Locating the Critical Point in the QCD Phase Diagram with Heavy-Ion Collisions

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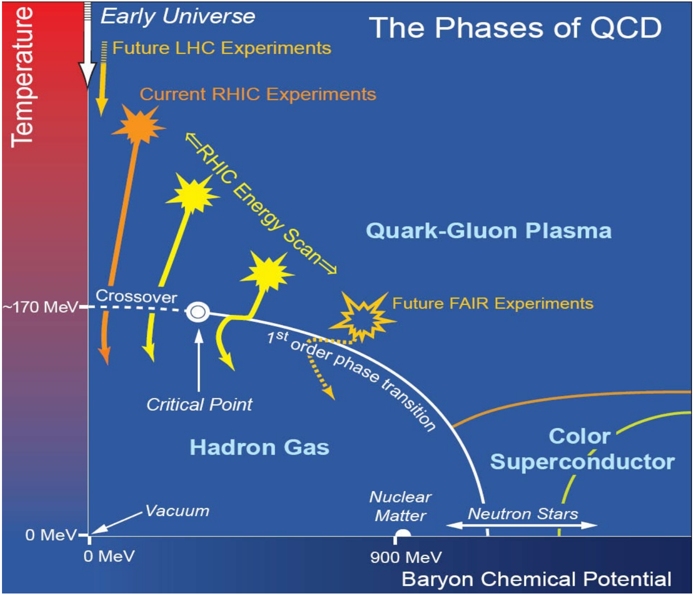
60’th Karpacz Winter School on Theoretical Physics
Wilhelm and Else Heraeus Physics School

Phases of QCD



Purpose and Goals

• Many model calculations predict the existence of a critical point in the QCD phase diagram at a value of the chemical potential where current lattice simulations are unreliable.

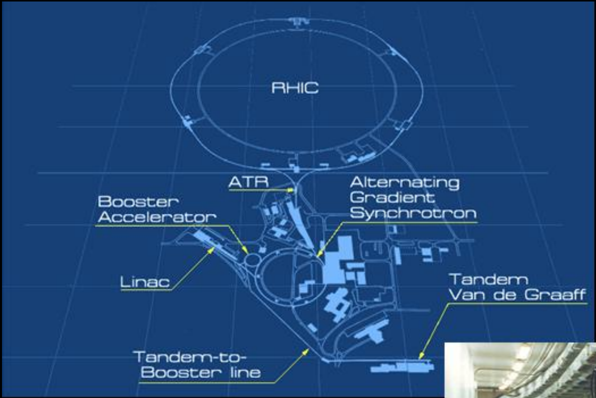
• How to combine or merge a critical equation of state with a smooth background is a long-standing problem in statistical physics with no unique solution.

• Our goal is to construct an equation of state in the same universality class as the liquid–gas phase transition and the 3D Ising model. It should have parameters which may be inferred by hydrodynamic modeling of heavy ion collisions in the Beam Energy Scan II at the Relativistic Heavy Ion Collider or in experiments at other accelerators.

• Such an equation of state is also needed for modeling neutron star mergers and closely related to the cold dense matter comprising neutron stars.

• We provide two very different mathematical constructions.

Relativistic Heavy Ion Collider



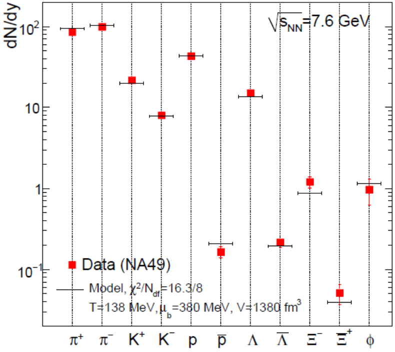
Colliding beams of 100 GeV/nucleon gold nuclei to create quark-gluon plasma.

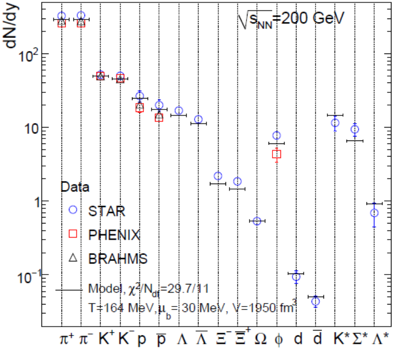
• The injection energy is 9.8 GeV/nucleon. Gold nuclei can be accelerated up to 100 GeV/nucleon ($\sqrt{s_{NN}}$ = 200 GeV).

• The Beam Energy Scan II ran at the injection energy. In addition, the energy was reduced to 7.3, 5.75, 4.59, and 3.85 GeV/nucleon, a very impressive feat.

• It will take up to 10 years to analyze all of the data.

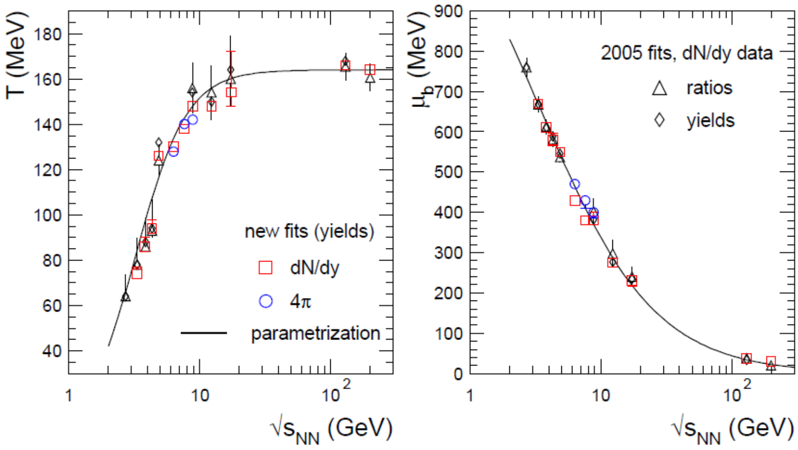
• There will be future programs at the Facility for Anti-proton and Ion Research (FAIR) at GSI and at the Japan Proton Accelerator Research Complex (J-PARC-HI). The status of the Nuclotron-based Ion Collider fAcility (NICA) at the Joint Institute for Nuclear Reseach (JINR) in Dubna is unknown.





Andronic, Braun-Munzinger, and Stachel, Acta Phys. Polon. B (2009)

Statistical Model Fits



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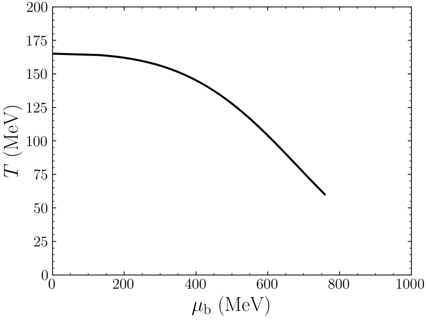
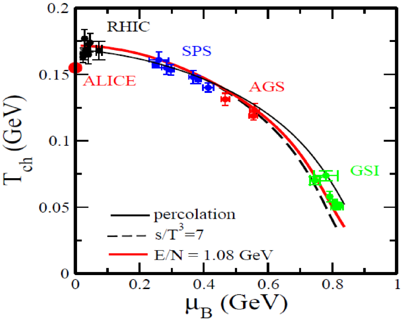




Andronic, Braun-Munzinger, and Stachel (2009) left panel
Castorina, J. Phys. Conf. Ser. (2016) right panel

Construction I (JK, Tom Welle, Chris Plumberg)

- Motivated by S-shaped curves in first order phase transitions and the cubic equation

$$Q_\pm(T,\mu) = \left\{ \left[(\Delta^2(T))^2 + r^2(T,\mu) \right]^{1/2} \pm r(T,\mu) \right\}^k$$

$$r(T,\mu) = \frac{\mu^4 - \mu_x^4(T)}{\mu^4 + \mu_x^4(T)} \qquad \Delta^2(T) \sim d_\pm |T/T_c - 1|^p \ \text{ for } \ T \to T_c^\pm$$

- Only two exponents k and p

$$P(T,\mu) = P_{BG}(T,\mu) R(T,\mu)$$

- For $T \geq T_c$

$$R(T,\mu) = 1 - a(T)\left(\sqrt{\Delta^4+1}+1\right)^k - a(T)\left(\sqrt{\Delta^4+1}-1\right)^k + a(T)(Q_+ + Q_-)$$

- For $T \leq T_c$ and $\mu \leq \mu_x(T)$

$$R_H = 1 + a(T) Q_-(T,\mu) - a(T)\left(\sqrt{\Delta^4+1}+1\right)^k$$

- For $T \leq T_c$ and $\mu \geq \mu_x(T)$

$$R_Q = 1 + a(T) Q_+(T,\mu) - a(T)\left(\sqrt{\Delta^4+1}+1\right)^k$$

• As $n \to n_c$ along the critical isotherm

$$P - P_c \sim \mathrm{sgn}(n - n_c)|n - n_c|^{\delta}\,, \quad \delta = 1/(k-1)$$

• As $t = (T - T_c)/T_c \to 0^+$ the susceptibility and heat capacity are

$$\chi_B \to \chi_+ t^{-\gamma}\,, \quad \gamma = (2-k)p$$

$$c_V \to c_+ t^{-\alpha}\,, \quad \alpha = 2 - kp$$

• As $t \to 0^-$ the susceptibility, heat capacity and density difference along the coexistence curve are

$$\chi_B \to \chi_- (-t)^{-\gamma}$$

$$c_V \to c_- (-t)^{-\alpha}$$

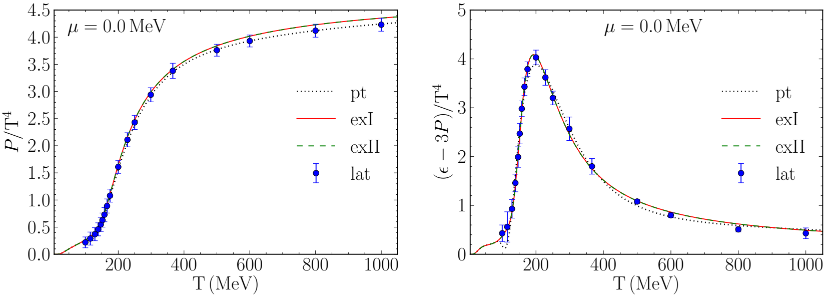
$$\Delta n \sim (-t)^{\beta}\,, \quad \beta = (k-1)p$$

• The critical exponents automatically satisfy the known relations $\alpha + 2\beta + \gamma = 2$ and $\gamma = \beta(\delta - 1)$.

• Predicts relation between universal ratios of critical amplitudes

$$\left(\frac{c_+}{c_-}\right)^{2-k} = 4\left(\frac{\chi_-}{\chi_+}\right)^{k} = 2^{2-k}\left(\frac{d_+}{d_-}\right)^{(2-k)k}$$

The background equation of state uses a switching function to transition smoothly from a hadron resonance gas, with excluded volume interactions, to a perturbative quark–gluon plasma. Two parameters in the QCD running coupling, two in the switching function, and an excluded volume parameter are adjusted and fixed by fitting to lattice QCD at $\mu = 0$.



Albright, Kapusta, and Young, Phys. Rev. C (2014)
Borsányi *et al.* JHEP (2010)

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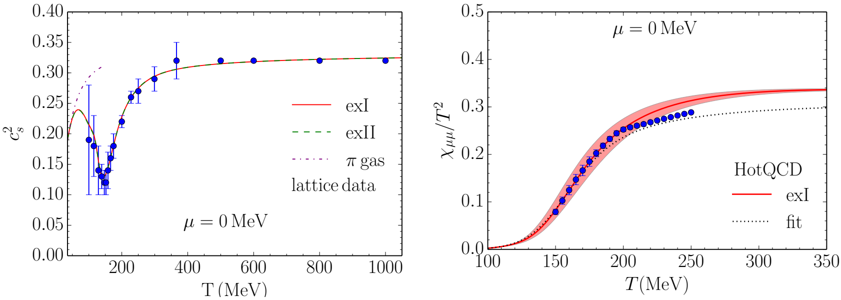


Albright, Kapusta, and Young, Phys. Rev. C (2015)

Borsányi *et al.* JHEP (2012) left panel

Borsányi *et al.* Phys. Rev. Lett. (2013) and Bazavov *et al.* Phys. Rev. D (2012) right panel

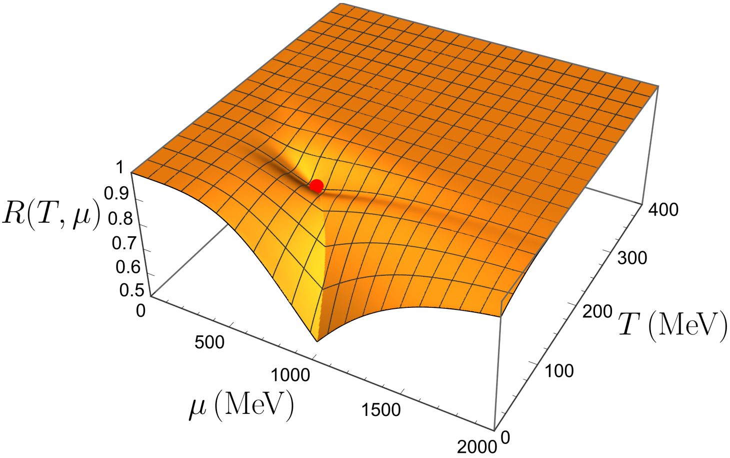
• In order to have an inverted U-shaped coexistence curve in the $T-n$ plane, as seen in the argon and carbon dioxide liquid-gas phase transitions, the function $\mu_x(T)$ is determined by $R(T,\mu_x(T))n_{BG}(T,\mu_x(T)) = n_c$.

• The critical parameters T_c, μ_c, n_c are related by $R(T_c,\mu_c)n_{BG}(T_c,\mu_c) = n_c$.

• 3D Ising model exponents give $k = 1.209, p = 1.564$ (mean field values are $k = 4/3, p = 3/2$). Then ratios of critical amplitudes give $d_+/d_- \approx 1/3$ (mean field value is $d_+/d_- = 1$).

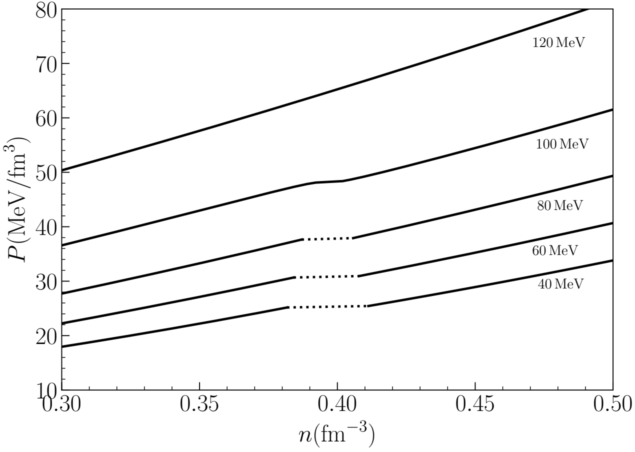
$$a(T) = a_0 \exp(-T/T_a)$$
$$\Delta^2(T) = d_+(T/T_c - 1)^p \exp(-T/T_d) \quad T \geq T_c$$
$$\Delta^2(T) = d_-(1 - T/T_c)^p \exp(-T/T_d) \quad T \leq T_c$$

$P(T,\mu) = P_{BG}(T,\mu)R(T,\mu)$ I (JK, TW, CP)



$T_c = 100$ MeV, $\mu_c = 750$ MeV, $n_c \approx 0.4\ \mathrm{fm}^{-3}$

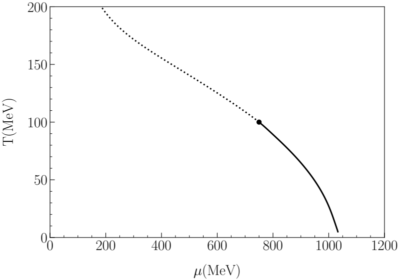
Isotherms of Pressure versus Density I (JK, Tom Welle, Chris Plumberg)

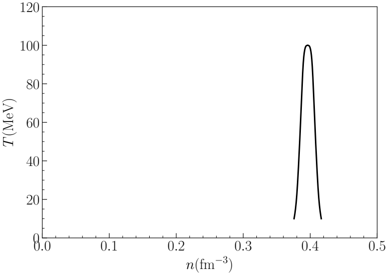


$T_c = 100$ MeV, $\mu_c = 750$ MeV, $n_c \approx 0.4$ fm^{-3}

Coexistence Curve I (JK, Tom Welle, Chris Plumberg)

$$R(T, \mu_x(T))n_{BG}(T, \mu_x(T)) = n_c$$





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Construction II (JK, Tom Welle)

- Adopt the Schofield parametric scaling equation of state (1969)

$$\text{temperature} \quad \frac{T - T_c}{T_c} = R(1 - \theta^2)$$

$$\text{magnetization} \quad M \to \frac{n - n_c}{n_c} = m_0 R^\beta \theta$$

$$\text{magnetic field} \quad H \to \frac{\mu - \mu_c}{\mu_c} = h_0 R^{\beta\delta} h(\theta)$$

$$h(\theta) = \theta(1 + h_3\theta^2 + h_5\theta^4)$$

$R \geq 0$ and $-\theta_0 \leq \theta \leq \theta_0$ where $h(\theta_0) = 0$ with $\theta_0 > 1$

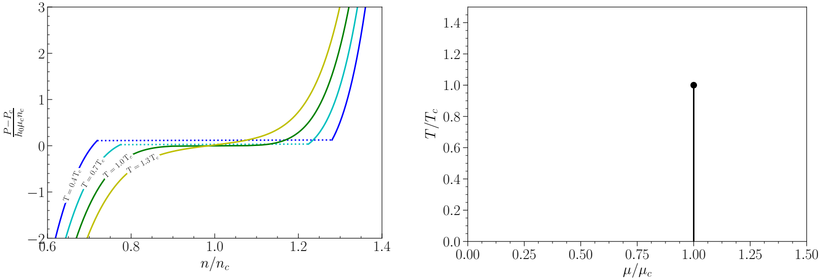
- The pressure must satisfy the condition $(\partial P/\partial\mu)_T = n$ implying

$$P = P_c + [\mu(R,\theta) - \mu_c]n(R,\theta) - m_0 h_0 \mu_c n_c R^{2-\alpha} g(\theta)$$

where $g(\theta)$ is determined by $h(\theta)$

- Critical curve is $\theta = \pm\theta_0$ and critical point is at $R = 0$
- h_3 and h_5 are determined by ratio of critical amplitudes
- h_0 and m_0 are positive parameters

Scaling equation of state only



$$m_0 = 0.2$$

• Modify the scaling variables

$$\frac{n - n_c}{n_c} = m_0 R^\beta \theta$$

$$\frac{\mu - \mu_x(T)}{\mu_c} = h_0 R^{\beta\delta} h(\theta)$$

This maintains the density as the order parameter

• Pressure

$$P(\mu, T) = P_{BG}(\mu, T) + W(\mu, T) P_*(R, \theta)$$

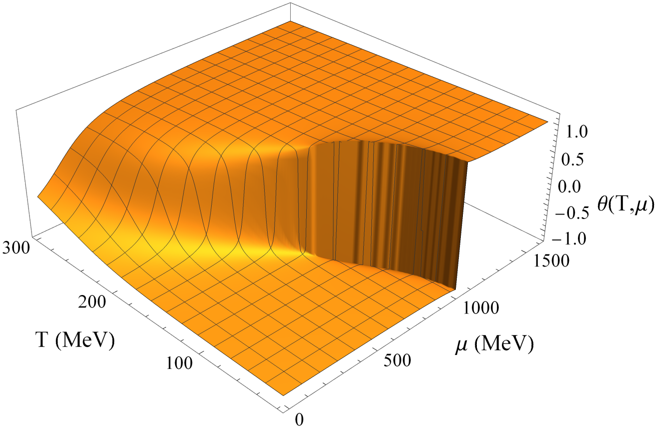
$$P_*(R, \theta) = P_0 + h_0 \mu_c n_0 R^{\beta\delta} h + m_0 h_0 \mu_c n_0 R^{2-\alpha} \left[\theta h(\theta) - g(\theta)\right]$$

• $W(\mu, T)$ is a window function that suppresses the critical contribution away from the coexistence curve

• Background is same as before except for use of point hadrons

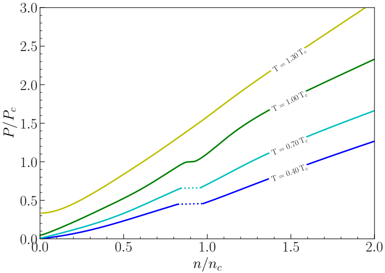
• In order to have an inverted U-shaped coexistence curve in the $T - n$ plane, as seen in the argon and carbon dioxide liquid-gas phase transitions, the function $\mu_x(T)$ is determined by $n_{BG}(\mu_x(T), T) = n_c - n_0$.

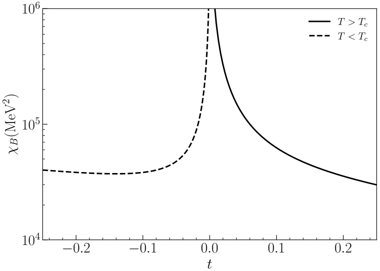
$\theta(T, \mu)$ II (JK, Tom Welle)



$T_c = 120$ MeV, $\mu_c = 750$ MeV, $n_c \approx 1.3$ fm^{-3}

Pressure and Susceptibility II (JK, Tom Welle)





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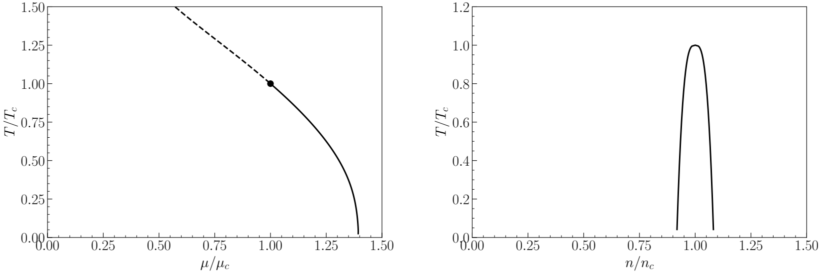
Coexistence Curve II (JK, Tom Welle)

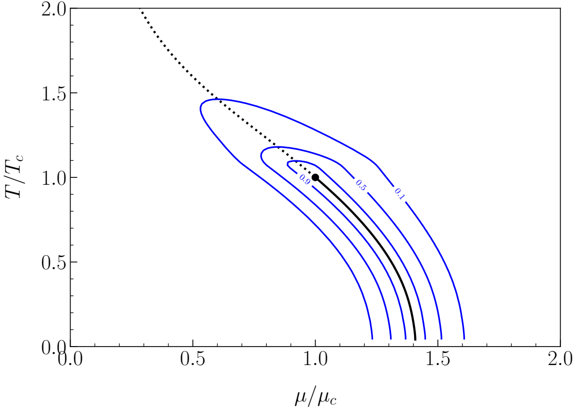
$$n_{BG}(\mu_x(T), T) = n_c - n_0$$



$$T_c = 120 \text{ MeV}, \mu_c = 750 \text{ MeV}, n_c \approx 1.3 \text{ fm}^{-3}$$

Window Function II (JK, Tom Welle)

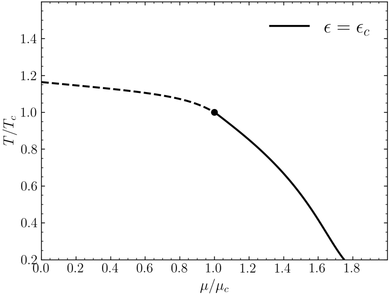
$$W(\mu, T) = \exp\left[-\left(\frac{\mu^{2j} - \mu_x^{2j}(T)}{c_* \mu_c^j \mu^j}\right)^2\right] \{1 - \exp[-(t_0/t)^2]\}$$

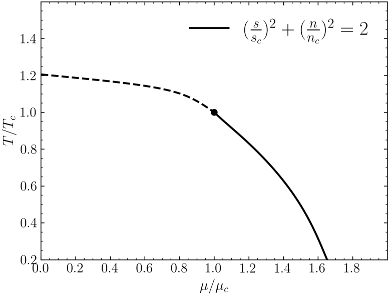


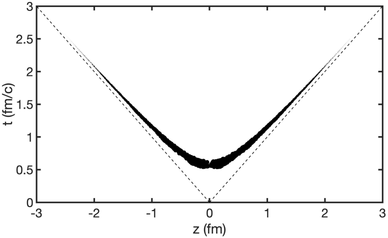
$T_c = 120$ MeV, $\mu_c = 750$ MeV, $n_c \approx 1.3$ fm^{-3}

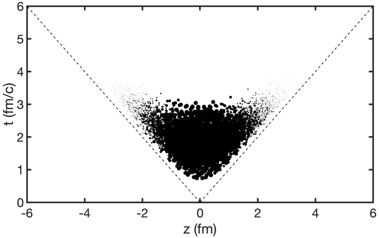
Coexistence Curve II (JK, Shensong Wan)

$\mu_x(T)$ determined by other conditions









$\sqrt{s_{NN}} = 200$ GeV (left) and $\sqrt{s_{NN}} = 11.5$ GeV (right).

Dynamics of the Transition in Heavy Ion Collisions (JK, Mayank Singh, in preparation)

• Nucleation is probably too slow for the Maxwell construction to be used. Spinodal decomposition is a more general and appropriate approach.

• Helmholtz free energy

$$F\{n(\mathbf{x},t)\} = \int d^3x \left[\tfrac{1}{2}K(\boldsymbol{\nabla} n)^2 + f(T,n)\right] \equiv \int d^3x \tilde{f}(T,n)$$

• Local chemical potential $\tilde{\mu} = \delta F/\delta n = \mu - K\nabla^2 n$

• Local isotropic pressure $\tilde{P} = P - Kn\nabla^2 n - \frac{1}{2}K(\boldsymbol{\nabla} n)^2$

• Entropy density $s = -\partial f/\partial T$

• Local energy density $\tilde{\epsilon} = \epsilon + \frac{1}{2}K(\boldsymbol{\nabla} n)^2$

• Local enthalpy density $\tilde{w} = \tilde{\epsilon} + \tilde{P} = w - Kn\nabla^2 n$

• Surface free energy $\sigma = K\int_{-\infty}^{\infty} dx \left(\frac{dn}{dx}\right)^2 = \frac{K\Delta n^2}{6\xi}$

• Correlation length $\xi^2 = \frac{2K}{\alpha\Delta n^2}$

Dynamics of the Transition in Heavy Ion Collisions (JK, Mayank Singh, in preparation)

- Stress–Energy–Momentum tensor

  $$T^{\mu\nu} = \tilde{P}(u^\mu u^\nu - g^{\mu\nu}) + \tilde{\epsilon} u^\mu u^\nu + K(D^\mu n)(D^\nu n)$$

- Gradient orthogonal to the flow velocity $D^\mu n \equiv \partial^\mu n - u^\mu u^\alpha \partial_\alpha n$
- Baryon current $J^\mu = n u^\mu + \sigma_B T D^\mu \left(\frac{\tilde{\mu}}{T}\right)$
- Baryon conductivity $\sigma_B = \frac{C_B}{3T}\left[\coth\left(\frac{\mu_B}{T}\right) - \frac{3Tn}{w}\right] n$
- Covariant expressions

  $$\tilde{\mu} = \mu + K D^2 n$$

  $$\tilde{P} = P + K n D^2 n + \frac{1}{2} K (D^\mu n)(D_\mu n)$$

  $$\tilde{\epsilon} = \epsilon - \frac{1}{2} K (D^\mu n)(D_\mu n)$$

  $$\tilde{w} = \tilde{P} + \tilde{\epsilon} = Ts + \tilde{\mu} n = w + K n D^2 n$$

1+1 Dimensional Non-Boost Invariant Hydrodynamic Flow (JK, Mayank Singh, in preparation)

• Space–time variables $t = \tau \cosh \xi$ and $z = \tau \sinh \xi$

• Dynamical equations to be solved numerically with some initial conditions

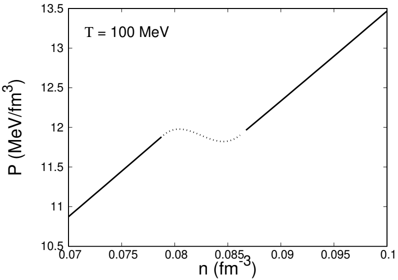
$$\frac{\partial \epsilon(n,T)}{\partial \tau} + \frac{w(n,T)}{\tau} + \frac{K}{\tau^2} \frac{\partial n}{\partial \xi} \frac{\partial^2 n}{\partial \tau \partial \xi} - \frac{K}{\tau^3} n \frac{\partial^2 n}{\partial \xi^2} = 0$$

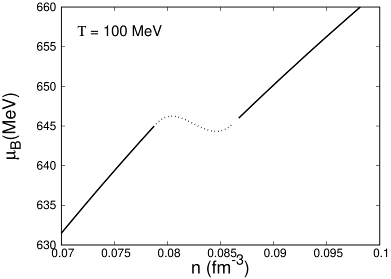
$$\frac{\partial}{\partial \tau}(\tau n) - \frac{\sigma_B T}{\tau} \frac{\partial^2}{\partial \xi^2} \left(\frac{\tilde{\mu}}{T} \right) - \frac{1}{\tau} \frac{\partial}{\partial \xi}(\sigma_B T) \frac{\partial}{\partial \xi} \left(\frac{\tilde{\mu}}{T} \right) = 0$$

$$\tilde{\mu} = \mu(n,T) - \frac{K}{\tau^2} \frac{\partial^2 n}{\partial \xi^2}$$

• Challenging to solve due to 4'th order derivatives. 3+1 dimensions even more challenging!

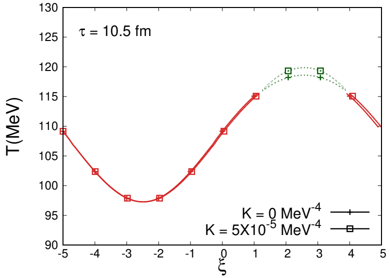
1+1 Dimensional Non-Boost Invariant Hydrodynamic Flow (JK, Mayank Singh, in preparation)

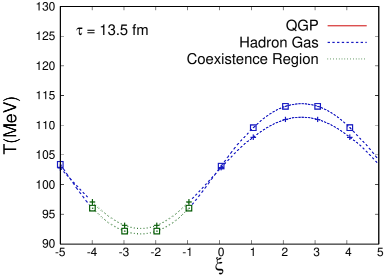




Isotherms of pressure and chemical potential.

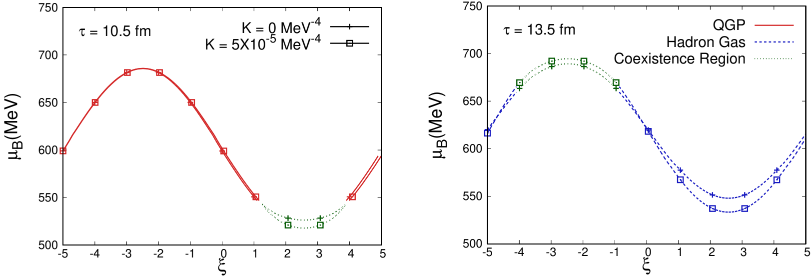
1+1 Dimensional Non-Boost Invariant Hydrodynamic Flow (JK, Mayank Singh, in preparation)



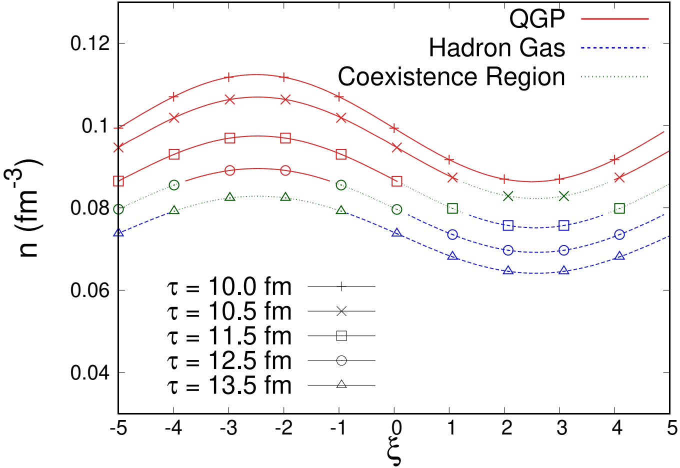


Temporal evolution of the temperature as a function of space-time rapidity (position). Non-boost invariance is necessary to get the spinodal dynamics started.

1+1 Dimensional Non-Boost Invariant Hydrodynamic Flow (JK, Mayank Singh, in preparation)



Temporal evolution of the chemical potential as a function of space-time rapidity (position). Non-boost invariance is necessary to get the spinodal dynamics started.



Temporal evolution of the baryon density as a function of space-time rapidity (position). Non-boost invariance is necessary to get the spinodal dynamics started.

Conclusion

• Lattice QCD simulations have shown unequivocally that the transition from hadrons to quarks and gluons is a crossover when the baryon chemical potential is zero or small. Using two different constructions, we show how to embed a critical point in a smooth background equation of state so as to yield the critical exponents and critical amplitude ratios expected of a transition in the same universality class as the liquid–gas phase transition and the 3D Ising model.

• Apart from the critical exponents and ratios of critical amplitudes (which are universal) and T_c and μ_c, construction I has 4 parameters while construction II has 6.

• The parameters might be inferred by hydrodynamic modeling of heavy ion collisions in the Beam Energy Scan II at the Relativistic Heavy Ion Collider or in experiments at other accelerators.

• With more realistic nuclear interactions, the equations of state may be used to model neutron star mergers.

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