Thermal correction to dark matter annihilation processes through real photon emission and absorption

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OUTLINE

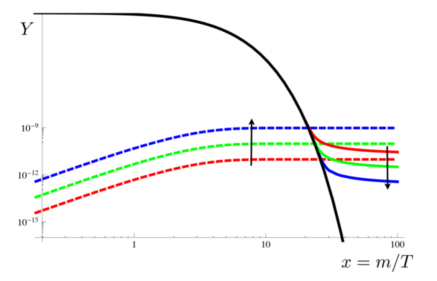
1. Introduction and Motivation
2. Model and Thermal Field Theory
3. Cancellation of soft and collinear divergences
4. Thermal correction to dark matter (DM) annihilation cross section

Introduction

- Boltzmann Equation

$$\frac{dn_\chi}{dt} + 3\mathcal{H}n_\chi = \langle \sigma_{\chi\bar{\chi}\to f\bar{f}} . v_{rel} \rangle (n_\chi^{eq}.n_{\bar{\chi}}^{eq} - n_\chi.n_{\bar{\chi}})$$

- Dark Matter Freeze-out scenario



- Details

1. Relic abundance of Dark Matter $0.1200 \pm 0.0012 (:= \Omega_{DM})$
   $\Omega = \rho/\rho_c$ (ρ_c : Critical Density for Our Universe)
2. $Y = n_\chi/T^3$
3. $x = m_\chi/T$
4. Coloured curves corresponds to different values of $\langle \sigma_{\chi\bar{\chi}\to f\bar{f}} . v_{rel} \rangle$

Courtesy (arXiv : 0911.1120)

Motivation

1. DM freezes-out in thermal plasma when expansion rate of the Universe starts dominating over the annihilation rate of DM particles
2. This is determined by $\langle \sigma_{\chi\bar{\chi}\rightarrow f\bar{f}} . v_{rel} \rangle$
3. Thermal corrections due to thermal fluctuations can be important, hence $\sigma_{\chi\bar{\chi}\rightarrow f\bar{f}}$ assumes importance due to same.

- Temperature dependence of dark matter annihilation cross-section, "σ", due to thermal fluctuation, utilizing Thermal Field Theory

$$\frac{dn_\chi}{dt} + 3\mathcal{H} n_\chi = \langle \sigma_{\chi\bar{\chi}\rightarrow f\bar{f}} . v_{rel} \rangle (n_\chi^{eq} . n_{\bar{\chi}}^{eq} - n_\chi . n_{\bar{\chi}})$$

Model

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{f}(i\not{D} - m_f)f + \frac{1}{2}\bar{\chi}(i\not{\partial} - m_\chi)\chi$$
$$+ (D_\mu\phi)^\dagger(D_\mu\phi) - m_\phi^2\phi^\dagger\phi + (\lambda\bar{\chi}P_L f^-\phi^+ + h.c.)$$

Details of Model[1]

1. $f \equiv (f^0, f^-)^T$ are SM fermions, doublets under SU(2)
2. DM : χ (Majorana fermion), singlets under SU(2) $\times$ U(1)
3. $\phi = (\phi^+, \phi^0)^T$, massive scalars, SU(2) doublets; assume $m_\phi \gg m_\chi$
4. $m_\chi/T \sim 20$ at freeze-out

[1] Beneke, M. et al. *JHEP* **2014**, *10*, [Erratum: JHEP 07, 106 (2016)], 045.

Thermal Field Theory (TFT)

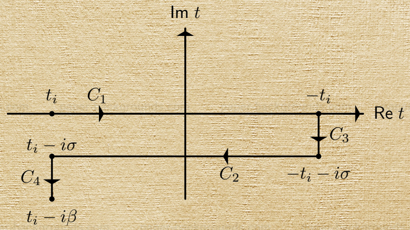
- Thermal field theory incorporates (inverse) temperature, β as the imaginary time axis
- Photon Propagator in TFT

$$iD^{t_a,t_b}_{\mu\nu}(k) = -g_{\mu\nu}\left(\begin{bmatrix} i\Delta_k & 0 \\ 0 & i\Delta^*_k \end{bmatrix} + 2\pi\delta(k^2)N_B(|k^0|)\begin{bmatrix} 1 & e^{\frac{k^0}{2T}} \\ e^{\frac{k^0}{2T}} & 1 \end{bmatrix}\right)$$

- Details

1. $i\Delta_k \quad = i/(k^2 \pm i\epsilon)$
2. $N_{\rm B}(|k^0|) \equiv \frac{1}{\exp\{|k^0|/T\}-1}$
3. $S = i/(\not{p} - m + i\epsilon)$
4. $S' = (\not{p} + m)$
5. $N_{\rm F}(|p^0|) \equiv \frac{1}{\exp\{|p^0|/T\}+1}$

- Time Path (Real Time Formalism)



- Fermion Propagator in TFT

$$i\mathcal{S}^{t_a,t_b}_{\rm f}(p,m) = \begin{bmatrix} S & 0 \\ 0 & S^* \end{bmatrix} - 2\pi S'\delta(p^2 - m^2)N_{\rm F}(|p^0|)\begin{bmatrix} 1 & \epsilon(p_0)e^{\frac{|p^0|}{(2T)}} \\ -\epsilon(p_0)e^{\frac{|p^0|}{(2T)}} & 1 \end{bmatrix}$$

Feynman Diagrams for $\chi\chi \to ff$ and $\chi\chi \to ff\gamma$

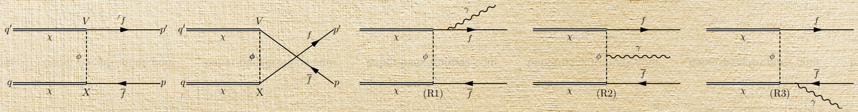


Figure: Tree level t- and u-channel, diagram along t-channel real photon emission diagram . If DM is Dirac type , then u-channel does not contribute.

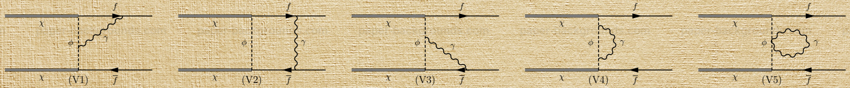
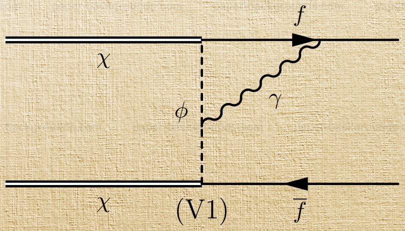


Figure: *The t-channel virtual photon corrections to the dark matter annihilation process at next to leading order (NLO). Diagrams are labelled from V1–V5.* *Analogous contributions from the u-channel diagrams exist.*

- NLO scattering process



- Details

1. Consider the sample diagram shown: $\chi(q')\,\overline{\chi}(q) \to f(p')\,\overline{f}(p)$
2. Thermal virtual photon is inserted between fermion and scalar
3. Photon Propagator is $iD^{t_\mu,t_\nu}_{\mu\nu,k} = -ig_{\mu\nu}\left[i/k^2 \pm 2\pi\delta(k^2)N_B(k)\right]$

- Soft and Collinear IR div. @ NLO

$$N_{\rm B}(|k^0|) \equiv \frac{1}{\exp\{|k^0|/T\}-1} \xrightarrow{k^0\to 0} \frac{T}{k^0}$$

$$-i\mathcal{M} = \int d^4k\{[\bar{u}_{p',m_f}(\gamma^\mu)(iS^{t_\mu,t_V}_{p'+k,m_f})(i\lambda^* P_R)u_{q',m_\chi}][i\Delta^{t_V,t_\nu}_{q-p+k,m_\phi}]$$

$$[(2q-2p+k)^\nu][i\Delta^{t_\nu,t_X}_{q-p,m_\phi}][\bar{v}_{q,m_\chi}(i\lambda P_L)v_{p,m_f}][iD^{t_\mu,t_\nu}_{\mu\nu,k}]\}$$

Cancellation of IR Div.{Grammer and Yennie(GY) Tech.}

- GY technique for Rearrangement of $-ig_{\mu\nu}$ (Virtual Photon)[2]

$$-ig_{\mu\nu} \to -i\{G_{\mu\nu} + K_{\mu\nu}\}$$

- Rearrangement of Photon polarization sum (Real Photon)

$$\sum_{\text{pol}} \epsilon^{*,\mu}(k)\,\epsilon^{\nu}(k) \to -g^{\mu\nu} \to [\widetilde{G}_k^{\mu\nu} + \widetilde{K}_k^{\mu\nu}]$$

- Details
  1. $iD^{t_\mu,t_\nu}_{\mu\nu,k} = g_{\mu\nu}(iD^{t_\mu,t_\nu}_{k})$
  2. IR Finite Contribution
     $G_{\mu\nu} := g_{\mu\nu} - b_k(p_i,p_f)k_\mu k_\nu$
     $\widetilde{G}_{\mu\nu} := g_{\mu\nu} - \widetilde{b}_k(p_i,p_f)k_\mu k_\nu$
  3. IR Divergent Contribution
     $K_{\mu\nu} := b_k(p_i,p_f)k_\mu k_\nu$
     $\widetilde{K}_{\mu\nu} := \widetilde{b}_k(p_i,p_f)k_\mu k_\nu$

- Structure of $b_k(p_f,p_i)$

$$\frac{1}{2}\left[\frac{(2p_f-k)\cdot(2p_i-k)}{((p_f-k)^2-m^2)((p_i-k)^2-m^2)} + (k \leftrightarrow -k)\right]$$

- Structure of $\widetilde{b}_k(p_f,p_i)$

$$\widetilde{b}_k(p_f,p_i) = b_k(p_f,p_i)|_{k^2\to 0}$$

- IR divergence cancellation takes place order by order between $K_{\mu\nu}$ & $\widetilde{K}_{\mu\nu}$
- We only need to compute $G_{\mu\nu}$ and $\widetilde{G}_{\mu\nu}$

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[2] Grammer Jr., G. et al. *Phys. Rev. D* **1973**, *8*, 4332–4344, Indumathi, D. *Annals Phys.* **1998**, *263*, 310–339, Sen, P. et al. *Eur. Phys. J. C* **2020**, *80*, 972.

Leading order (LO) annihilation cross section (σ_{LO})

- LO scattering processes

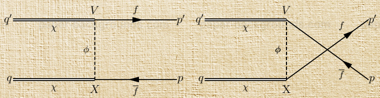


Figure: *The t- and u-channel DM annihilation processes at LO.*

- Heavy scalar approximation
  $i\Delta \to i/(-m_\phi^2)$
- Dynamical scalar approxmation (dsa)
  $i\Delta_{l+k} \approx \frac{i}{l^2-m_\phi^2}\left(1+\frac{2l\cdot k}{l^2-m_\phi^2}\right)$
- $H = \sqrt{s}/2$

$$\sigma_{LO}^{\text{heavy scalar}} = \frac{1}{12\pi s}\frac{P'}{P}\frac{\lambda^4}{m_\phi^4}\left[8H^2(H^2-m_\chi^2)+m_f^2(5m_\chi^2-2H^2)\right] \; . \qquad (1)$$

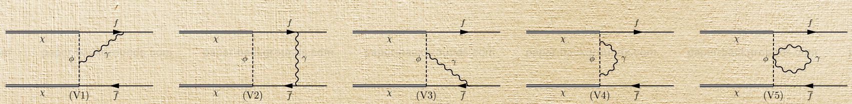
$$\sigma_{LO}^{dsa} \xrightarrow{v \text{ small}} \frac{\lambda^4}{4\pi s}\frac{P'}{P}\left[\frac{m_\chi^2 m_f^2}{(m_\chi^2+m_\phi^2-m_f^2)^2}+\mathcal{O}(v^2)\right] \; . \qquad (2)$$

- Momenta in CM frame [3]

[3] $q'^\mu = (H,0,0,P)\;;\; p'^\mu = (H, P'\sin\theta, 0, P'\cos\theta)$
$\mathrm{q}^\mu = (H,0,0,-P)\;;\; p^\mu = (H,-P'\sin\theta,0,-P'\cos\theta)$

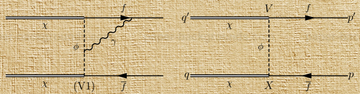
Thermal correction to annihilation cross section ($\sigma^{virtual}_{NLO(T)}$)

- t-channel virtual photon correction at NLO



$$\sigma^{NLO} \propto \left[\frac{1}{4} \sum_{spins} \left(\mathcal{M}^t_{LO} - \mathcal{M}^u_{LO} \right)^\dagger \left(\mathcal{M}^t_{NLO} - \mathcal{M}^u_{NLO} \right) + h.c. \right] ,$$

- Feynman Diagram for Sample Calculation



1. Thermal contribution from ϕ is negligible ($m_\phi \gg m_\chi$)
2. Heavy scalar approximation $i\Delta \to i/(-m_\phi^2)$
3. Dynamical scalar approxmation (dsa) $i\Delta_{l+k} \approx \frac{i}{l^2 - m_\phi^2} \left(1 + \frac{2l \cdot k}{l^2 - m_\phi^2} \right)$

Thermal correction to annihilation cross section ($\sigma^{virtual}_{NLO(T)}$)

• It can be shown that the contribution with fermion and photon propagator simultaneously thermal, vanishes.

• Consider the contribution when the thermal part of the photon propagator is included.

• The photon propagator appearing in $\mathcal{M}^t_{NLO}$ term contains the $\delta - function$ term, which gives:

$$\begin{aligned}
\int \mathrm{d}^4k \left(2\pi\delta(k^2)\right) F(k) &= 2\pi \int_{-\infty}^{\infty} \mathrm{d}k^0 \int_0^{\infty} K^2 \mathrm{d}K \int \mathrm{d}\Omega_k \left[\delta((k^0)^2 - K^2)\right] F(k^0, K, \Omega_k) \ , \\
&= 2\pi \int \mathrm{d}k^0 \int \mathrm{d}\Omega_k \int K^2 \mathrm{d}K \, \frac{\left[\delta(k^0 - K) + \delta(k^0 + K)\right]}{|2k^0|} F(k^0, K, \Omega_k) \ , \\
&= \pi \int \mathrm{d}\Omega_k \left[\int_0^{\infty} |k^0| \mathrm{d}k^0 F(k^0, k^0, \Omega_k) + \int_{-\infty}^{0} |k^0| \mathrm{d}k^0 F(k^0, -k^0, \Omega_k)\right] \ , \\
&\equiv \pi \int_0^{\infty} \omega \mathrm{d}\omega \int \mathrm{d}\Omega_k \left[F_+(\omega, \omega, \Omega_k) + F_-(-\omega, \omega, \Omega_k)\right] \ ,
\end{aligned}$$

and the angular integrals can be performed analytically.

• If there is no explicit ω dependence in this angular integral, we have

$$\int_0^{\infty} \omega \mathrm{d}\omega \, N_B(\omega) = \frac{\pi^2 T^2}{6} \ .$$

Thermal correction to annihilation cross section ($\sigma^{virtual}_{NLO(T)}$)

- Result for Sample Case

$$\sigma^{t,1\gamma}_{NLO} = \frac{1}{128s(2\pi)^4}\,\frac{p'}{p}\int \omega d\omega\, n_B(\omega)\, Int^{t,1\gamma}_{NLO}\,,$$

$$= \frac{1}{128s(2\pi)^4}\,\frac{p'}{p}\,\frac{\pi^2 T^2}{6} \times Int^{t,1\gamma}_{NLO}\,,$$

$$Int^{tt+uu-tu,1\gamma}_{NLO} = \frac{256\pi e^2\lambda^4}{3{m_\phi}^6}\left(8H^4 - 2\,H^2\left(4{m_\chi}^2 + {m_f}^2\right) + 5{m_\chi}^2\,{m_f}^2\right)\,,$$

• Collinear IR Divergences get cancelled after inclusion of real emission and absorption of photon at NLO

$$Int^{tt+uu-tu,1\gamma}_{NLO,complete} = \frac{64\pi e^2\lambda^4}{3{m_\phi}^6\,p'}\Big[4p'\left(8H^4 - 2\,H^2\left(4{m_\chi}^2 + {m_f}^2\right) + 5{m_\chi}^2\,{m_f}^2\right)$$
$$+3\log\frac{H-p'}{H+p'}\left(8H^5 - 4H^3\left(2{m_\chi}^2 + {m_f}^2\right) + 5H{m_\chi}^2{m_f}^2\right)\Big]$$

Thermal correction to annihilation cross section ($\sigma^{virtual}_{NLO(T)}$)

- Contribution to σ_{NLO} from one NLO process (photon thermal)

$$Int^{tt+uu-tu,1\gamma}_{NLO} = \frac{256\pi e^2\lambda^4}{3{m_\phi}^6}\left(8H^4 - 2\,H^2\left(4{m_\chi}^2 + {m_f}^2\right) + 5{m_\chi}^2\,{m_f}^2\right) \tag{3}$$

- Contribution to σ_{NLO} from all NLO processes (photon thermal)

$$\begin{aligned} Int^{\gamma T}_{NLO} &= Int^{tt+uu-tu,1\gamma}_{NLO} + Int^{tt+uu-tu,2\gamma}_{NLO} + Int^{tt+uu-tu,3\gamma}_{NLO} + Int^{tt+uu-tu,4\gamma}_{NLO} + Int^{tt+uu-tu,5\gamma}_{NLO} \\ &= \frac{512\pi e^2\lambda^4}{15{m_\phi}^8}\Big[216H^6 - 4H^4\left(68\,{m_\chi}^2 + 7{m_f}^2\right) \\ &\quad + H^2\left(56{m_\chi}^4 + 86{m_\chi}^2\,{m_f}^2 + 5{m_\phi}^4\right) - 28{m_\chi}^4{m_f}^2 - 5{m_\chi}^2\,{m_\phi}^4\Big] \end{aligned} \tag{4}$$

- NLO scattering cross section for process (photon / fermion thermal)

$$\sigma_{NLO} = \frac{1}{128s(2\pi)^4}\,\frac{\sqrt{H^2 - m_f^2}}{\sqrt{H^2 - m_\chi^2}}\,\frac{\pi^2 T^2}{6} \times \left[Int^{\gamma T}_{NLO} + \frac{1}{2} Int^{fT}_{NLO}\right] \tag{5}$$

Thermal correction to annihilation cross section ($\sigma^{virtual}_{NLO(T)}$)

- Thermal correction to annihilation cross section in dynamical scalar approximation ; $\sigma_{LO} \propto 1/D^2$

Diagram	γ/f	Int^a_{NLO} (T^2 contribution)	Int^a_{NLO} (T^4 contribution)
1	γ	$-8m_\chi^2 m_f^2(m_f^2 - m_\phi^2)/D^4$	0
	f	$4m_\chi^2 m_f^2(5m_\chi^2 - 5m_f^2 + m_\phi^2)/D^4$	0
	Total$_{\gamma+f}$	$2m_\chi^2 m_f^2(5m_\chi^2 - 9m_f^2 + 5m_\phi^2)/D^4$	0
2	γ	$-8m_\chi^2 m_f^2/D^3$	0
	f	$-6m_f^2(2m_\chi^2 - m_f^2)/D^3$	$-\frac{21\pi^2 T^2}{10m_\chi^2 D^3} m_f^2(2m_\chi^2 - m_f^2)$
	Total$_{\gamma+f}$	$-m_f^2(14m_\chi^2 - 3m_f^2)/D^3$	$-\frac{21\pi^2 T^2}{10m_\chi^2 D^3} m_f^2(2m_\chi^2 - m_f^2)$
3	γ	$-8m_\chi^2 m_f^2(m_f^2 - m_\phi^2)/D^4$	0
	f	$4m_\chi^2 m_f^2(3m_\chi^2 - 2m_f^2 + m_\phi^2)/D^4$	0
	Total$_{\gamma+f}$	$2m_\chi^2 m_f^2(3m_\chi^2 - 6m_f^2 + 5m_\phi^2)/D^4$	0
4	γ	$32m_\chi^4 m_f^2/D^4$	$-\frac{56\pi^2 T^2}{15D^5} m_\chi^2 m_f^2(m_\chi^2 - m_f^2)$
5	γ	$-16m_\chi^2 m_f^2/D^3$	0
All	Total$_{\gamma+f}$	$\frac{1}{D^3} m_f^2(2m_\chi^2 + 3m_f^2) +$ $\frac{2}{D^4} m_f^2 m_\chi^2(10m_\phi^2 + 24m_\chi^2 - 15m_f^2)$	$-\frac{21\pi^2 T^2}{10m_\chi^2 D^3} m_f^2(2m_\chi^2 - m_f^2) +$ $-\frac{56\pi^2 T^2}{15D^5} m_\chi^2 m_f^2(m_\chi^2 - m_f^2)$

Table: *The $v \to 0$ contributions from various diagrams to the NLO cross section (the so-called "a" terms in the non-relativistic cross section). Here D is defined as $D = (m_\chi^2 - m_f^2 + m_\phi^2)$.*

Thermal correction to annihilation cross section ($\sigma^{real}_{(T)}$)

- Photon phase space factor is

$$d\phi_k = \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2)[\theta(k^0) + N_B(|k^0|)\{\theta(k^0) + \theta(-k^0)\}] \quad (6)$$

- The thermal part simplifies to

$$\int d\phi_k \tilde{F}(k^0, K, \Omega_k) \propto \int d\omega N_B(|k^0| \left[\widetilde{F_+}(\omega, \omega, \phi) + \widetilde{F_-}(-\omega, \omega, \phi)\right]_\theta$$

- Diagrams for real photon correction

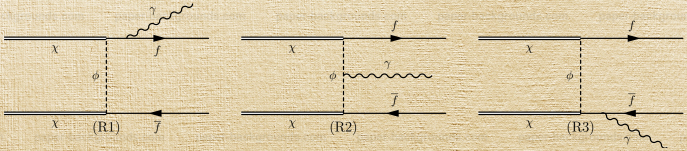


Figure: t-channel real photon emission diagram (R1-R3).

- Considering DM (χ) to be Dirac type, we have following result for real photon correction upto $\mathcal{O}(T^2)$

$$\sigma^{real}_{(T)} \propto \Bigg[\frac{64\pi e^2 \lambda^4 \left(H^2 - m_\chi^2\right)}{3 m_\phi^4} - \frac{128\pi e^2 H^2 \lambda^4 \left(10H^2 - 7m_\chi^2\right)}{3 m_\phi^6}$$
$$\frac{256\pi e^2 H^2 \lambda^4 \left(54H^4 - 43H^2 m_\chi^2 + 4m_\chi^4\right)}{15 m_\phi^8} \Bigg] \times \frac{\pi^2 T^2}{6}$$

Summary

1. We investigated thermal correction to DM annihilation process utilizing thermal field throry,
2. We used Grammer and Yennie's approach for IR Div. Cancellation , and obtain Finite Remainder for σ_{NLO} ,
3. We obtain $\mathcal{O}(T^2)$ contribution to $\sigma_{NLO}^{virtual}$ in heavy scalar limit,
4. We present $\mathcal{O}(T^2)$ and $\mathcal{O}(T^4)$ contribution to $\sigma_{NLO}^{virtual}$, in dynamical scalar approximation, which are helicity supressed.
5. We present $\mathcal{O}(T^2)$ contribution to σ_{NLO}^{real} in heavy scalar limit, considering DM particle to be Dirac type (Calculations with Majorana DM in progress) ,
6. Thermal correction to $\langle \sigma v \rangle$ in the Boltzmann equation will alter ρ_{DM}.

References

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2. Pritam Sen et al. , Eur.Phys.J.C 80 (2020) 10, 972
3. Martin Beneke et al. , JHEP 09 (2016) 031 & JHEP 10 (2014) 045
4. D. Indumathi, Annals Phys. 263 (1998) 310-339
5. G. Grammer, Jr. ,et al., Phys.Rev.D 8 (1973) 4332-4344
6. D.R. Yennie, et al., Annals Phys. 13 (1961) 379-452

Relative size of NLO contribution

- Considering only the corresponding "*a*" (v^0) terms in the non-relativistic limit, the relative size of the NLO contribution *for each* flavour of fermion pair is given by,

$$\frac{\sigma^a_{NLO}}{\sigma^a_{LO}} = \frac{\pi\alpha T^2}{6m_\phi^2}\frac{m_f^2(22m_\chi^2 + 3m_f^2)}{m_f^2 m_\chi^2} ,$$

$$\approx \frac{11\pi\alpha}{3}\frac{T^2}{m_\phi^2} ,$$