# The structure of the Proton

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# Outline

Lecture I: How do we know the nucleon has a structure

$e^-\mu^-$ elastic scattering

$e^-p^+$ elastic and inelastic scattering

Parton model e Bjorken scaling

The proton structure

Lecture II: Factorization

The operator product expansion

The Collins way

The formal definition of PDFs

One loop-corrections

PDFs from first principles?

Lecture IIII: PDFs and the path towards a 3-D picture

PDFs and quasi-PDFs

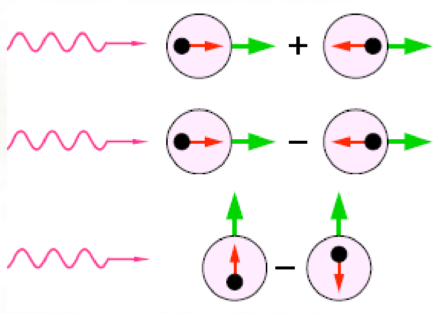
GPDs

TMD-PDFs

Recent results

# Twist-2 Parton Distribution Functions (PDFs)

Complete set of twist-2 parton distribution functions



Unpolarized PDF $\gamma^+$: $f_1(x)$

Helicity PDF $\gamma^+\gamma_5$: $g_1(x)$

Transversity PDF $\sigma^{+j}$: $h_1(x)$

Light-cone coordinates

$$\gamma^+ = \frac{\gamma^0+\gamma^3}{\sqrt{2}}$$

$$x = x_{Bj} = \frac{-q^2}{2P.q}$$

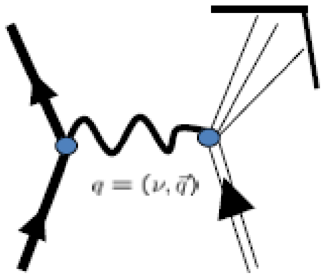
is the momentum fraction carried by a given parton

Example:

$$f_1(x) = \frac{1}{4\pi}\int dz^- e^{-iP^+z^-}\langle P|\bar{\psi}(z^-)\gamma^+\psi(0)|P\rangle$$

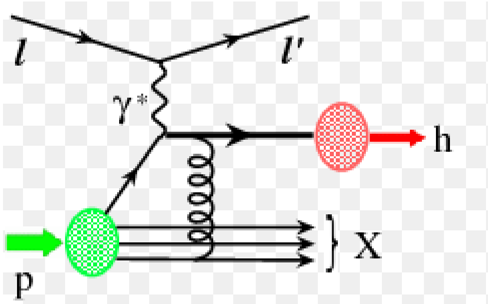
Cross sections are measured:

Totally inclusive




Have access to the chiral-even distributions $f_1(x)$ (unpolarized) and $g_1(x)$ (helicity)

Semi-inclusive




Have access to the chiral-odd distribution $h_1(x)$ (transversity). Naturally more difficult to obtain data on transversity

# Twist-3 PDFs

Twist expansion: $f_i(x) = f_i^{(0)} + \frac{f_i^{(1)}}{Q} + \frac{f_i^{(2)}}{Q^2} + \cdots$

Twist-2 + Twist-3 +Twist-4

Twist-3:

$\hat{1}$ : $e(x)$

$\gamma^j \gamma_5$ : $g_T(x)$

$\sigma^{jk}$ : $h_L(x)$

No density interpretation;

Contain information of quark-gluon-quark correlations;

Hard do determine experimentally.

Examples: $g_T(x) \equiv g_1(x) + g_2(x)$

on the

Pion-nucleon sigma term

$$\sigma^{\pi N} = m_q \langle N|\bar{u}u + \bar{d}d|N\rangle$$

But

$$e(x) = \frac{1}{4\pi}\int dz^- e^{-ixP^+z^-}\langle N|\bar{\psi}(z^-)\psi(0)|N\rangle$$

and

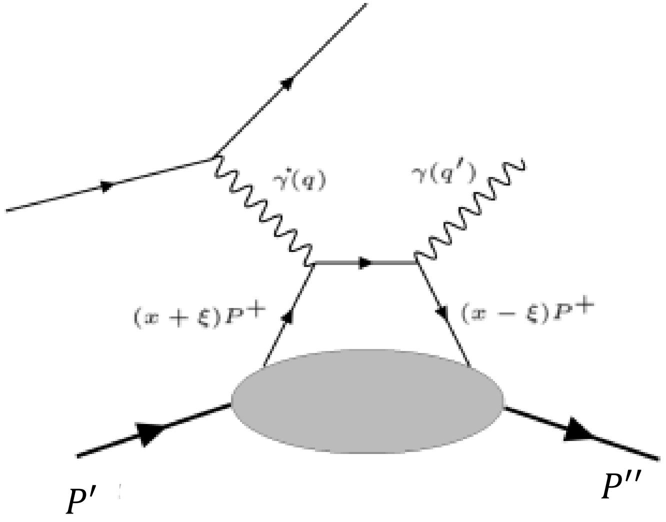
$$\int_0^1 dx(e^u(x) + e^d(x)) = \frac{\sigma^{\pi N}}{m}$$

$g_T(x)$ has information on the transverse spin and its twist-3 part can be related to orbital angular momentum

$$\int_0^1 dx g_2(x) \propto L_q$$

Burkardt-Polyakov-Diehl relation

# Generalised PDFs (GPDs)




A virtual photon is exchanged, with a real photon measured in the final state

Momentum transfer: $\Delta \equiv P'' - P', \qquad t \equiv \Delta^2,$

Fraction of the momentum transfer: $\xi \equiv -\frac{P''^+ - P'^+}{P''^+ + P'^+} = -\frac{2\Delta^+}{P^+}$ , $\xi$ is called skewness, which is the fraction of longitudinal momentum transfer relative to the sum of the longitudinal momentum of the initial and final states

GPDs are multidimensional objects, depending on $x, t, \xi$

Example: Matrix elements of $\gamma^+$give the unpolarized GPDs

$$\mathcal{F}_1^q(x,t,\xi) = \frac{1}{4\pi}\int dz^- e^{-ixP^+z^-}\left\langle P''\middle|\bar{\psi}(z^-)\gamma^+ W\,(z^-,0)\psi(0)\middle|P'\right\rangle$$

Wilson line connecting the two quark fields. It ensures gauge invariance of the distributions

$$\equiv \frac{\bar{u}(P'')}{2p^+}\left(H^q(x,t,\xi)\gamma^+ + E^q(x,t,\xi)\frac{\sigma^{+\mu}\Delta_\mu}{2M}\right)u(P')$$

Generalised PDFs

$\gamma^+\gamma_5$ for the helicity case, there are 2 GPDs

$$\tilde{H}^q(x,t,\xi) \quad \tilde{E}^q\,(x,t,\xi)$$

$\sigma^{+j}$ for the transversty case, there are 4 GPDs

$$H_T^q(x,t,\xi) \;\; E_T^q(x,t,\xi) \;\; \tilde{H}_T^q\,(x,t,\xi) \;\; \tilde{E}_T^q(x,t,\xi)$$

Some of the physics involved

$$q(x) = H^q(0 < x < 1,0,0)$$

$$\overline{q}(x) = H^q(-1 < x < 0,0,0)$$

PDFs (deep inelastic and semi-inclusive scattering)

$$F_1^q(t) = \int_{-1}^{+1} dx\, H^q(x,t,\xi)$$

$$F_2^q(t) = \int_{-1}^{+1} dx \ldots$$

Elastic form factors (elastic scattering)

GPDs unify momentum, spin,

and spatial structure of hadrons

$$\int_{-1}^{+1} dx x H^q\,(x,\xi, \ldots$$

$$\int_{-1}^{+1} dx x E^q\,(x,\xi,t) = B^q(t) - 4\xi^2 D^q(t)$$

Gravitational form factors

$$\langle p''|T^{\mu\nu}|p'\rangle = \bar{u}(p'')\left[A(t)\gamma^{\{\mu}P^{\nu\}} + B(t)\frac{i\sigma^{\{\mu\alpha}q_\alpha P^{\nu\}}}{2M} + D(t)\frac{q^\mu q^\nu - g^{\mu\nu}q^2}{4M}\right]u(p')$$

$$2J^{q,g} = A^{q,g}(o) + B^{q,g}\,(0)$$

$= \langle x\rangle^{q,g}$

Ji sum rule for total angular momentum

- Historically the internal structure of th nucleon comes from FF and PDFs;

- GPDs encode in themselves both PDFs and FFs, but have even more information burried in them, such as angular momentum and the gravitational form factors;

- GPDs depend not only on $x$ but also on the momentum transfer;

- The impact parameter $(\vec{b}_T)$ is the conjugated variable to the transverse part of the momentum transfer $(\vec{\Delta}_\mathrm{T})$ ;

- Thus GPDs do not have information on the transverse momentum of the partons;

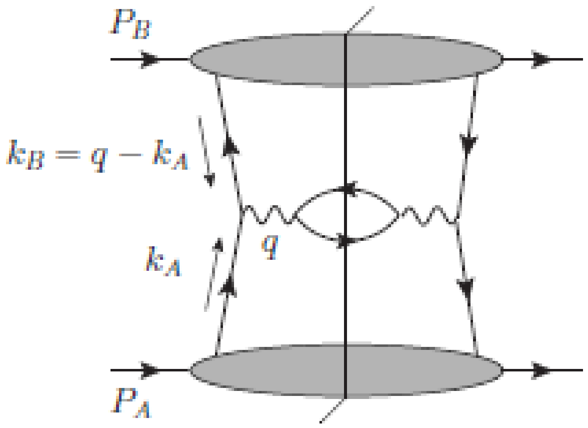
To access the transverse momentum we need the transferse momentum dependent PDFs:TMDPDFs

# Transverse momentum dependent PDFs (TMDPDFs)

Why TMDPDFs?

If we measure only the invariant mass of the final lepton pair:

$$\frac{d\sigma}{dQ^2} = \sum_{i,j} \int dx_1 dx_2 f_i(x_1) f_j(x_2) H(x_1, x_2) \left(1 + \mathcal{O}\left(\frac{m^2}{Q^2}\right)\right)$$




If we measure the transverse momentum $\vec{q}_T$ of the lepton pair, we have access to the transverse momentum of the quarks!

$$\frac{d^2\sigma}{dQ^2 dq_T^2} = \sum_{i,j} \int dx_1 dx_2 \int d^2 b_T e^{i\,\vec{r}_T \cdot \vec{q}_T} f_i(x_1, \vec{r}_T\ ) f_j(x_2, \vec{r}_T) H(x_1, x_2) \left(1 + \mathcal{O}\left(\frac{\Lambda_{QCD}}{Q^2}, \frac{q_T^2}{Q^2}\right)\right), \qquad q_T \ll Q$$

Transverse momentum dependent PDFs

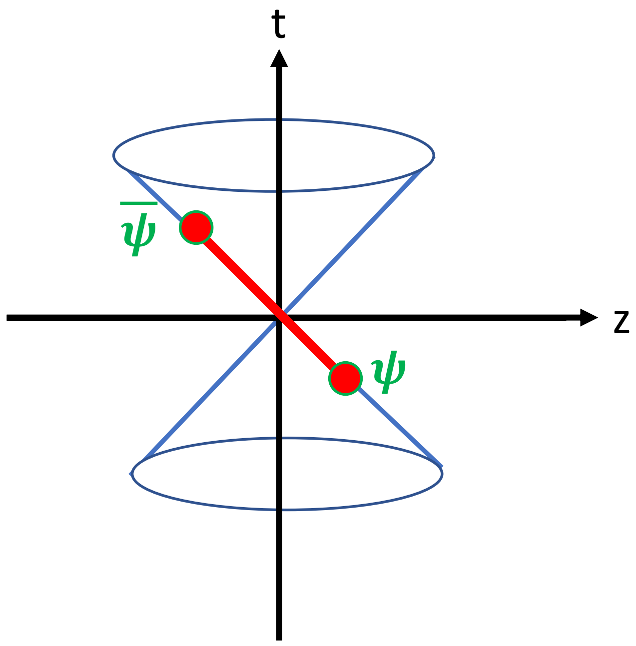
# Light-cone PDFs and quasi PDFs

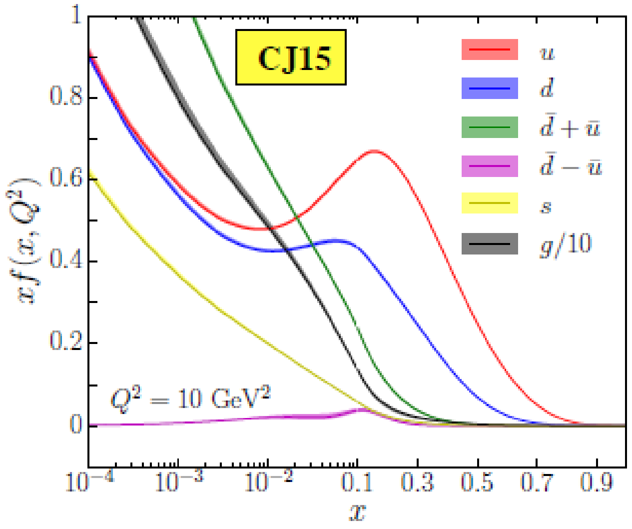
$$q(x) = \frac{1}{4\pi}\int dz^- e^{-iP^+z^-}\langle P|\bar{\psi}(z^-)\gamma^+ W(z^-,0)\psi(0)|P\rangle$$

Dirac Structure　　Wilson line

Quark distribution is given by a light-front correlation

$$z^- = \frac{t-z}{\sqrt{2}}, P^+ = \frac{E+P^z}{\sqrt{2}}$$

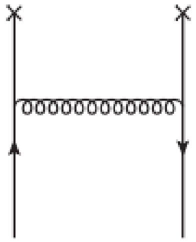


Our focus: $q(x) \equiv u(x) - d(x)$

Why? Because isovector distributions avoid complications from mixing with Gluons

We have seen:



$$= 2\alpha_s C_F (1-x) \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{\overline{u}(p)\gamma^+ u(p)}{k_\perp^2} = \frac{\alpha_s}{2\pi} 4p^+ (1-x) \left( \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} + \ln\left(\frac{\mu^2}{\mu_F^2}\right) \right)$$

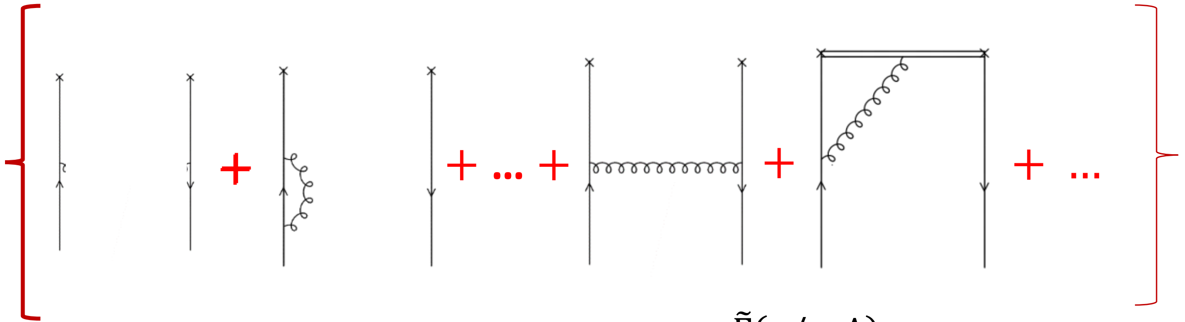
Support in the physical region only, $\mathbf{0 < x < 1}$

With an equivalence between LC and IMF

$$\Longrightarrow \quad q(x, \Lambda) = q_{bare}(x) \left\{ 1 + \frac{\alpha_s}{2\pi} \Pi(\Lambda) \right\} + \frac{\alpha_s}{2\pi} \int_x^1 \Gamma\left(\frac{x}{y}, \Lambda\right) q_{bare}(y) \frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

# What if $p_3$ is kept finite?

$\tilde{q}(x,\Lambda)=$

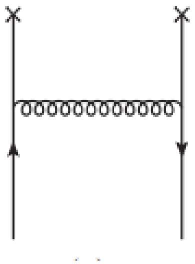



$\otimes q_{bare}(y) + \mathcal{O}(\alpha_s^2)$

Regulator of IR and UV divergences

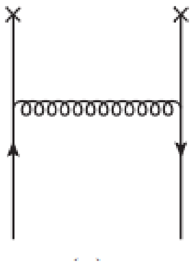
$$\tilde{q}(x,\Lambda) = q_{bare}(x)\left\{1+\frac{\alpha_s}{2\pi}\tilde{\Pi}(\Lambda)\right\} + \frac{\alpha_s}{2\pi}\int_x^1 \tilde{\Gamma}\left(\frac{x}{y},\Lambda\right) q_{bare}(y)\frac{dy}{y} + \mathcal{O}(\alpha_s^2)$$

## Keeping $p_3$ finite



$$= -ig^2 C_F \int \frac{dk^0 dk^3 d^2 k_\perp}{(2\pi)^4} \frac{\bar{u}(p)\gamma^\mu k\cdot\gamma\, \gamma^3\, k\cdot\gamma\gamma_\mu u(p)}{(k^2)^2 (p-k)^2}\, \delta\left(x - \frac{k^3}{p^3}\right)$$

Integrating in $k^0$ and keeping $p_3$ finite, we have an integral over $k_T$ which is UV finite! But has an IR divergence. Using Dimensional Regularization:



$$= \frac{\alpha_s}{2\pi} 4p_3 \left( (1-x)\left( -\frac{1}{\epsilon_{IR}} + ln\left(\frac{p_3^2}{\mu_F^2}\right) + \ln(4x(1-x)) \right) + 1 \right), \qquad 0 < x < 1$$

$$+ \frac{\alpha_s}{2\pi} 4p_3 \left( (1-x) \ln\left(\frac{x}{x-1}\right) + 1 \right), \qquad x > 1$$

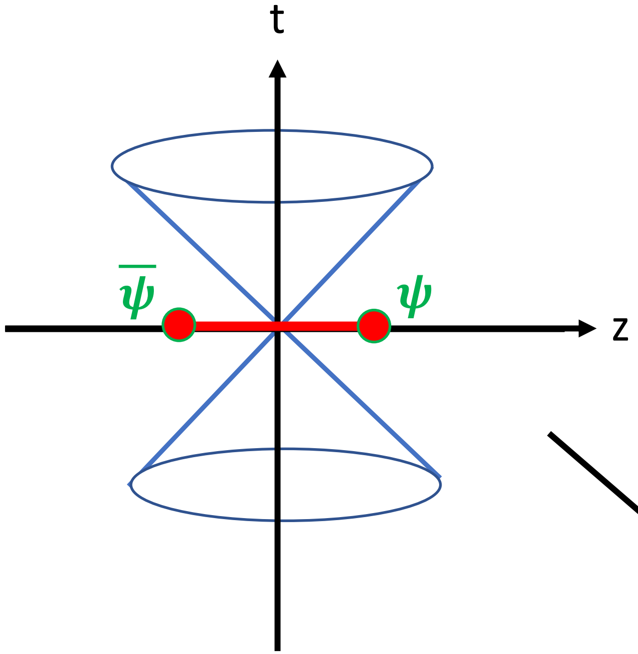
$$+ \frac{\alpha_s}{2\pi} 4p_3 \left( (1-x) \ln\left(\frac{x-1}{x}\right) - 1 \right), \qquad x < 0$$

Support outside the physical region!

Same IR pole as in the LC a

No UV divergence appears explicitly

UV divergence appears only when integrating over all parton momentum fraction $x$

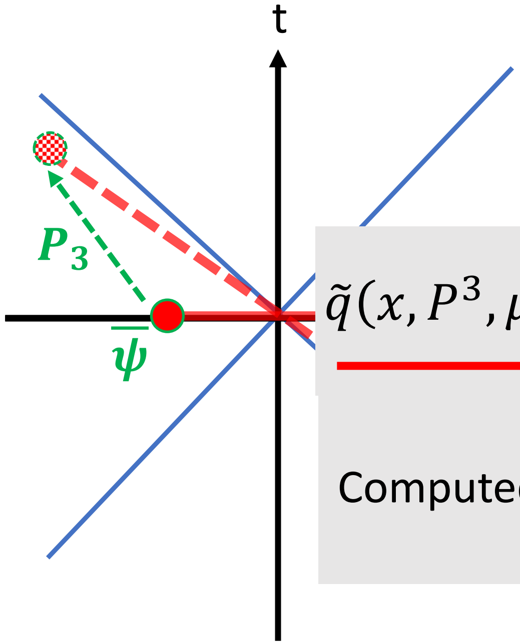



$$\frac{1}{2}\int \frac{dz^3}{2\pi}\, e^{ik^3z^3}\left\langle P\left|\bar\psi\left(-\frac{z^3}{2}\right)\Gamma\mathcal{W}\left(-\frac{z^3}{2},\frac{z^3}{2}\right)\psi\left(\frac{z^3}{2}\right)\right|P\right\rangle$$

Purely spatial correlation

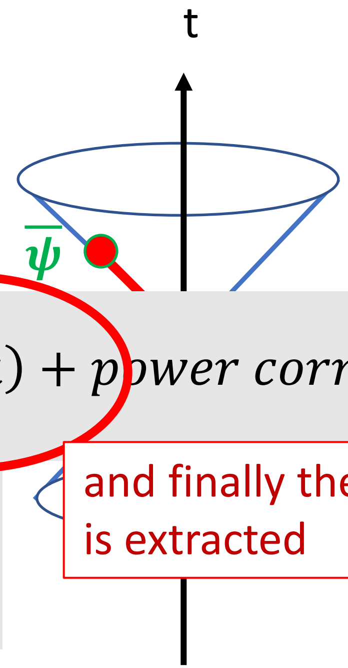
X. Ji, PRL 110 (2013) 262002.

We want to go from a purely spatial correlation to a light-front correlation




$$\tilde q(x,P^3,\mu)=\int\frac{dy}{|y|}\,C\left(\frac{x}{y},\frac{\mu}{yP^3}\right)q(y,\mu)+power\ corrections$$

Computed in LQCD    Computed in pQCD




and finally the LC PDF is extracted

# Results for Twist-2

ETMC

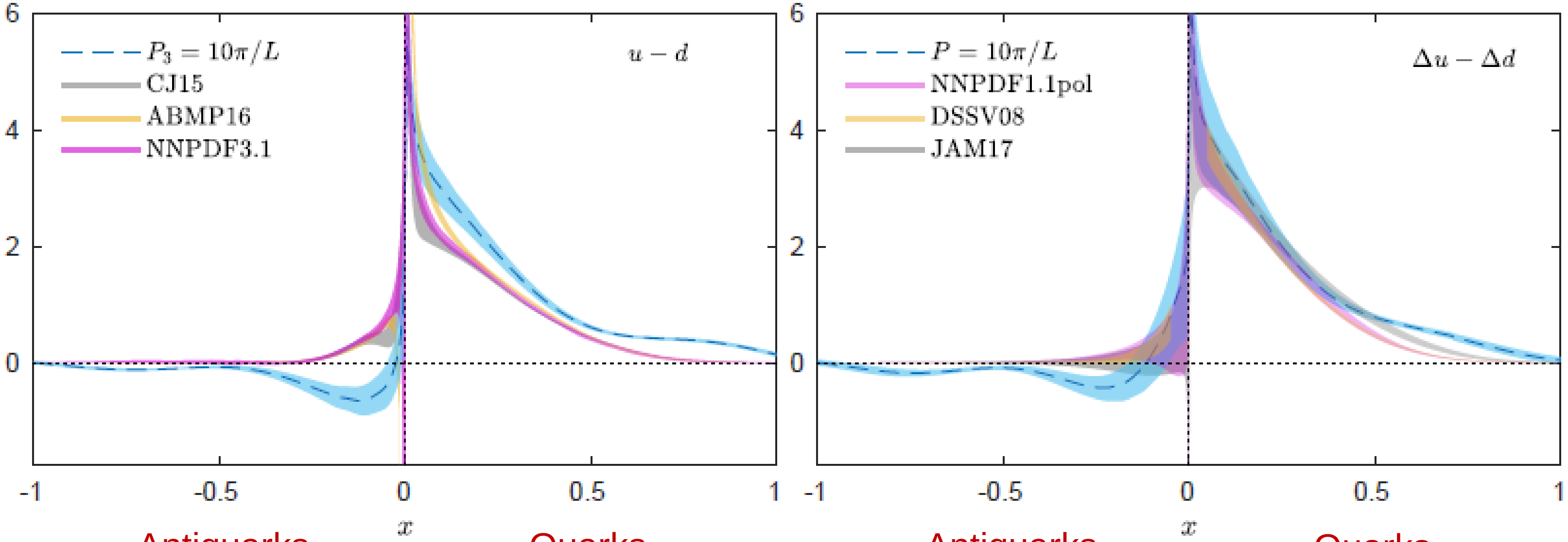
ETMC, NPB 923, 394 (2017) - Nonperturbative renormalization

ETMC, PRL 121, 112001 (2018) - Unpolarized and helicity quasi

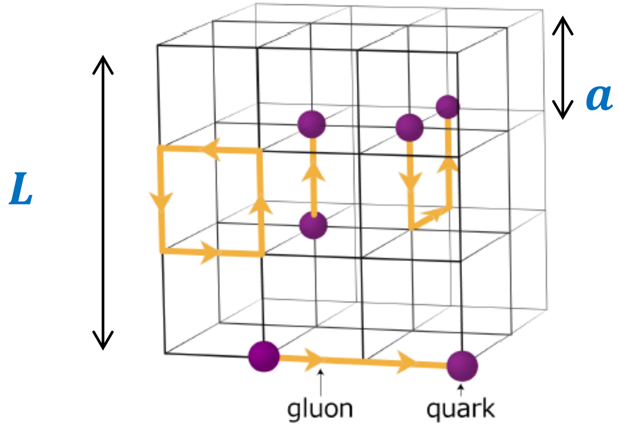
ETMC, PRD 103, 034510 (2021) - Unpolarized and helicity pseudo

# Quasi-PDF approach

Unpolarized $P_3 = 1.38\ GeV$ **Helicity**




Antiquarks Quarks Antiquarks Quarks




Source: JICFuS, Tsukuba

C. Alexandrou, K. Cichy, M. Constantinou, K. Hadjiyiannakou, K. Jansen, A. Scapellato and F. Steffens, PRL 121, 112001 (2018)

$m_\pi \cong 130$ MeV

$48^3 \times 96$ lattice

$a \cong 0.093$ fm

# Results for Twist-3

ETMC + Temple

PRD 102 (2020) 11, 111501 - Lattice $g_T(x)$

PRD 102 (2020) 3, 034005 - Matching $g_T(x)$

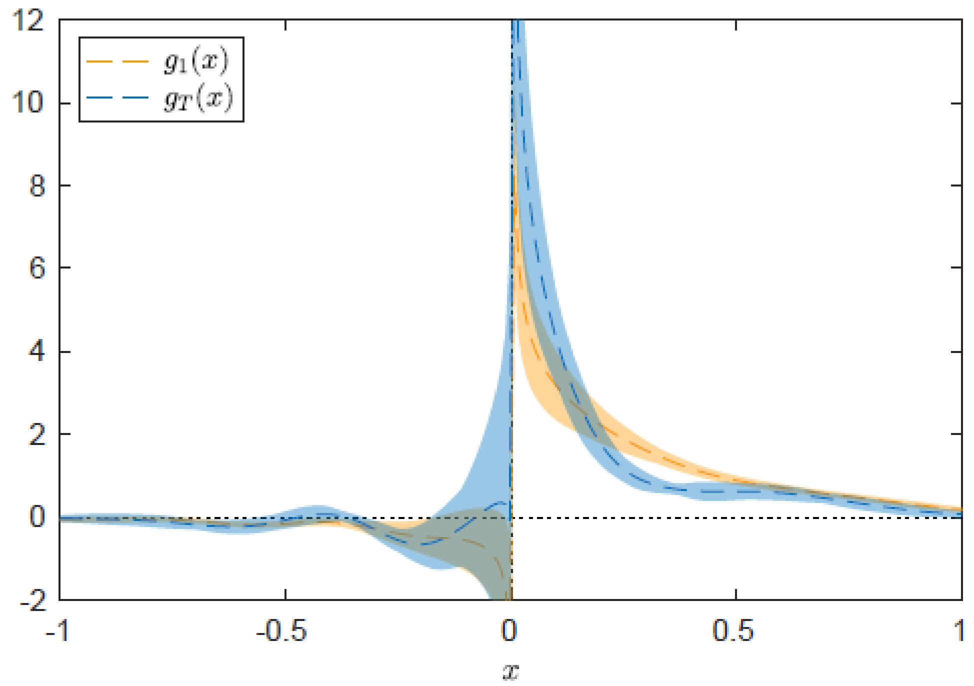
PRD 102 (2020) 2, 24025 - Matching $e(x)$ and $h_L(x)$

PRD 104 (2021) 11, 114510 - Lattice $h_L(x)$

| Name | $\beta$ | $N_f$ | $L^3 \times L_T$ | $a$ [fm] | $M_\pi$ | $m_\pi L$ |
|---|---|---|---|---|---|---|
| cA211.32 | 1.726 | $u, d, s, c$ | $32^3 \times 64$ | 0.093 | 260 MeV | 4 |

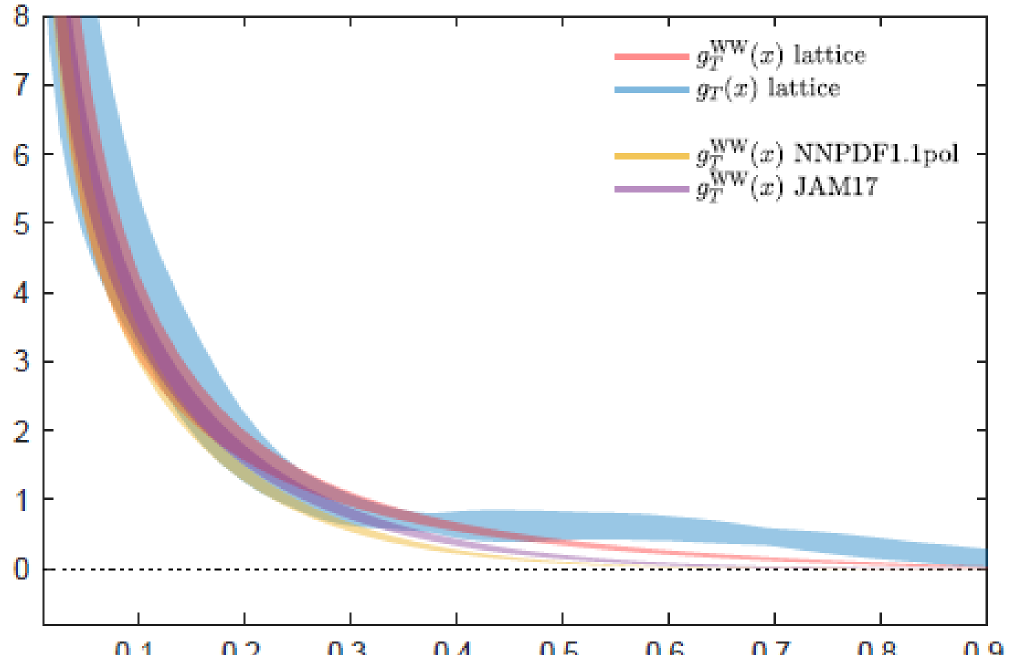
The $g_T(x) = g_1(x) + g_2(x)$ distribution

$P_3 = 1.67\ GeV$




The BC sum rule is verified:

$$\int_{-1}^{+1} dx\ g_T(x) - \int_{-1}^{+1} dx\ g_1(x) = 0.01(20)$$




The WW approximation

$$g_T^{ww}(x) = \int_x^{+1} dy\ g_1(y)$$

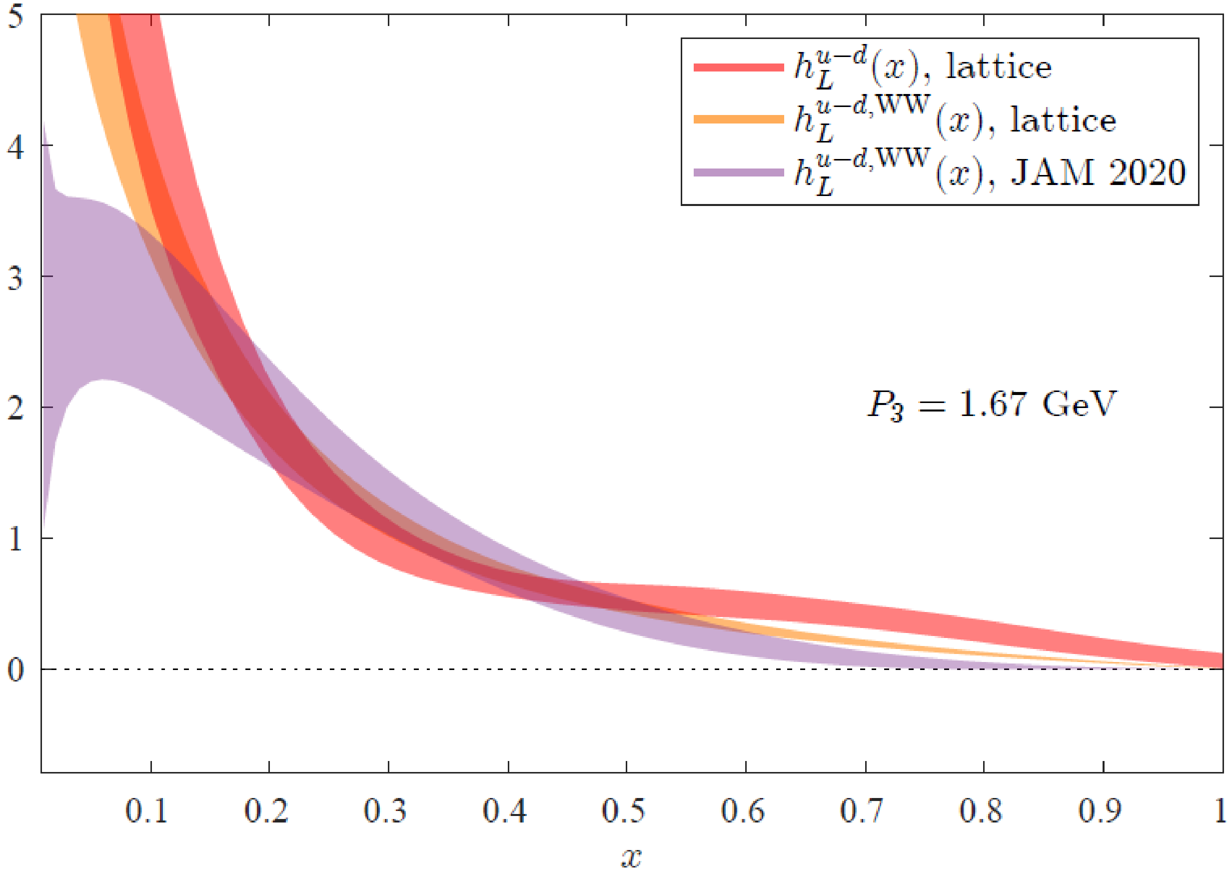
Up to $x < 0.5$, $g_T(x)$ agrees with $g_T^{ww}(x)$

Violations of 30-40% possible

# The chiral-odd twist-3 distribution $h_L(x)$

The WW approximation relates $h_L(x)$ to its twist-2 counterpart $h_1(x)$

$$h_L^{ww}(x) = 2x \int_x^{+1} \frac{dy}{y^2} h_1(x)$$




Suggests that the twist-3 distribution can be determined From its twis-2 counterpart

# Twist-2 GPDs

ETMC

PRL 125, 262001 (2020) - Unpolarized ane helicity

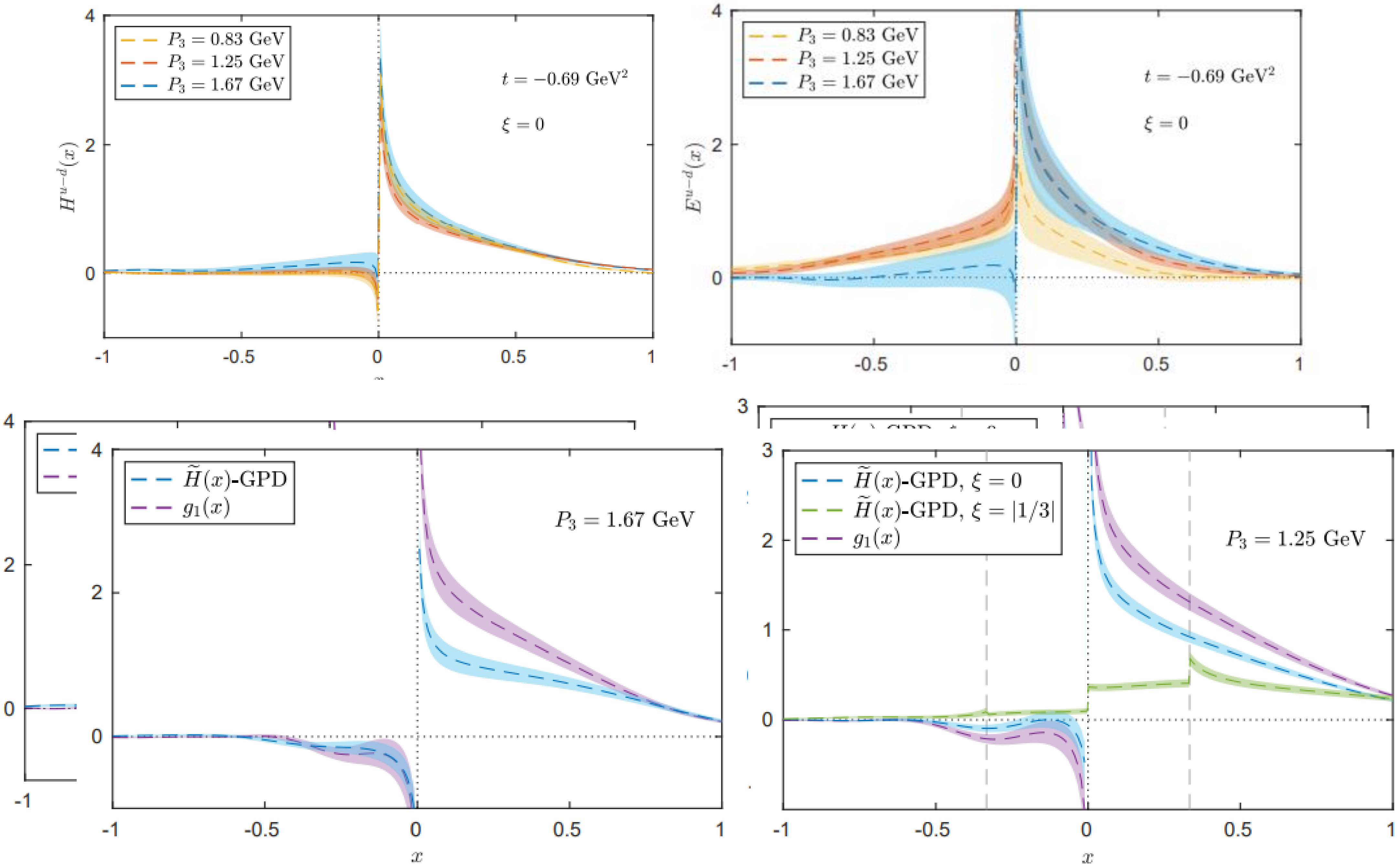
PRD 105, 034501 (2022) - Transversity

PRD 108, 014507 (2023) – Moments of GPDs

PRD 108, 054501 (2023) – Axial twist-3 GPDs

JHEP 01 (2025) 146 – Moments of GPDs

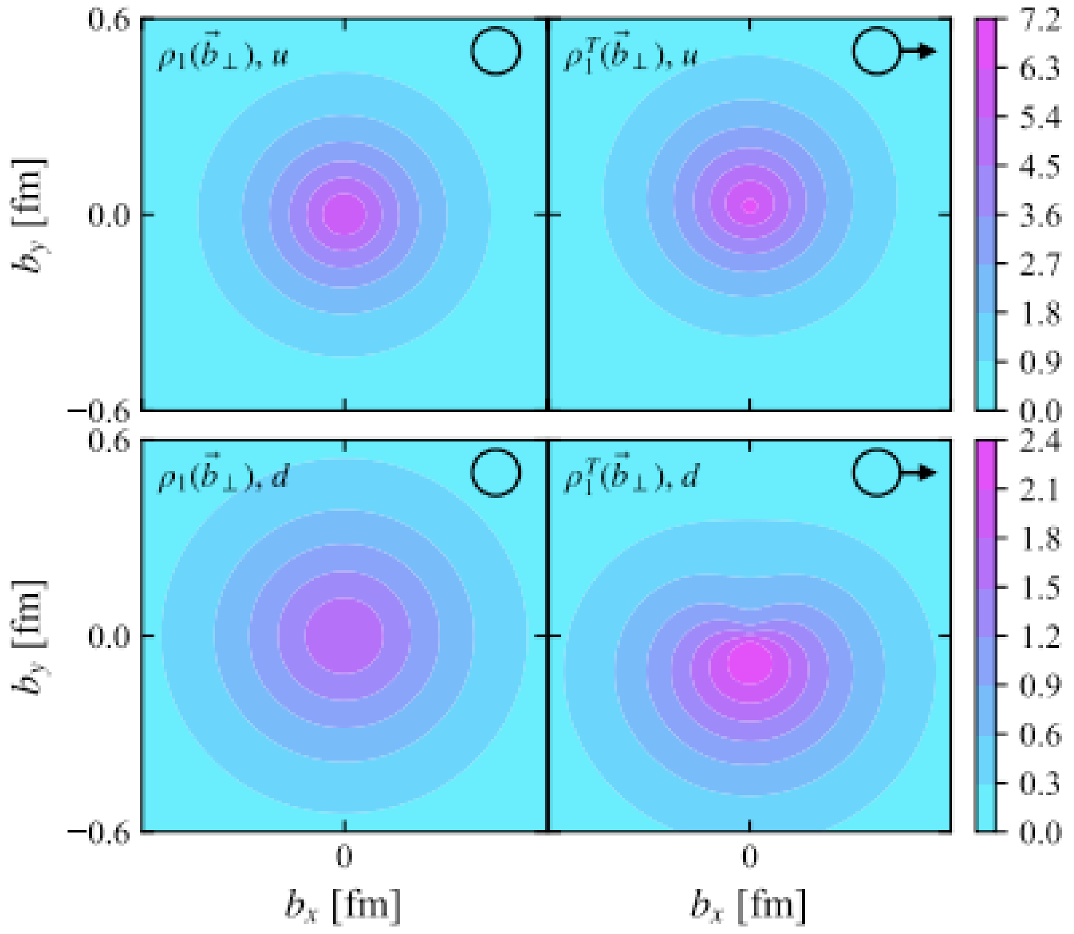
| Name | $\beta$ | $N_f$ | $L^3 \times L_T$ | $a$ [fm] | $M_\pi$ | $m_\pi L$ |
|---|---|---|---|---|---|---|
| cA211.32 | 1.726 | $u, d, s, c$ | $32^3 \times 64$ | 0.093 | 260 MeV | 4 |



$P_3 = 0.83$ GeV
$P_3 = 1.25$ GeV
$P_3 = 1.67$ GeV
$t = -0.69$ GeV$^2$
$\xi = 0$
$H^{u-d}(x)$
$P_3 = 0.83$ GeV
$P_3 = 1.25$ GeV
$P_3 = 1.67$ GeV
$t = -0.69$ GeV$^2$
$\xi = 0$
$E^{u-d}(x)$
$\widetilde{H}(x)$-GPD
$g_1(x)$
$P_3 = 1.67$ GeV
$x$
$\widetilde{H}(x)$-GPD, $\xi = 0$
$\widetilde{H}(x)$-GPD, $\xi = |1/3|$
$g_1(x)$
$P_3 = 1.25$ GeV
$x$

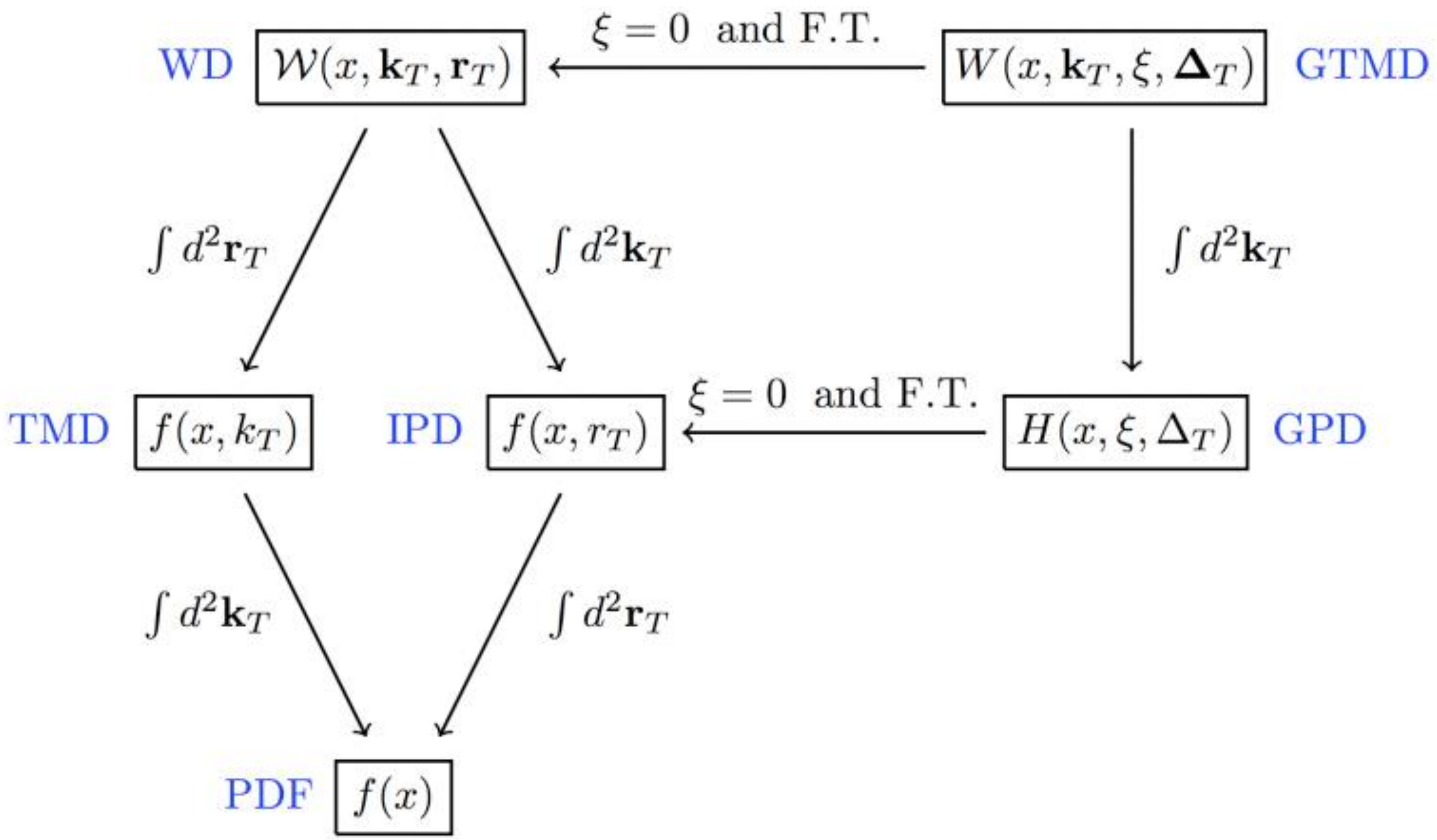
The first moment of the impact parameter PDF
in the transverse plane

Unpolarized
Proton




Transversely polarized
Proton in the x direction

# Summary



WD
$\mathcal{W}(x, \mathbf{k}_T, \mathbf{r}_T)$
$\xi = 0$ and F.T.
$W(x, \mathbf{k}_T, \xi, \mathbf{\Delta}_T)$
GTMD
$\int d^2\mathbf{r}_T$
$\int d^2\mathbf{k}_T$
$\int d^2\mathbf{k}_T$
TMD
$f(x, k_T)$
IPD
$f(x, r_T)$
$\xi = 0$ and F.T.
$H(x, \xi, \Delta_T)$
GPD
$\int d^2\mathbf{k}_T$
$\int d^2\mathbf{r}_T$
PDF
$f(x)$

# Last 5 years witnessed enormous progress on first principles computations of both PDFs and GPDs

**Theoretical papers**

X. Ji, PRL 110, 262002 (2013) - Quasi
A.V. Radyushkin, PRD 96, 034025 (2017) - Pseudo
A. J. Chambers et al., PRL 118, 242001 (2017) - OPE without OPE
Yan-Qing Ma and Jian-Wei Qiu, PRL 120, 022003 (2018) - Good lattice cross sections

**Exploratory studies**

LP3, PRD 91, 054510 (2015)
ETMC, PRD 92, 014502 (2015)

**Nucleon PDFs at physical pion mass using Quasi**

ETMC, NPB 923, 394 (2017) - Nonperturbative renormalization
ETMC, PRL 121, 112001 (2018) - Unpolarized and helicity
LP3, PRL 121, 242003 (2018) - Helicity

**Nucleon PDFs at physical pion mass using Pseudo**

ETMC, PRD 103, 034510 (2021)
HadStruc., PRL 125, 232003 (2020) - Extrapolated to physical pion mass

**Nucleon GPDs**

ETMC, PRL 125, 262001 (2020) - Unpolarized and helicity
ETMC+Temple+BNL+ANL, PRD 106 125, 115412 (2022) Symmetric and asymmetric frames

**Twist-3**

ETMC/Temple, PRD 102, 111501 (2020)
ETMC/Temple, PRD 104 115410 (2021)

List restricted to physical pion mass results or exploratory studies. There are many more works on the subject and I apologize to authors of works not listed

# TMDPDs just starting

**Theoretical papers**

M. A. Ebert, I. W. Stewart, Y. Zhao, PRD 99, 034505 (2019)
M. A. Ebert et al., JHEP 37, 2019 (2019)
X. Ji, Y. Liu, Yu-Sheng Liu, Phys. Lett. B 811, 135956 (2020)
X. Ji, Y. Liu, Yu-Sheng Liu, Nucl. Phys. B 955, 115054 (2020)
P. Shanahan, M. Wagman, Y. Zhao, PRD 102, 014511 (2020)
M. A. Ebert et al., arXiv:2201.08401

**Exploratory studies – Soft function**

LPC, PRL 125, 192001 (2020)
ETMC, PRL 128, 062002 (2022)

**Exploratory studies – Collins-Sopper kernel**

ETMC, PRL 128, 062002 (2022)
LPC, arXiv: 2204.00200

**Exploratory studies – Beam functions and TMDPDFs**

ETMC+PKU, *PoS* Lattice2022 (2023) 123
ETMC+PKU, *PoS* Lattice2022 (2023) 733
LPC, arXiv: 2211.02340

## Many more works already done,

- Pion and Kaon PDFs
- Meson DA
- Delta PDF
- Gluon PDF
- Transversity GPDs
- Synergy between lattice and phenomenology

## Many improvements can be made:

- Higher boost
- Discretization effects
- Finite volume effects
- Higher twist contamination
- Truncation effects in the matching
- The problem of x reconstruction

## Road towards precision is open!