**Emergence of dark symmetry as well as neutrino mass scales from** *A*<sup>4</sup> **flavor symmetry (arXiv:2406.00188)**

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# **Introduction: Massive Neutrino !!**

- ✐ Neutrinos are massless in the Standard Model.
- ➨ Neutrino Oscillation =⇒ Neutrinos are massive.
- **►** Neutrinos tiny mass and mixing are long standing puzzle  $\implies$  "Leptonic flavor puzzle".



Non-abelian groups "Flavor symmetries" provide a deeper understanding of this puzzle.  $(S_3, A_4, S_4, A_5, \Delta(27),$  *etc.*) [Ma and Rajasekaran,](#page-30-0) hep-ph/0106291 [Babu, Ma, and Valle,](#page-30-1) hep-ph/0206292 [Altarelli and Feruglio,](#page-30-2) hep-ph/0512103

# **The Dark Matter ??**

- ✐ Discrepancy between observed galaxy rotation curves and the theoretical prediction.
- **The Planck satellite data tells that** there is  $\sim$  26.8 % dark matter (DM). [Aghanim et al.,](#page-30-3) 10.1051/0004-6361/201833910



**Estimated matter-energy content of the Universe** 

The absence of a cosmological DM particle in the SM raises another significant concern.

 $\cdots$ 



# **The Scotogenic Model**

- $\rightarrow$  There are various theoretical models to generate the neutrino mass and explain the DM stability.
- **►** The Scotogenic model proposed by Ma is minimal extension of SM which provides a possible connection between neutrino mass generation and DM stability. [Ma,](#page-30-4) hep-ph/0601225
- $\blacktriangleright$  Two BSM particles scalar  $\eta$ and fermion *N* are introduced.
- $\rightarrow$  The  $Z_2$  symmetry on top of SM symmetry  $\implies$  DM stability.



# **Neutrino Oscillations: Two mass scales**

- ✐ From the neutrino oscillations we infer neutrinos are massive but no information about mass ordering: Normal or Inverted.
- **■** Till now we only know that there are two mass scales: atmospheric and solar.



The typical neutrino mass models or scotogenic models fail to provide an understanding of these two different mass scales.

**Recently, scoto-seesaw mass mechanism has been proposed which explains the two different mass scales observed in neutrino oscillation data while preserving the outcomes of the scotogenic model.** [Rojas, Srivastava, and Valle,](#page-30-5) 10.1016/j.physletb.2018.12.014

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# **A Unified Model ??**

- ✘ **The scoto-seesaw models do not address the flavor structure of neutrino oscillation: The pattern of mixing angles !!**
- ➨ **Hence, even the most simple and elegant BSM models require the addition of at least a new dark as well as a flavor symmetry, along with the expansion of the BSM particle content, to account for DM and to explain neutrino mass and flavor structure !!**

**We develop a simple framework with minimal particle content, that can explain the DM stability, neutrino mass generation, and flavor structure of the lepton sector along with the two mass scales of neutrino oscillation experiments using only a single flavor symmetry.**

# **Our Proposal**

✌ **We have employed a** *A*<sup>4</sup> **flavor symmetry on top of SM symmetry.**



- $\rightarrow$  The particle content of our model is similar to the scotogenic.
- $\rightarrow$  The BSM particles, scalar  $\eta$  and fermion N are triplet under  $A_4$ while all the SM particles are singlets of *A*4.
- $\rightarrow$  We have utilized the fact that  $Z_2$  is a subgroup of  $A_4$  which can serve as a dark symmetry.

# **Yukawa Lagrangian and Scalar Potential**

$$
-\mathcal{L}_y = y_{11}(\overline{L_1})_1 \phi(e_{1R})_1 + y_{22}(\overline{L_2})_{1''} \phi(e_{2R})_{1'} + y_{33}(\overline{L_3})_{1'} \phi(e_{2R})_{1''}
$$
  
+  $y_1(\overline{L_1})_1(\eta N)_1 + y_2(\overline{L_2})_{1''}(\eta N)_{1'} + y_3(\overline{L_3})_{1'}(\eta N)_{1''}$   
+  $M(\overline{N^c}N)_1 + h.c.$ ,

$$
V = \mu_{\eta}^{2} \eta^{\dagger} \eta + \mu_{\phi}^{2} \phi^{\dagger} \phi + \lambda_{1} [\phi^{\dagger} \phi]_{1}^{2} + \lambda_{2} [\eta^{\dagger} \eta]_{1}^{2} + \lambda_{3} [\eta^{\dagger} \eta]_{1} \nu [\eta^{\dagger} \eta]_{1} \nu + \lambda_{4} [\eta^{\dagger} \eta^{\dagger}]_{1} \nu [\eta \eta]_{1} \nu
$$
  
+  $\lambda_{4} \nu [\eta^{\dagger} \eta^{\dagger}]_{1} \nu [\eta \eta]_{1} \nu + \lambda_{5} [\eta^{\dagger} \eta^{\dagger}]_{1} [\eta \eta]_{1} + \lambda_{6} ([\eta^{\dagger} \eta]_{31} [\eta^{\dagger} \eta]_{31} + h.c.) + \lambda_{7} [\eta^{\dagger} \eta]_{31} [\eta^{\dagger} \eta]_{32}$   
+  $\lambda_{8} [\eta^{\dagger} \eta^{\dagger}]_{31} [\eta \eta]_{32} + \lambda_{9} [\eta^{\dagger} \eta]_{1} [\phi^{\dagger} \phi] + \lambda_{10} [\eta^{\dagger} \phi]_{31} [\phi^{\dagger} \eta]_{31} + \lambda_{11} ([\eta^{\dagger} \eta^{\dagger}]_{1} \phi \phi + h.c.)$   
+  $\lambda_{12} ([\eta^{\dagger} \eta^{\dagger}]_{31} [\eta \phi]_{31} + h.c.) + \lambda_{13} ([\eta^{\dagger} \eta^{\dagger}]_{32} [\eta \phi]]_{31} + h.c.) + \lambda_{14} ([\eta^{\dagger} \eta]_{31} \eta^{\dagger} \phi + h.c.)$   
+  $\lambda_{15} ([\eta^{\dagger} \eta]_{32} \eta^{\dagger} \phi + h.c.)$ .

## Scalar Mass Spectrum

$$
\begin{split} m_{H_1,H_2}^2&=\lambda_1v_1^2+\Lambda v_2^2\mp\sqrt{(\lambda_1v_1^2+\Lambda v_2^2)^2+v_1^2v_2^2(\alpha^2-4\Lambda\lambda_1)}\,,\\ m_{A}^2&=-2\lambda_{11}\left(v_1^2+v_2^2\right),\quad m_{G}^2=0\,,\\ m_{H^\pm}^2&=-(\lambda_{10}+2\lambda_{11})\left(v_1^2+v_2^2\right)/2,\quad m_{G^\pm}^2=0\,,\\ m_{\eta_2^R}^2&=\left(\kappa_1v_2^2-3\zeta v_1v_2\right)/2,\quad m_{\eta_2^I}^2=\left(\kappa_2v_2^2-4\lambda_{11}v_1^2-\zeta v_1v_2\right)/2\,,\\ m_{\eta_3^R}^2&=m_{\eta_2^R}^2+3\zeta v_1v_2,\quad m_{\eta_3^I}^2=m_{\eta_2^I}^2+\zeta v_1v_2\,,\\ m_{\eta_2^\pm}^2&=\left(\kappa_3v_2^2-(\lambda_{10}+2\lambda_{11})v_1^2-\zeta v_1v_2\right)/2,\quad m_{\eta_3^\pm}^2=m_{\eta_2^\pm}^2+\zeta v_1v_2\,. \end{split}
$$

 $\lambda$ 's have perturbativity limit,  $\lambda \leq \sqrt{4\pi}$ . It implies scalar masses in this model,  $m_S \sim 600$  GeV so as DM mass.

$$
\begin{aligned} \Lambda&=\lambda_2+\lambda_3+2\lambda_4+\lambda_5\,,\\ \kappa_1&=\left(-3\lambda_3-6\lambda_4+2\lambda_6+\lambda_7+\lambda_8\right),\\ \kappa_2&=\left(-3\lambda_3-2\lambda_4-4\lambda_5-2\lambda_6+\lambda_7+\lambda_8\right),\\ \kappa_3&=\left(-3\lambda_3-4\lambda_4-2\lambda_5+\lambda_8\right),\\ \alpha&=\lambda_9+\lambda_{10}+2\lambda_{11},\\ \zeta&=\lambda_{12}+\lambda_{13}+\lambda_{14}+\lambda_{15}\,. \end{aligned}
$$

# $A_4 \rightarrow Z_2$  **Breaking**

- $\rightarrow$  The **A**<sub>4</sub> symmetry is broken to residual  $Z_2$  symmetry by the VEV of  $\eta$ .
- $\rightarrow$  All the SM particles behave even under  $Z_2$ . Since they are singlets of *A*4.
- ➨ The BSM particles being triplets of *A*<sup>4</sup> can have odd charges under Z<sub>2</sub> symmetry.



The VEV alignment  $\langle \eta \rangle = (\mathsf{v}_2, \mathsf{0}, \mathsf{0})^\mathsf{T}$  leads to the  $\mathsf{A}_4 \to \mathsf{Z}_2$ breaking. The triplet components have the following transformation under the residual symmetry

$$
\begin{array}{ccc}\nN_1 & \rightarrow & +N_1, & \eta_1 & \rightarrow & +\eta_1, \\
N_{2,3} & \rightarrow & -N_{2,3}, & \eta_{2,3} & \rightarrow & -\eta_{2,3}.\n\end{array}
$$

## **Neutrino Mass Generation**

#### **Once the** η **develops VEV the loop breaks into two pieces as shown: tree level seesaw and scoto loop.**

 $\rightarrow$  Neutrino mass is generated from this hybrid mass mechanism known as scoto-seesaw mass mechanism.



# **Neutrino Mass Matrix**

At the tree level the seesaw contribution is given by

$$
m_D = \begin{pmatrix} y_1v_2 & 0 & 0 \\ y_2v_2 & 0 & 0 \\ y_3v_2 & 0 & 0 \end{pmatrix}, \ \mathcal{M} = \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix} \quad \Rightarrow \quad m_{\nu}^{(1)} = -\frac{1}{M} \begin{pmatrix} y_1^2v_2^2 & y_1y_2v_2^2 & y_1y_3v_2^2 \\ * & y_2^2v_2^2 & y_2y_3v_2^2 \\ * & * & y_3^2v_2^2 \end{pmatrix}
$$

- ➨ Rank of matrix *m* (1) <sup>ν</sup> is 1 ⇒ **only one neutrino is massive at tree level.**
- **►** Remaining two neutrinos masses can be generated at one loop level given by  $m_\nu^{(2)}$

$$
m^{(2)}_{\nu} = \begin{pmatrix} y_1^2d_1 & y_1y_2d_2 & y_1y_3d_3 \\ * & y_2^2d_3 & y_2y_3d_1 \\ * & * & y_3^2d_2 \end{pmatrix}
$$

# **Neutrino Mass Matrix**

Combining the tree level and loop level mass matrices the final form of neutrino mass matrix is given by

$$
m_{\nu}^{(TOT)} = \begin{pmatrix} A & C & \tilde{C}^* \\ C & B & D \\ \tilde{C}^* & D & \tilde{B}^* \end{pmatrix}
$$

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$$

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Cont...

$$
A = y_1^2 \left( d_1 - \frac{v_2^2}{2M} \right), \qquad D = y_2 y_3 \left( d_1 - \frac{v_2^2}{2M} \right),
$$
  
\n
$$
B = y_2^2 \left( d_3 - \frac{v_2^2}{2M} \right), \qquad \tilde{B} = y_3^2 \left( d_2 - \frac{v_2^2}{2M} \right),
$$
  
\n
$$
C = y_1 y_2 \left( d_2 - \frac{v_2^2}{2M} \right), \qquad \tilde{C} = y_1 y_3 \left( d_3 - \frac{v_2^2}{2M} \right).
$$
  
\n
$$
d_1 = c_1 + c_2 + c_3, \quad d_2 = c_1 + \omega c_2 + \omega^2 c_3, \quad d_3 = c_1 + \omega^2 c_2
$$

In the  $y_2 = y_3$  limit it produce the exact  $\mu - \tau$  reflection symmetry which predicts,  $\theta_{23} = 45^\circ$  and  $\delta_{\rm CP} = \pm \pi/2$ .

 $+\omega c_3$ .

## **Nu Sector: Generalized** µ − τ **reflection symmetry**

- Our model aligns with normal ordering of neutrino masses.
- The generalized  $\mu \tau$ reflection symmetry shown by blue and green color.
- The octant of the mixing angle  $\theta_{23}$

 $y_2 \le y_3 \implies \theta_{23} \le 45^\circ$  $y_2 \ge y_3 \implies \theta_{23} \ge 45^\circ.$ 

◦ Majorana phases are also correlated in our model with the Dirac CP phase.



#### **Nu sector: Lightest Neutrino Mass and** 0ν*ee* **Decay**

- The *m*<sup>1</sup> has a lower limit in our model while concerning the mixing angle  $\theta_{12}$  values in its  $3\sigma$  value.
- The 0ν*ee* decay is a robust way of searching for lepton number violation and Majorana nature of neutrinos.
- This narrow region of ⟨*mee*⟩ is due to the constraint value of neutrino masses and the Majorana phases.



 $|m_{ee}| = |c_{12}^2 c_{13}^2 m_1 + s_{12}^2 c_{13}^2 e^{2i\phi_{12}} m_2 + s_{13}^2 e^{2i\phi_{13}} m_3|$ 

# **Dark Sector: Fermionic DM** ×

◦ In our analysis we have found that the fermionic DM case is not allowed if we consider the  $3\sigma$  allowed ranges for the two mass squared differences of the neutrino oscillations.



**►** Here, we can infer that both the mass squared differences observed in neutrino oscillations can not be fit together.

## **Dark Sector: Scalar DM** ✓

The model prediction for scalar DM



- $\rightarrow$  The magenta points which lie below the LZ line are allowed points and the range is from 20 GeV to 80 GeV.
- **►** Imposing the collider constraints from LEP and LHC will further constraint this allowed mass region of DM candidate in  $55$  GeV  $\leq$   $m_{\eta_2^R} \leq 80$  GeV.

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- ✓ **The allowed parameter space for scalar DM mass to be in the**  $55\,\textsf{GeV} \leq m_{\eta_2^R} \leq 80\,\textsf{GeV}$  range.
- ✌ **In summary, our explicit model is highly predictive with its predictions testable in various neutrino sectors as well as dark matter experiments.**

# *Thank you for your attention !!*

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# Back Up (Fermionic DM)



## Back Up (Loop Mass)

$$
\label{eq:mass} (\mathcal{M}_{\nu})_{ij} = \frac{Y_{ik}Y_{jk}}{32\pi^2}M_k \left[ \ \frac{\left(m_{\eta_l}^R\right)^2}{\left(m_{\eta_l}^R\right)^2-M_k^2} \ln\left(\frac{m_{\eta_l}^R}{M_k}\right)^2 - \frac{\left(m_{\eta_l}^I\right)^2}{\left(m_{\eta_l}^I\right)^2-M_k^2} \ln\left(\frac{m_{\eta_l}^I}{M_k}\right)^2 \right] \ .
$$

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Because of  $A_4$  symmetry in our model, we have  $l = k$ . We express the mass matrix as

$$
(\mathcal{M}_{\nu})_{ij} = \sum_{k=1}^{3} Y_{ik} Y_{jk} c_k ,
$$

where,  $Y_{ik}$  and  $Y_{jk}$  are Yukawa couplings at one-loop level, and

$$
\label{eq:ck} \begin{split} c_k = \frac{M_k}{32\pi^2} \left[ \frac{(m_{\eta_k}^R)^2}{(m_{\eta_k}^R)^2 - M_k^2} \ln\left(\frac{m_{\eta_k}^R}{M_k}\right)^2 - \frac{(m_{\eta_k}'^P)^2}{(m_{\eta_k}'^P)^2 - M_k^2} \ln\left(\frac{m_{\eta_k}^I}{M_k}\right)^2 \right] \\ d_1 = c_1 + c_2 + c_3, \quad d_2 = c_1 + \omega c_2 + \omega^2 c_3, \quad d_3 = c_1 + \omega^2 c_2 + \omega c_3. \end{split}
$$

$$
A = y_1^2 \left( d_1 - \frac{v_2^2}{2M} \right), \qquad D = y_2 y_3 \left( d_1 - \frac{v_2^2}{2M} \right),
$$
  
\n
$$
B = y_2^2 \left( d_3 - \frac{v_2^2}{2M} \right), \qquad \tilde{B} = y_3^2 \left( d_2 - \frac{v_2^2}{2M} \right),
$$
  
\n
$$
C = y_1 y_2 \left( d_2 - \frac{v_2^2}{2M} \right), \qquad \tilde{C} = y_1 y_3 \left( d_3 - \frac{v_2^2}{2M} \right).
$$

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Back Up (A<sub>4</sub> generators and Z<sub>2</sub> subgroup)

$$
S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix};
$$

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ -a_2 \\ -a_3 \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{v}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{v}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix}
$$