are equivalent, we can in principle compute PDFs with finite momentum and try to subtract the spurious contamination

$$(i\gamma^- p^+ + i\gamma^+ p^- - m)\psi = V\psi$$

$$\psi_- = \frac{m - V}{2ip^+}\gamma^+\psi_+$$