भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Observing the Unseen: Faraday Rotation Signatures and Parker bounds on Primordial Magnetic Black Holes

Based on : Primordial magnetic relics and their signatures (2406.08728)

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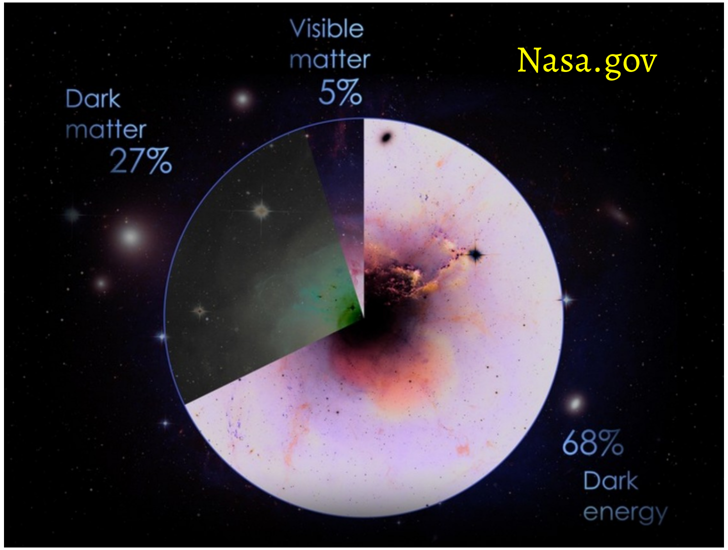
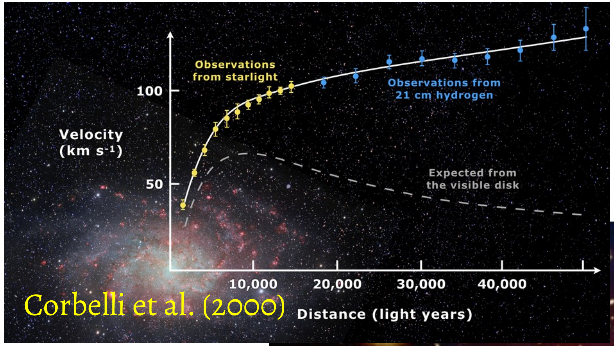
Outline

- Motivation
- Parker bounds on magnetic black holes from intergalactic magnetic fields
  - Cosmic voids magnetic fields
  - Cosmic web magnetic fields
- Faraday Rotation signatures
  - Primordial magnetic black holes (MBHs)
  - Neutron Stars

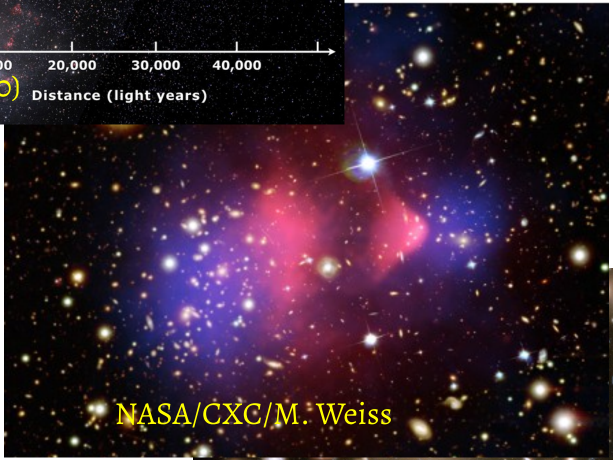
Motivation



Nasa.gov



Corbelli et al. (2000)

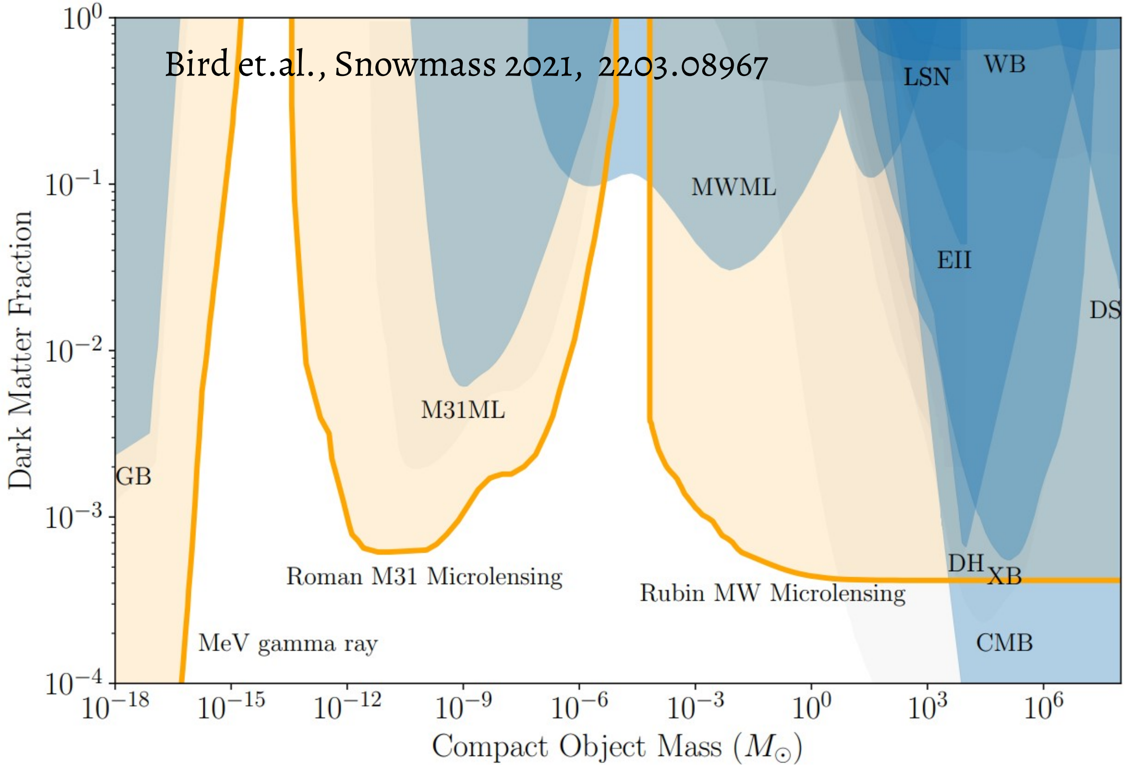

NASA/CXC/M. Weiss


NASA, ESA, J. Lotz and the HFF Team

Motivation

- Primordial black holes (PBHs) are exotic compact objects that originated in the early moments of the universe, born during the primordial era.
- Possible origin of PBHs
  - Large primordial fluctuation on small scales [Carr et. al., MNRAS 168 (1974) 399-415]
  - Phase transition [Hawking et. al., PRD 26 (1982) 2681]
  - Preheating after inflation [Bellido et. al., 9605094]
  - ...
- Recent works on PBHs
  - gravitational lensing [Niikura et. al., 1701.02151 , 1901.07120]
  - signatures from gravitational waves [Sasaki et. al., 1801.05235]
  - mass distribution [Carr, Astro. J. 201 (1975) 1-19]
  - ...

PBH constraint



$$T^{\mathrm{MBH}} = \frac{\sqrt{M_{\mathrm{BH}}^2 - Q_{\mathrm{BH}}^2}}{4\pi\left(M_{\mathrm{BH}} + \sqrt{M_{\mathrm{BH}}^2 - Q_{\mathrm{BH}}^2}\right)^2} .$$

For extremal MBH, $M_{\mathrm{BH}}^{\mathrm{Ext.}} = Q_{\mathrm{BH}}$,

$$T^{\mathrm{MBH}} \to 0 ,$$

Parker bounds

PHYSICAL REVIEW D VOLUME 26, NUMBER 6 15 SEPTEMBER 1982

Magnetic monopoles and the survival of galactic magnetic fields

Michael S. Turner, E. N. Parker, and T. J. Bogdan
Astronomy and Astrophysics Center, The University of Chicago, Chicago, Illinois 60637
(Received 1 June 1982)

THE ASTROPHYSICAL JOURNAL, **321**: 349–354, 1987 October 1

MAGNETIC MONOPOLE PLASMA OSCILLATIONS AND THE SURVIVAL OF GALACTIC MAGNETIC FIELDS[1]

E. N. PARKER
Enrico Fermi Institute and Departments of Physics and Astronomy, University of Chicago
Received 1987 February 6; accepted 1987 March 17

Parker bounds on MBHs

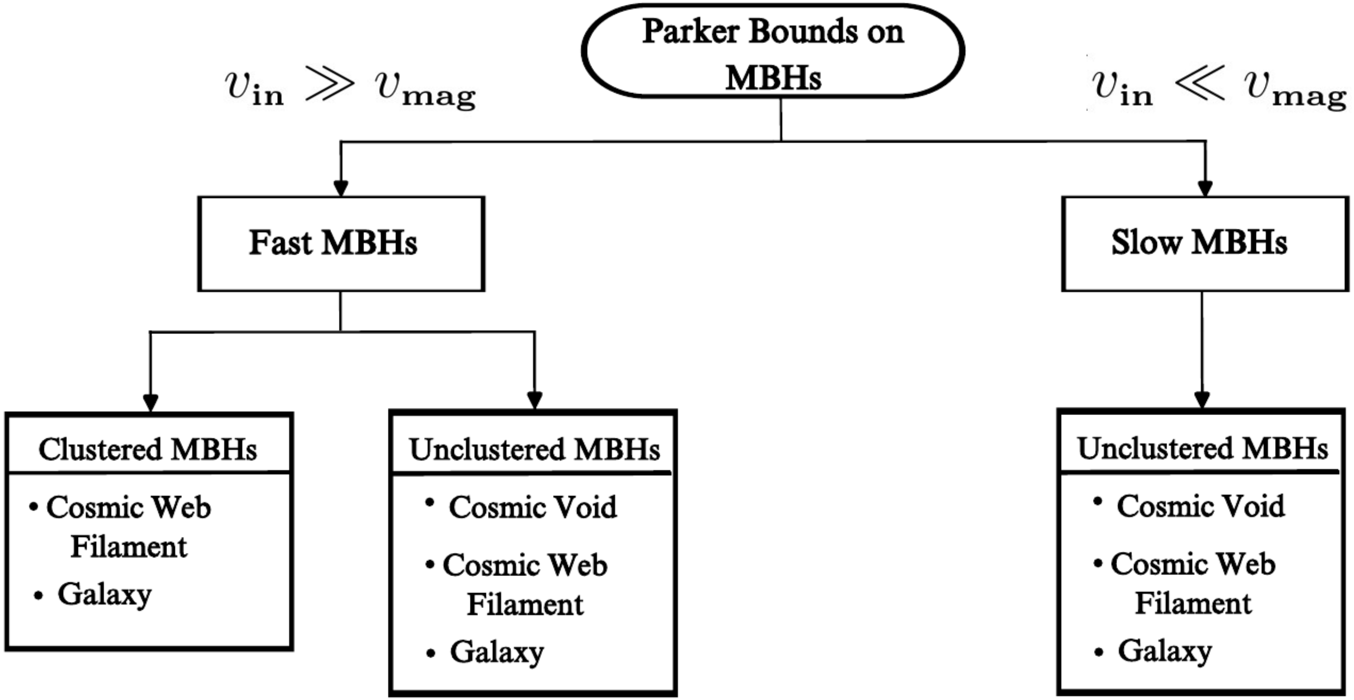
- Parker introduced an idea focusing on estimating bounds on the flux of the magnetic monopoles based on the survival of galactic magnetic fields.

- Magnetic monopoles traveling in the galactic magnetic fields undergo acceleration and subsequently deplete the energy stored in the magnetic fields.

- It can also be used to find the bounds on the dark matter fraction $f_{\rm DM}$ contained in MBHs.

- We estimate stringent bounds on $f_{\rm DM}$ based on survival of intergalactic magnetic fields in cosmic web filaments and cosmic voids .

Parker bounds on MBHs (cont.)



$$v_{\rm mag} = \sqrt{\frac{2Q_{\rm BH} l_{\rm c} B_{\rm c}}{M_{\rm BH}}}$$

Parker bounds on MBHs (cont.)

- For the fast unclustered MBHs, bounds are

$$f_{\rm DM} \leq \frac{B_{\rm c}^2 l_{\rm c} M_{\rm BH}}{6\mu_0 \Delta\mathcal{E}_{\rm k} t_{\rm reg} v \rho_{\rm DM}} \,. \qquad \Delta\mathcal{E}_{\rm k}^{\rm fast,unclust} \simeq Q_{\rm BH} B_{\rm c} l_{\rm c} \cos\alpha + \frac{Q_{\rm BH}^2 B_{\rm c}^2 l_{\rm c}^2}{2 M_{\rm BH} v_{\rm in}^2}$$

- Bounds from magnetic fields in cosmic void,

[Bai et.al. 2007.03703]

$$f_{\rm DM} \lesssim \mathcal{O}(10^{-3}) \,.$$

$$\boxed{f_{\rm DM}^{\rm fast,unclust} \lesssim 10^{-8}} \qquad \boxed{f_{\rm DM}^{\rm fast,unclust} \lesssim 10^{-9} \left(\frac{M_{\rm BH}}{\rm kg}\right)\left(\frac{\text{A-m}}{Q_{\rm BH}}\right)}$$

$$B^{\rm void} \gtrsim \mathcal{O}(10^{-15})\,{\rm G}\,, \quad l_{\rm c}^{\rm void} \sim \mathcal{O}(1-10)\,{\rm Mpc}\,, \quad t_{\rm reg}^{\rm void} \sim \mathcal{O}(10)\,{\rm Gyr}\,.$$ [H.E.S.S. ,Fermi Lat collab. , 2308.16717]

- Bounds from magnetic fields in cosmic web filaments,

$$\boxed{f_{\rm DM}^{\rm fast,unclust} \lesssim 10^{-4}} \qquad \boxed{f_{\rm DM}^{\rm fast,unclust} \lesssim 10^{-5} \left(\frac{M_{\rm BH}}{\rm kg}\right)\left(\frac{\text{A-m}}{Q_{\rm BH}}\right)}$$

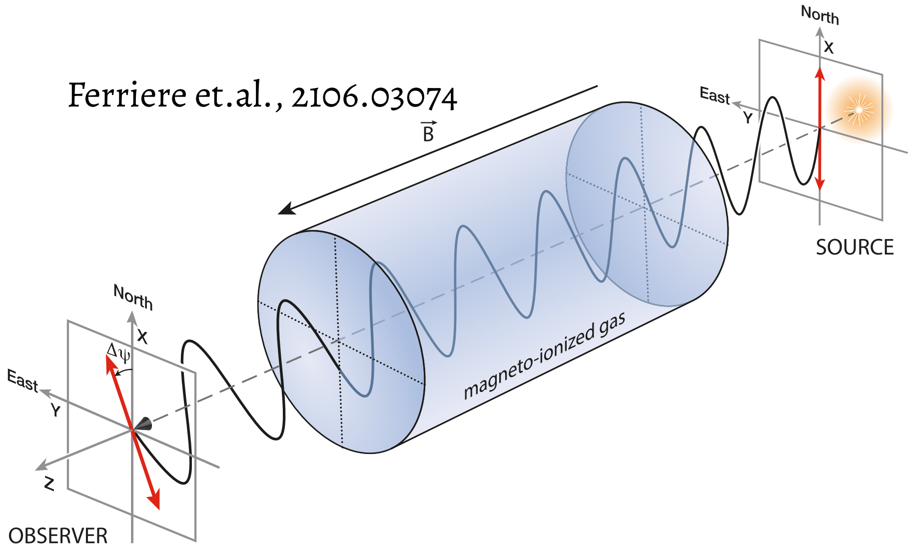
$$B^{\rm fil} \sim \mathcal{O}(10^{-9})\,{\rm G}\,, \quad l_{\rm c}^{\rm fil} \sim \mathcal{O}(1)\,{\rm Mpc}\,, \quad t_{\rm reg}^{\rm fil} \sim \mathcal{O}(10)\,{\rm Gyr}\,.$$ [Caretti et.al., 2202.04607]

Faraday Rotation

- A linearly polarized electromagnetic wave passing through a magneto-ionized medium experiences a rotation of its polarization plane, known as Faraday rotation.

- Faraday rotation observations from point sources are widely used to probe astrophysical magnetic fields along the line of sight.

Ferriere et.al., 2106.03074



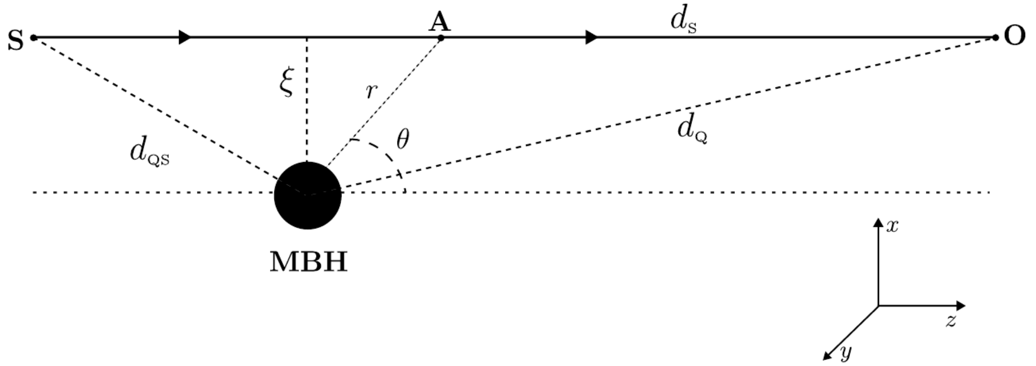
- The spatial variation of the polarization angle in the plane perpendicular to the observer's line of sight depends on the configuration of the source's magnetic field.

- We observed a distinct Faraday rotation signature characteristic of magnetic black holes (MBHs) and compared it to that of a neutron star.

Faraday Rotation due to an MBH

- Magnetic field by a magnetic black hole,

$$\vec{B}_{\rm BH}(r) \equiv \frac{\mu_0}{4\pi}\frac{Q_{\rm BH}}{r^2}\hat{r}\ .$$



- Refractive index for a wave ,

$$n^{\rm BH}_{\rm L,(R)} = \begin{cases} \left(1 - \frac{\omega_{\rm p}^2(\omega^2-\omega_{\rm p}^2)}{\omega^2\left(\omega^2-\omega_{\rm p}^2-\frac{1}{2}\tilde{\omega}_{\rm BH}^2\sin^2\theta \pm \left(\frac{1}{4}\tilde{\omega}_{\rm BH}^4\sin^4\theta+(\omega^2-\omega_{\rm p}^2)^2\frac{\tilde{\omega}_{\rm BH}^2}{\omega^2}\cos^2\theta\right)^{1/2}\right)}\right)^{1/2} , & 0<\theta<\pi/2\ , \\ \left(1 - \frac{\omega_{\rm p}^2(\omega^2-\omega_{\rm p}^2)}{\omega^2\left(\omega^2-\omega_{\rm p}^2-\frac{1}{2}\tilde{\omega}_{\rm BH}^2\sin^2\theta \mp \left(\frac{1}{4}\tilde{\omega}_{\rm BH}^4\sin^4\theta+(\omega^2-\omega_{\rm p}^2)^2\frac{\tilde{\omega}_{\rm BH}^2}{\omega^2}\cos^2\theta\right)^{1/2}\right)}\right)^{1/2} , & \pi/2<\theta<\pi\ . \end{cases}$$

$$\tilde{\omega}_{\rm BH}(r) \equiv \frac{e}{m_{\rm e}}\vec{B}_{\rm BH}(r)\ .$$

$$\omega_{\rm p}^2 \equiv \frac{e^2\mathcal{N}_{\rm e}(r)}{\epsilon_0 m_{\rm e}}\ .$$

- Condition for propagation,

$$\omega > \frac{\tilde{\omega}_{\rm BH}}{2} + \frac{1}{2}\sqrt{\tilde{\omega}_{\rm BH}^2 + 4\omega_{\rm p}^2}\ ,$$

$$r \gtrsim r^{\rm BH}_{\rm cut} \simeq \sqrt{\frac{\mu_0 e Q_{\rm BH}}{4\pi m_{\rm e}}\frac{\omega}{(\omega^2-\omega_{\rm p}^2)}}\ .$$

Faraday Rotation due to an MBH

- Change in polarisation and Rotation measure,

$$\psi^{\rm BH}_{\rm pol.} = \frac{\omega}{2c}\int_{-d_{\rm QS}}^{d_{\rm Q}} dz\,\left(n^{\rm BH}_{\rm L}(r) - n^{\rm BH}_{\rm R}(r)\right)\ , \qquad \mathrm{RM}^{\rm BH}(\lambda) = \frac{d}{d\lambda^2}\left(\psi^{\rm BH}_{\rm pol.}\right)\ .$$

- In a high frequency limit, $\omega \gg \max\left(\omega_{\rm p}, \Omega\right)$

$$\psi^{\rm BH}_{\rm pol.} \simeq \frac{e^3\lambda^2}{8\pi^2\epsilon_0 m_{\rm e}^2 c^3}\int_{-d_{\rm QS}}^{d_{\rm Q}} dz\ \mathcal{N}_{\rm e}(r) B_{\rm BH,z}(\vec{r}) + O\left(B^3_{\rm BH}\right)\ ,$$

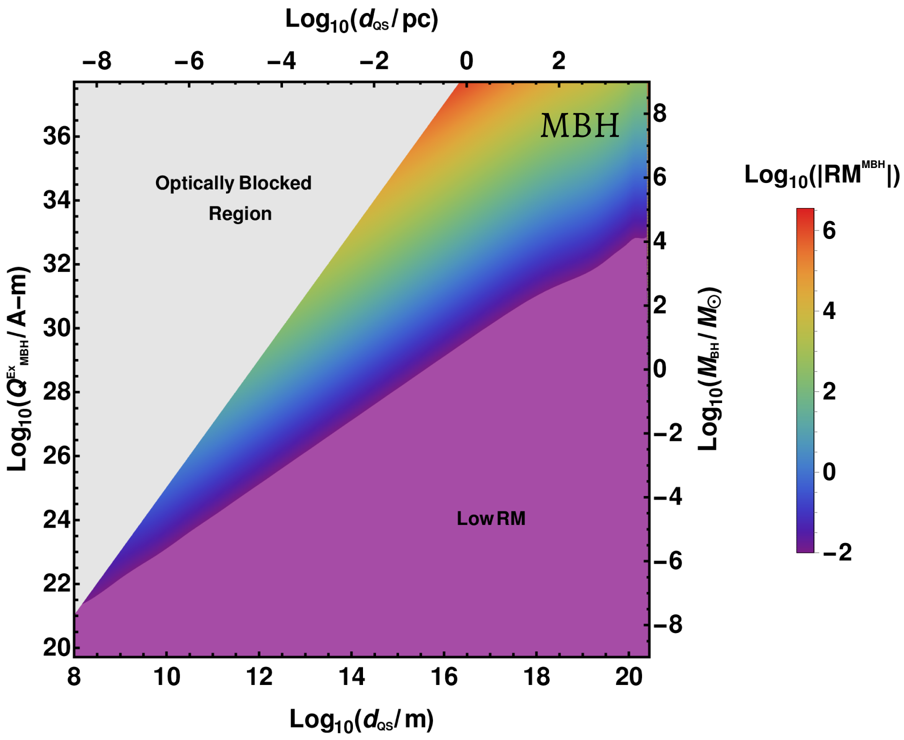
$$\mathrm{RM}^{\rm BH}(\lambda) \simeq \frac{e^3}{8\pi^2\epsilon_0 m_{\rm e}^2 c^3}\int_{-d_{\rm QS}}^{d_{\rm Q}} dz\ \mathcal{N}_{\rm e}(r) B_{\rm BH,z}(\vec{r}) + O\left(B^3_{\rm BH}\right)\ .$$

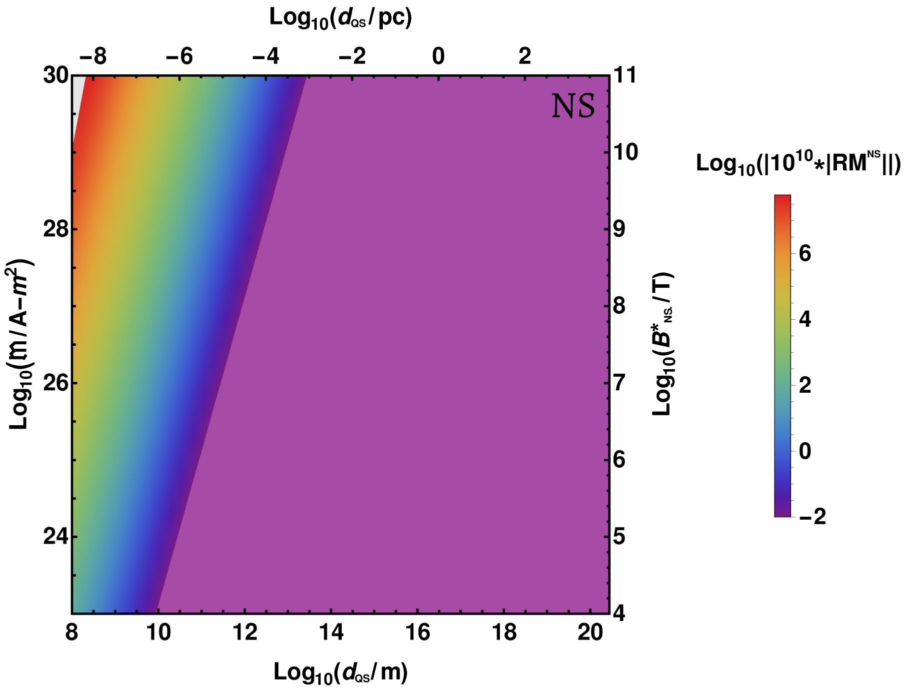
- Galactic profile for plasma density,

$$\mathcal{N}^{\rm MW}_{\rm e}(r) = N^{\rm MW}_{\rm e,1} e^{-\left((z-d_{\rm QS})/A^{\rm MW}_1\right)^2} + N^{\rm MW}_{\rm e,2} e^{-\left((z-d_{\rm QS}-2A^{\rm MW}_2)/A^{\rm MW}_2\right)^2}\ ,$$

[Cordes et. al. Nature 354, 121–124 (1991)]

Rotation measure





$$\vec{B}_{\rm BH}(r) \equiv \frac{\mu_0}{4\pi}\frac{Q_{\rm BH}}{r^2}\hat{r}\ .$$

$$Q_{\rm BH} \simeq 10^{29}\ \text{A-m}$$

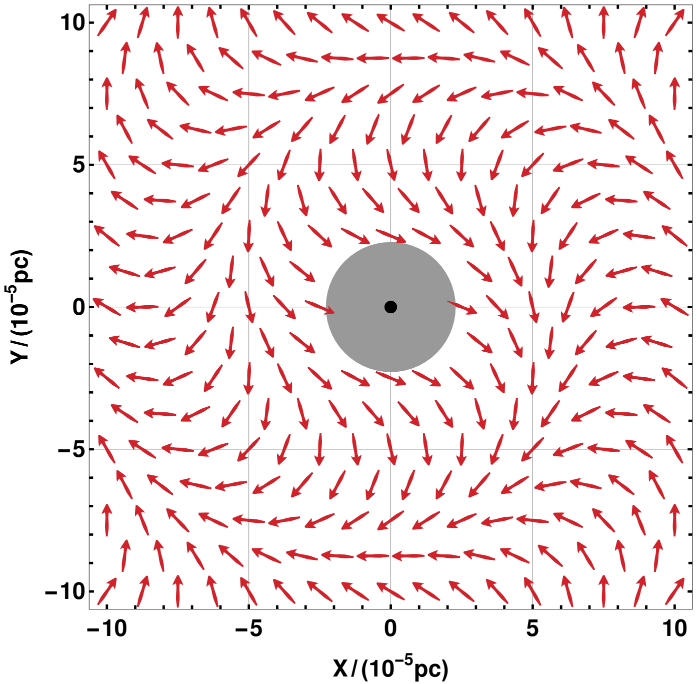
$$\vec{B}_{\rm NS}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{\mathfrak{m}}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{\mathfrak{m}}}{r^3}\right)$$

$$\mathfrak{m}=10^{30}\ \ \text{A-m}^2$$

Fig : Rotation measure due to an MBH and a neutron star located inside the Milky Way galaxy .

Lalit S. Bhandari, PPC 2024

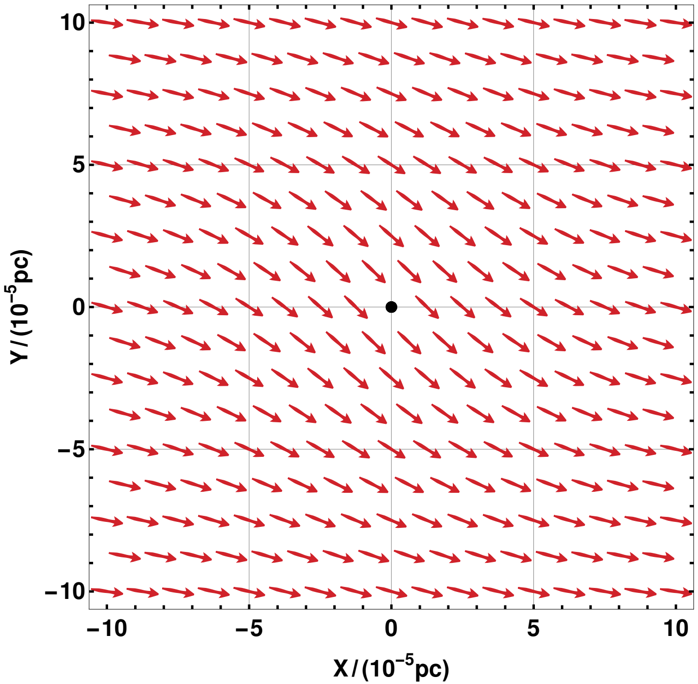
Change in polarisation



MBH

$$\vec{B}_{\rm BH}(r) \equiv \frac{\mu_0}{4\pi}\frac{Q_{\rm BH}}{r^2}\hat{r}\ .$$

$$Q_{\rm BH} \simeq 10^{29}\ \text{A-m}$$



NS

$$\vec{B}_{\rm NS}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{\mathfrak{m}}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{\mathfrak{m}}}{r^3}\right)$$

$$\text{m}=10^{30}\ \ \text{A-m}^2$$

Fig : Change in polarisation due to an MBH and a neutron star located inside the Milky Way galaxy .

Comparison between an MBH and a neutron star

- Integral measure $\mathcal{M}$

$$\mathcal{M}^{\rm NS} \equiv \frac{1}{2\pi\psi^{\rm NS}_{\rm pol.}(\phi_0)}\left[\left(\oint_C d\phi\, \psi^{\rm NS}_{\rm pol.}(\phi)\right) - 2\pi\psi^{\rm NS}_{\rm pol.}(\phi_0)\right] \geq 0\,.$$

$$\mathcal{M}^{\rm BH} \equiv \frac{1}{2\pi\psi^{\rm BH.}_{\rm pol.}}\left[\left(\oint_C d\phi\, \psi^{\rm BH}_{\rm pol.}(\phi)\right) - 2\pi\psi^{\rm BH}_{\rm pol.}\right] = 0\,.$$

$B^*_{\rm BH(NS)}(T)$	NS or BH	$Q_{\rm BH}(A-m)$ or $\mathfrak{m}(A-m^2)$	θ_{dp}	$\psi_{\rm pol.}(r = r^{\rm BH(NS)}_{\rm cut})$	$\mathcal{M}^{\rm BH(NS)}$
10^5	BH	10^{20}	–	2.4×10^{-7}	0
	NS	10^{24}	0	3.7×10^{-15}	0
			$\pi/4$	2.6×10^{-15}	0.9×10^{-4}
			$\pi/2$	3.5×10^{-19}	1
10^8	BH	10^{23}	–	2.4×10^{-4}	0
	NS	10^{27}	0	3.7×10^{-12}	0
			$\pi/4$	2.6×10^{-12}	3.0×10^{-3}
			$\pi/2$	3.3×10^{-14}	1
10^{11}	BH	10^{26}	–	0.24	0
	NS	10^{30}	0	3.5×10^{-9}	0
			$\pi/4$	2.7×10^{-9}	0.86×10^{-1}
			$\pi/2$	3.3×10^{-10}	1

$d_{\rm QS} = 10^{-5}$ pc, $d_{\rm S} = 10$ kpc, $\lambda = 1$ m, $N^{\rm MW}_{e,1} = 0.025$ cm^{-3}, $N^{\rm MW}_{e,2} = 0.2$ cm^{-3}, $A^{\rm MW}_1 = 20$ kpc, and $A^{\rm MW}_2 = 2$ kpc .

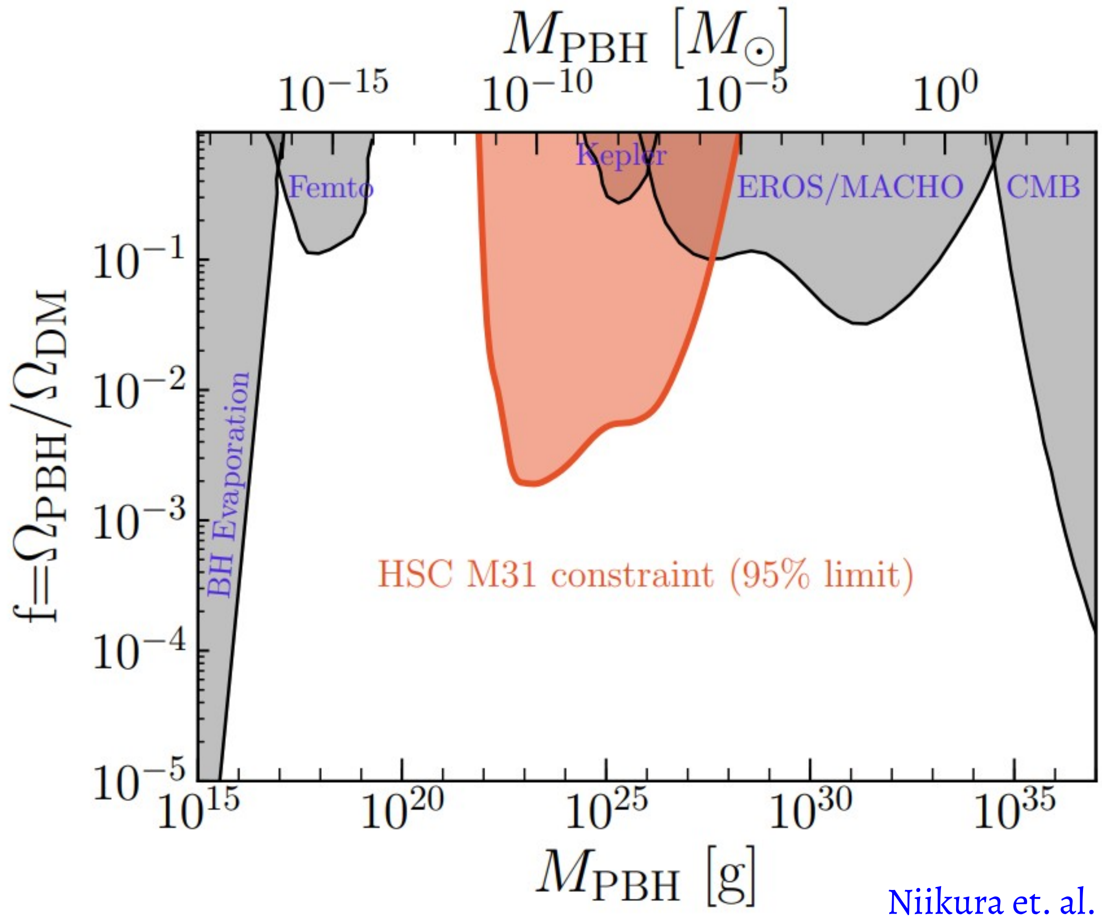
Summary

- MBHs may evade Hawking radiation due to zero hawking temperature, leading to their significant population even for masses, $M_{\rm BH} \lesssim 10^{15}{\rm g}$ which is not possible for Kerr black holes.
- The most stringent constraints on the population of MBHs originate from magnetic fields in cosmic voids $(f_{\rm DM} \lesssim 10^{-8})$ and cosmic web filaments $(f_{\rm DM} \lesssim 10^{-4})$, which are considerably stronger than previous limits set by galactic fields.
- For extremal MBHs with a magnetic charge $\rm Q^{Ex.}_{BH} \gtrsim 10^{22}$ A-m or mass $\rm M^{Ex.}_{BH} \gtrsim 10^{-6}\ M_\odot$, exhibit rotation measure values that are detectable by Earth-based radio telescopes.
- For $r < r^{\rm BH}_{\rm cut}$, the magnetic black hole will be completely opaque to the observer.
- There is an integral measure for Faraday rotation that distinguishes MBHs from other astrophysical sources.

Thank You

Backup Slides

PBH constraint



Niikura et. al. (2019)

$$T^{\rm Sch.} = \frac{1}{8\pi M_{\rm BH}} \,,$$

$$T^{\rm MBH} = \frac{\sqrt{M_{\rm BH}^2 - Q_{\rm BH}^2}}{4\pi\left(M_{\rm BH} + \sqrt{M_{\rm BH}^2 - Q_{\rm BH}^2}\right)^2} \,.$$

For extremal MBH, $M_{\rm BH}^{\rm Ext.} = Q_{\rm BH}$,

$$T^{\rm MBH} \to 0 \,,$$

MBH properties

- Metric: $$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$

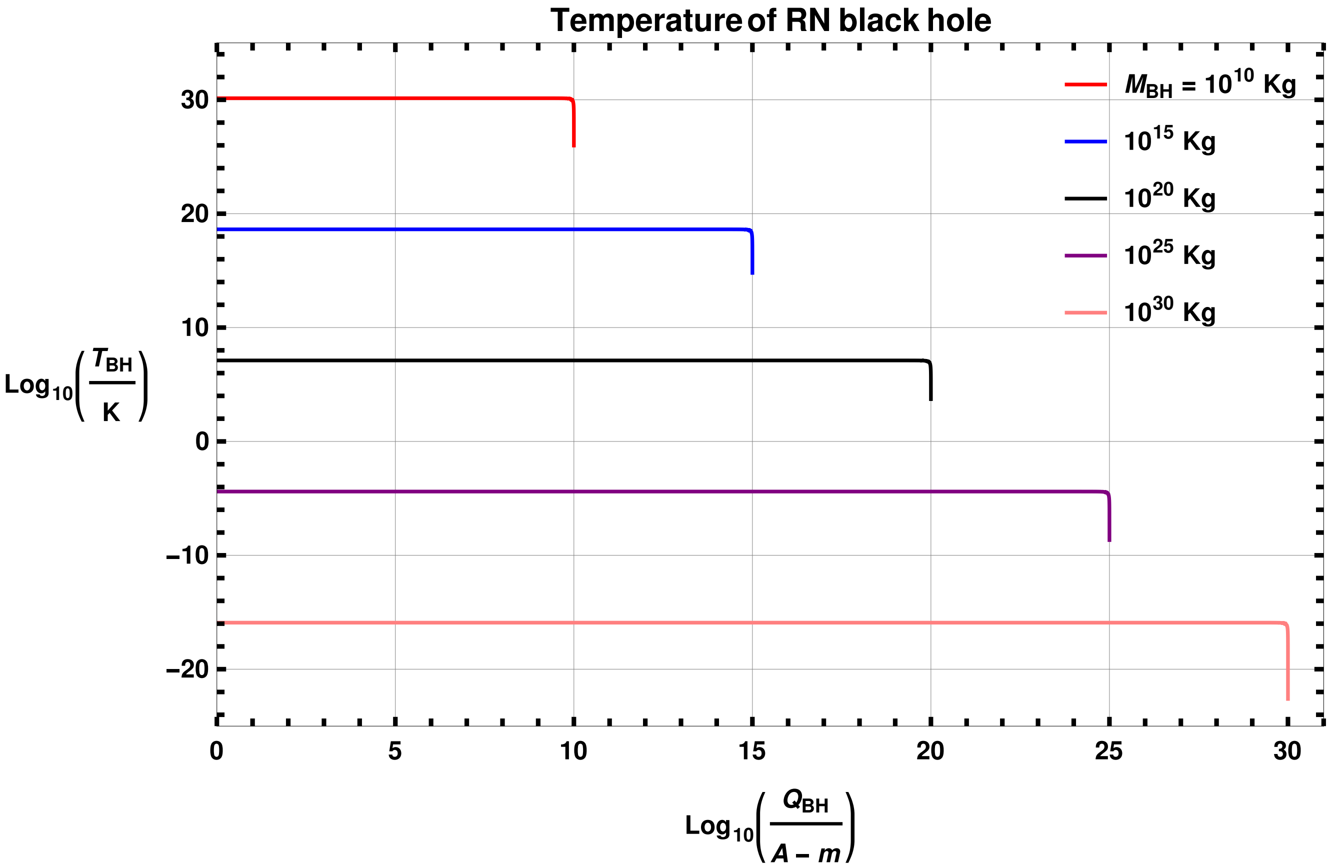
$$f(r) = 1 - \frac{2M_{\text{BH}}}{r} + \frac{Q^2_{\text{m,BH}}}{r^2}$$

- Outer Horizon : $$r^{\text{RN}}_{\text{out.}} = M_{\text{BH}} + \sqrt{M^2_{\text{BH}} - Q^2_{\text{BH}}}$$

- Extremal Condition: $$Q_{\text{BH}} = \sqrt{4\pi G/\mu_0} M_{\text{BH}} (\text{in S.I. units})$$

- Temperature : $$T = \frac{\sqrt{M^2_{\text{BH}} - Q^2_{\text{BH}}}}{2\pi \left(M_{\text{BH}} + \sqrt{M^2_{\text{BH}} - Q^2_{\text{BH}}}\right)^2}$$

MBH Temperature



Parker Bound

- Equation of motion: $$\frac{d\vec{v}}{dt} = \frac{Q_{\rm BH}\vec{B}_{\rm c}}{M_{\rm BH}}$$ $$v_{\rm mag} = \sqrt{\frac{2Q_{\rm BH}l_{\rm c}B_{\rm c}}{M_{\rm BH}}}$$
- Gain in Energy: $$\Delta\mathcal{E}_{\rm k} = Q_{\rm BH}\langle\vec{B}_{\rm c}\cdot\vec{v}_{\rm in}\rangle\Delta t + \frac{Q_{\rm BH}^2B_{\rm c}^2}{2M_{\rm BH}}\Delta t^2$$
- Depletion of fields energy: $$\left|\frac{{\rm d}\mathcal{E}_{\rm field}}{{\rm d}t}\right| = 4\pi l_{\rm c}^2\Delta\mathcal{E}_{\rm k}F_{\rm BH}$$
- Bounds on MBH's flux: $$F_{\rm BH} \leq \frac{B_{\rm c}^2 l_{\rm c}}{6\mu_0\Delta\mathcal{E}_{\rm k}t_{\rm reg}}$$ $$t_{\rm reg} \leq t_{\rm dep}$$
- Bounds on f_{DM} : $$f_{\rm DM} \leq \frac{B_{\rm c}^2 l_{\rm c}M_{\rm BH}}{6\mu_0\Delta\mathcal{E}_{\rm k}t_{\rm reg}v\rho_{\rm DM}}$$ $$F_{\rm BH} = \frac{v\rho_{\rm DM}f_{\rm DM}}{M_{\rm BH}}$$

Parker Bound Cases

- Fast MBHs $v \sim v_{\rm in} \gg v_{\rm mag}$: $\Delta\mathcal{E}_{\rm k}^{\rm fast} \simeq Q_{\rm BH} l_{\rm c} \langle \vec{B}_{\rm c} \cdot \hat{v}_{\rm in} \rangle + \dfrac{Q_{\rm BH}^2 B_{\rm c}^2 l_{\rm c}^2}{2 M_{\rm BH} v_{\rm in}^2}$ $\qquad v_{\rm mag} = \sqrt{\dfrac{2 Q_{\rm BH} l_{\rm c} B_{\rm c}}{M_{\rm BH}}}$

- Fast clustered MBHs : $\Delta\mathcal{E}_{\rm k}^{\rm fast,clust} \simeq \dfrac{Q_{\rm BH}^2 B_{\rm c}^2 l_{\rm c}^2}{2 M_{\rm BH} v_{\rm in}^2}$

  Bounds on f_{DM} : $f_{\rm DM}^{\rm fast,clust} \lesssim \dfrac{M_{\rm BH}^2 v_{\rm in}}{3 \mu_0 Q_{\rm BH}^2 l_{\rm c} t_{\rm reg} \rho_{\rm DM}}$

- Fast unclustered MBHs: $\Delta\mathcal{E}_{\rm k}^{\rm fast,unclust} \simeq Q_{\rm BH} B_{\rm c} l_{\rm c} \cos\alpha + \dfrac{Q_{\rm BH}^2 B_{\rm c}^2 l_{\rm c}^2}{2 M_{\rm BH} v_{\rm in}^2}$

  Bounds on f_{DM} : $f_{\rm DM}^{\rm fast,unclust} \lesssim \dfrac{M_{\rm BH}^2 v_{\rm in}}{3 \mu_0 Q_{\rm BH}^2 l_{\rm c} t_{\rm reg} \rho_{\rm DM}} \dfrac{1}{\left(1 + 4 \frac{v_{\rm in}^2}{v_{\rm mag}^2} \cos\alpha\right)}$

Parker Bounds

- Cosmic voids magnetic field:

  - Primordial or void galaxy : $B^{\text{void, prim}} \gtrsim \mathcal{O}(10^{-15})\,\text{G}\,,\quad l_c^{\text{void, prim}} \sim \mathcal{O}(1-10)\,\text{Mpc}\,,\quad t_{\text{reg}}^{\text{void, prim}} \sim \mathcal{O}(10)\,\text{Gyr}$

    Bounds on f_{DM} : $f_{\text{DM}}^{\text{fast,unclust}} \lesssim 10^{-8}$

  - Galactic flux leakage : $\mathcal{O}(10^{-12})\,\text{G} \lesssim B^{\text{void, out}} \lesssim \mathcal{O}(10^{-8})\,\text{G}\,,\quad l_c^{\text{void, out}} \sim \mathcal{O}(1)\,\text{Mpc}\,,\quad t_{\text{reg}}^{\text{void, out}} \sim \mathcal{O}(10)\,\text{Gyr}\,.$

    Bounds on f_{DM} : $f_{\text{DM}}^{\text{fast,unclust}} \lesssim 10^{-1} - 10^{-5}$

- Cosmic web filaments magnetic fields: $B^{\text{fil}} \sim \mathcal{O}(10^{-9})\,\text{G}\,,\quad l_c^{\text{fil}} \sim \mathcal{O}(1)\,\text{Mpc}\,,\quad t_{\text{reg}}^{\text{fil}} \sim \mathcal{O}(10)\,\text{Gyr}$

  Bounds on f_{DM} : $f_{\text{DM}}^{\text{fast,unclust}} \lesssim 10^{-4}$

Faraday Rotation general

- Equation of motion : $$m_e \frac{d^2\vec{r}_e}{dt^2} = -e\left(\vec{E} + \frac{d\vec{r}_e}{dt} \times \vec{B}\right)$$
- Magnetic Field : $$\vec{\mathcal{B}}_{\text{ext.}}(\vec{r}) = B_{\text{ext,x}}(\vec{r})\ \hat{x} + B_{\text{ext,y}}(\vec{r})\ \hat{y} + B_{\text{ext,z}}(\vec{r})\ \hat{z}$$
- Perturbations : $$\begin{aligned} \vec{r}_e &= \vec{r}_e^{\,(0)} + \vec{r}_e^{\,(1)} e^{-i\omega t} , \\ \vec{E}(\vec{r}) &= 0 + \vec{E}^{\,(1)}(\vec{r}) e^{-i\omega t} , \\ \vec{B}(\vec{r}) &= \vec{\mathcal{B}}_{\text{ext.}}(\vec{r}) + \vec{B}^{\,(1)}(\vec{r}) e^{-i\omega t} , \end{aligned}$$
- EOM after perturbations : $$\frac{e\vec{E}^{\,(1)}}{m_e} = \omega^2 \vec{r}_e^{\,(1)} + \frac{ie\omega}{m_e}\left(\vec{r}_e^{\,(1)} \times \vec{\mathcal{B}}_{\text{ext.}}(\vec{r})\right)$$
- Oscillations amplitude : $$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \frac{e}{m_e\left(\omega^2 - \tilde{\omega}^2\right)} \begin{pmatrix} 1 - \frac{\tilde{\omega}_x^2}{\omega^2} & -\frac{\tilde{\omega}_x\tilde{\omega}_y + i\omega\tilde{\omega}_z}{\omega^2} & \frac{-\tilde{\omega}_x\tilde{\omega}_z + i\omega\tilde{\omega}_y}{\omega^2} \\ \frac{-\tilde{\omega}_x\tilde{\omega}_y + i\omega\tilde{\omega}_z}{\omega^2} & 1 - \frac{\tilde{\omega}_y^2}{\omega^2} & -\frac{\tilde{\omega}_y\tilde{\omega}_z + i\omega\tilde{\omega}_x}{\omega^2} \\ -\frac{\tilde{\omega}_x\tilde{\omega}_z + i\omega\tilde{\omega}_y}{\omega^2} & \frac{-\tilde{\omega}_y\tilde{\omega}_z + i\omega\tilde{\omega}_x}{\omega^2} & 1 - \frac{\tilde{\omega}_z^2}{\omega^2} \end{pmatrix} \begin{pmatrix} E_{1,x} \\ E_{1,y} \\ E_{1,z} \end{pmatrix} .$$

Faraday Rotation general

- Dielectric tensor:

$$\epsilon_{\rm r} = \begin{pmatrix} 1-\frac{\omega_{\rm p}^2(\omega^2-\tilde\omega_{\rm x}^2)}{\omega^2(\omega^2-\tilde\omega^2)} & \frac{\omega_{\rm p}^2(\tilde\omega_{\rm x}\tilde\omega_{\rm y}+i\omega\tilde\omega_{\rm z})}{\omega^2(\omega^2-\tilde\omega^2)} & \frac{\omega_{\rm p}^2(\tilde\omega_{\rm x}\tilde\omega_{\rm z}-i\omega\tilde\omega_{\rm y})}{\omega^2(\omega^2-\tilde\omega^2)} \\ \frac{\omega_{\rm p}^2(\tilde\omega_{\rm x}\tilde\omega_{\rm y}-i\omega\tilde\omega_{\rm z})}{\omega^2(\omega^2-\tilde\omega^2)} & 1-\frac{\omega_{\rm p}^2(\omega^2-\tilde\omega_{\rm y}^2)}{\omega^2(\omega^2-\tilde\omega^2)} & \frac{\omega_{\rm p}^2(\tilde\omega_{\rm y}\tilde\omega_{\rm z}+i\omega\tilde\omega_{\rm x})}{\omega^2(\omega^2-\tilde\omega^2)} \\ \frac{\omega_{\rm p}^2(\tilde\omega_{\rm x}\tilde\omega_{\rm z}+i\omega\tilde\omega_{\rm y})}{\omega^2(\omega^2-\tilde\omega^2)} & \frac{\omega_{\rm p}^2(\tilde\omega_{\rm y}\tilde\omega_{\rm z}-i\omega\tilde\omega_{\rm x})}{\omega^2(\omega^2-\tilde\omega^2)} & 1-\frac{\omega_{\rm p}^2(\omega^2-\tilde\omega_{\rm z}^2)}{\omega^2(\omega^2-\tilde\omega^2)} \end{pmatrix} .$$

$$\vec P \equiv -\mathcal{N}_{\rm e} e \vec r_{\rm e}^{\,(1)}$$

$$\vec P = \epsilon_0(\epsilon_{\rm r} - I)\cdot \vec E^{\,(1)}$$

- Using Maxwell's equation :

$$\left(\nabla\cdot\vec E^{\,(1)}\right) - \nabla^2\vec E^{\,(1)} - \frac{\omega^2}{c^2}\epsilon_{\rm r}\cdot\vec E^{\,(1)} = 0$$

$$\vec E^{\,(1)}(\vec r) \propto e^{i\psi_{\rm ph.}(\vec r)}$$

$$\left(\vec k\cdot\vec E^{\,(1)}\right)\vec k - k^2\vec E^{\,(1)} + \frac{\omega^2}{c^2}\epsilon_{\rm r}\cdot\vec E^{\,(1)} \approx 0$$

$$\vec k = \nabla\psi_{\rm ph.}(\vec r)$$

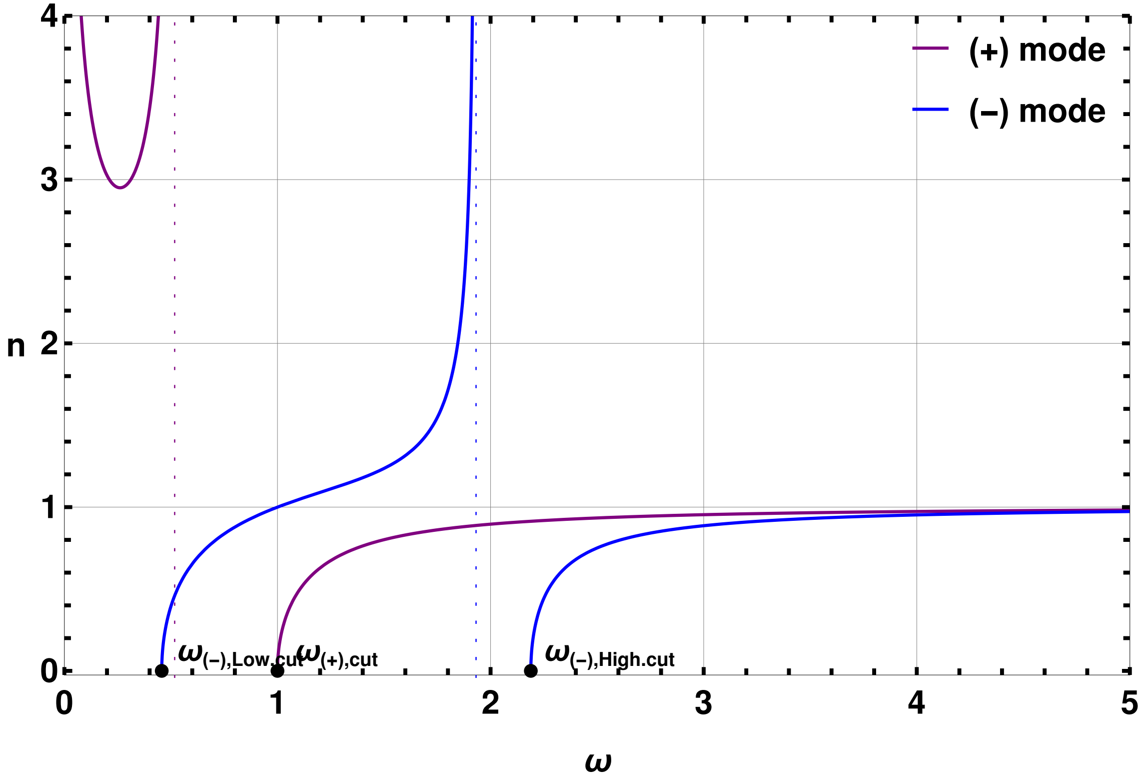
- Phase of electric field:

$$\psi_{\rm ph.}(\vec r) = \int dz\ k(\vec r) = \frac{c}{\omega}\int dz\ n(\vec r)$$

- Refractive index :

$$n_{(\pm)}(\vec r) = \left(1 - \frac{\omega_{\rm p}^2\left(\omega^2-\omega_{\rm p}^2\right)}{\omega^2\left(\omega^2-\omega_{\rm p}^2-\frac{1}{2}\left(\tilde\omega_{\rm x}^2+\tilde\omega_{\rm y}^2\right)\pm\left(\frac{1}{4}\left(\tilde\omega_{\rm x}^2+\tilde\omega_{\rm y}^2\right)^2+(\omega^2-\omega_{\rm p}^2)^2\frac{\tilde\omega_{\rm z}^2}{\omega^2}\right)^{1/2}\right)}\right)^{1/2}$$

Faraday Rotation (mode cut offs)



$$\omega > \omega_{(-),\text{High. cut.}} = \frac{\tilde{\omega}}{2} + \frac{1}{2}\sqrt{\tilde{\omega}^2 + 4\omega_{\text{p}}^2}$$

Faraday Rotation general

- Characteristics modes: $$\left(\frac{E_{\mathrm{x}}^{(1)}}{E_{\mathrm{y}}^{(1)}}\right)_{(\pm)} = \frac{i\left(\omega(\tilde{\omega}_{\mathrm{x}}^2-\tilde{\omega}_{\mathrm{y}}^2)\pm\sqrt{4\tilde{\omega}_{\mathrm{z}}^2\left(\omega^2-\omega_{\mathrm{p}}^2\right)^2+\omega^2\left(\tilde{\omega}_{\mathrm{x}}^2+\tilde{\omega}_{\mathrm{y}}^2\right)^2}\right)}{2\left((\omega^2-\omega_{\mathrm{p}}^2)\tilde{\omega}_{\mathrm{z}}+i\omega\tilde{\omega}_{\mathrm{x}}\tilde{\omega}_{\mathrm{y}}\right)}$$

- Left/Right circular polarization condition i.e. $\mathrm{E}_{\mathrm{x}}^{(1)}/E_{\mathrm{y}}^{(1)} \simeq \pm i\,\mathrm{sgn}(\tilde{\omega}_{\mathrm{z}})$

$$\frac{\omega\left(\tilde{\omega}_{\mathrm{x}}^2+\tilde{\omega}_{\mathrm{y}}^2\right)}{2(\omega^2-\omega_{\mathrm{p}}^2)\tilde{\omega}_{\mathrm{z}}} \ll 1$$

- Refractive index : $$n_{\mathrm{L(R)}} = \begin{cases} n_{+(-)}\ ; & \tilde{\omega}_{\mathrm{z}} > 0\ , \\ n_{-(+)}\ ; & \tilde{\omega}_{\mathrm{z}} < 0\ . \end{cases}$$

- Polarisation angle and RM measure:

$$\psi_{\mathrm{pol.}} \equiv \tan^{-1}\left(\vec{E}_{\mathrm{tot.,y}}^{(1)}/\vec{E}_{\mathrm{tot.,x}}^{(1)}\right) = \frac{1}{2}\left(\psi_{\mathrm{ph.\ (L)}}(\vec{r}) - \psi_{\mathrm{ph.\ (R)}}(\vec{r})\right) = \frac{\omega}{2c}\int dz\,\left(n_{\mathrm{L}}(\vec{r}) - n_{\mathrm{R}}(\vec{r})\right)$$

$$\mathrm{RM}(\lambda) \equiv \frac{d\psi_{\mathrm{pol.}}(\lambda)}{d\lambda^2}\ ,$$

Faraday Rotation general

- Expansion of polarisation angle in the limit $\omega \gg \max\left(\omega_{\mathrm{p}}, \tilde{\omega}\right)$:

$$\begin{aligned}\psi_{\mathrm{pol.}} \quad &\simeq \quad \frac{e^3\lambda^2}{8\pi^2\epsilon_0 m_{\mathrm{e}}^2 c^3}\int dz\ \mathcal{N}_{\mathrm{e}}(r)B_{\mathrm{z}}(\vec{r}) \\ &+\frac{e^5\lambda^4}{32\pi^4\epsilon_0 m_{\mathrm{e}}^4 c^5}\int dz\ \mathcal{N}_{\mathrm{e}}(r)B_{\mathrm{z}}(\vec{r})\left(B_{\mathrm{x}}(\vec{r})^2+B_{\mathrm{y}}(\vec{r})^2+\frac{B_{\mathrm{x}}(\vec{r})^2B_{\mathrm{y}}(\vec{r})^2}{4B_{\mathrm{z}}(\vec{r})^2}\right)\ .\end{aligned}$$

- Expansion of RM measure in the limit $\omega \gg \max\left(\omega_{\mathrm{p}}, \tilde{\omega}\right)$:

$$\begin{aligned}\mathrm{RM}(\lambda) \quad &\simeq \quad \frac{e^3}{8\pi^2\epsilon_0 m_{\mathrm{e}}^2 c^3}\int dz\ \mathcal{N}_{\mathrm{e}}(r)B_{\mathrm{z}}(\vec{r}) \qquad (1)\\ &+\frac{e^5\lambda^2}{16\pi^4\epsilon_0 m_{\mathrm{e}}^4 c^5}\int dz\ \mathcal{N}_{\mathrm{e}}(r)B_{\mathrm{z}}(\vec{r})\left(B_{\mathrm{x}}(\vec{r})^2+B_{\mathrm{y}}(\vec{r})^2+\frac{B_{\mathrm{x}}(\vec{r})^2B_{\mathrm{y}}(\vec{r})^2}{4B_{\mathrm{z}}(\vec{r})^2}\right)\ .\end{aligned}$$

Faraday Rotation due to MBH

- Magnetic field : $\vec{B}_{\rm BH}(\vec{r}) \equiv \frac{\mu_0}{4\pi}\frac{Q_{\rm BH}}{r^2}\hat{r}$ $\qquad \tilde{\omega}_{\rm BH}(r) \equiv \frac{e}{m_{\rm e}} B_{\rm BH}(r)$

- Refractive index :

$$n^{\rm BH}_{\rm L,(R)} = \begin{cases} \left(1 - \frac{\omega_{\rm p}^2(\omega^2-\omega_{\rm p}^2)}{\omega^2\left(\omega^2-\omega_{\rm p}^2-\frac{1}{2}\tilde{\omega}_{\rm BH}^2\sin^2\theta \pm\left(\frac{1}{4}\tilde{\omega}_{\rm BH}^4\sin^4\theta+(\omega^2-\omega_{\rm p}^2)^2\frac{\tilde{\omega}_{\rm BH}^2}{\omega^2}\cos^2\theta\right)^{1/2}\right)}\right)^{1/2} , & 0<\theta<\pi/2 \, , \\ \left(1 - \frac{\omega_{\rm p}^2(\omega^2-\omega_{\rm p}^2)}{\omega^2\left(\omega^2-\omega_{\rm p}^2-\frac{1}{2}\tilde{\omega}_{\rm BH}^2\sin^2\theta \mp\left(\frac{1}{4}\tilde{\omega}_{\rm BH}^4\sin^4\theta+(\omega^2-\omega_{\rm p}^2)^2\frac{\tilde{\omega}_{\rm BH}^2}{\omega^2}\cos^2\theta\right)^{1/2}\right)}\right)^{1/2} , & \pi/2<\theta<\pi \, . \end{cases}$$

- Cut off radius :

$$r^{\rm BH}_{\rm cut} \simeq \sqrt{\frac{\mu_0 e Q_{\rm BH}}{4\pi m_{\rm e}}\frac{\omega}{(\omega^2-\omega_{\rm p}^2)}}$$

Faraday Rotation due to an MBH in constant density plasma

- Expansion of polarisation angle in the limit $\omega \gg \max\left(\omega_{\rm p}, \tilde{\omega}\right)$:

$$\psi^{\rm BH}_{\rm pol.} \simeq -\frac{e^3 Q_{\rm BH} N^{\rm MW}_{\rm e,0} \lambda^2}{32\pi^3 \epsilon_0 m_{\rm e}^2 c^3} \left(\frac{1}{\left(\xi^2 + d_{\rm Q}^2\right)^{1/2}} - \frac{1}{\left(\xi^2 + d_{\rm QS}^2\right)^{1/2}} \right) .$$

- Expansion of RM measure in the limit $\omega \gg \max\left(\omega_{\rm p}, \tilde{\omega}\right)$:

$$\mathrm{RM}(\lambda) \simeq -\frac{e^3 Q_{\rm BH} N^{\rm MW}_{\rm e,0}}{32\pi^3 \epsilon_0 m_{\rm e}^2 c^3} \left(\frac{1}{\left(\xi^2 + d_{\rm Q}^2\right)^{1/2}} - \frac{1}{\left(\xi^2 + d_{\rm QS}^2\right)^{1/2}} \right) .$$