The Statistical Mechanics of Swarm Formation

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Two physicists are having coffee on the Piazza San Marco in Venice. The Piazza is densely populated by pigeons, moving here and there in a random way, picking up bits of food.

The set of pigeons has rotational symmetry.

Suddenly there is a loud bang, the pigeons all fly straight up and then depart in a big swarm in one direction.

The rotational symmetry is spontaneously broken.

The physicists protest: the birds are not allowed to do that, the Mermin-Wagner theorem forbids it!

A continuous symmetry in 2D cannot be spontaneously broken.

How did the pigeons manage to get around Mermin-Wagner?

$\rightarrow$ are birds smarter then nerds?

The physicists protest: the birds are not allowed to do that, the Mermin-Wagner theorem forbids it:

A continuous symmetry in 2D cannot be spontaneously broken.

How did the pigeons manage to get around Mermin-Wagner?

$\rightarrow$ are birds smarter then nerds?

The answer was given 1995 by Tamás Vicsek and collaborators, in Phys. Rev. Lett. 75 (1995) 1226 (> 4500 citations):

NOVEL TYPE OF PHASE TRANSITION IN A SYSTEM

OF SELF-DRIVEN PARTICLES

Tamás Vicsek, András Czirók, Eshel Ben-Jacob, Inon Cohen

and Ofer Shochet

That will be the subject of this talk.

Recall critical behavior in 2D spin systems

- Ising model: N^2 lattice, $s_i = \pm 1$, $N \to \infty$, n.n. interaction

$$\mathcal{H} = -J \sum_{\{i,j\}} s_i s_j - H \sum_i s_i;$$

$$Z(T,H,N) = \prod_{i=1}^{N^2} \sum_{s_i=\pm 1} \exp\{-\mathcal{H}/\mathcal{T}\}$$

aligned spins energetically favored; for $H = 0$ with decreasing T continuous transition from disordered (paramagnetic) to ordered (ferromagnetic) state; order parameter

$$m(T,N) = \frac{1}{Z(T,N)} \prod_{i=1}^{N^2} \sum_i \left[\frac{\sum_i s_i}{N^2}\right] \exp\{-(J/T) \sum_{i,j}^{nn} s_i s_j\}$$

defines transition point $T = T_c = 2J/\ln[1+\sqrt{2}]$

$$m(T) = \begin{cases} (1-(T/T_c))^\beta & \forall\, T < T_c \\ 0 & \forall\, T > T_c \end{cases}$$

NB: interaction range vs. correlation range

interaction range: nearest neighbor interactions

correlation range: define $\Gamma_{i,j}(t, H = 0) = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$

average over all i, j with $i - j = r$ to get

$$\text{correlation function} \quad \Gamma(r, T) \sim \frac{e^{-r/\xi}}{r^p}$$

with correlation length $\xi \sim |1 - (T/T_c)|^{-\nu} \gg$ interaction range.

ξ specifies how many spins can "see" each other, for a given two-body nearest neighbor interaction range.

At critical point, $T = T_c$, correlation length ξ diverges,

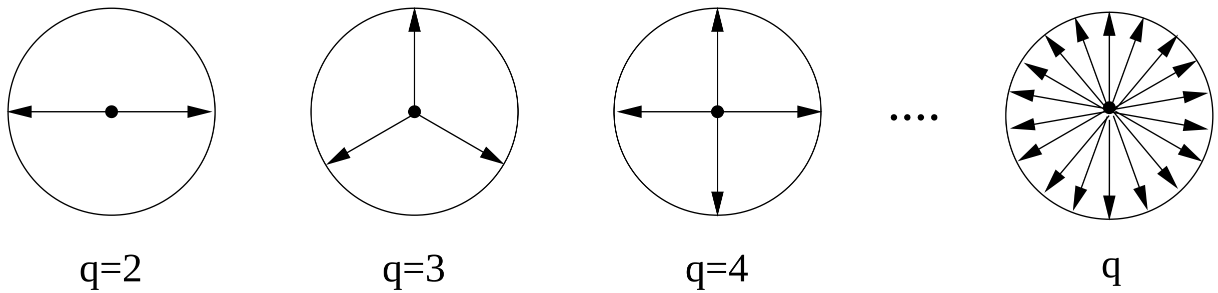
$$\Gamma(r, T_c) \sim \frac{1}{r^p}$$

scale-free system, everybody is connected to everybody (swarm!).

- q-state Potts model, N^2 lattice, $\theta_i(n) = 2\pi n/q,\ n = 1, 2, ..., q$

$$\mathcal{H} = -J \sum_{\{i,j\}} \cos(\theta_i - \theta_j) - H \sum_i \cos(\theta_i);$$

For $q = 2$: Ising model



In general, discrete Z_q symmetry, spontaneously broken
at transition temperature $T_c = 2J/\ln(1 + \sqrt{q})$ (for 2D),

for $q \leq 4$, continuous transition; for $q > 4$, first order.

The larger q, the lower the transition temperature,
the smaller the temperature range of long-range order

consider transition in discrete 2D spin models:

Peierls formulation

Ising model: island of down spins in sea of up spins, N sites

change up spin to down spin: $\Delta E = 2J$

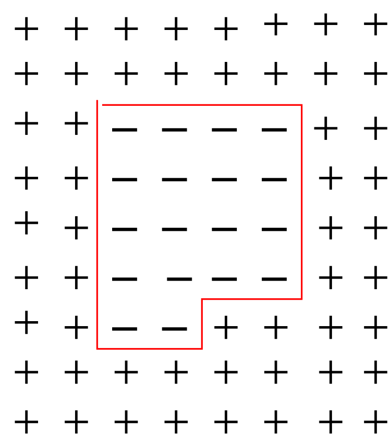
perimeter ("domain wall") Γ

build Γ from given site:

three possibilities (can't go back)

hence $\sim\ 3^{\Gamma} N$ possibilities for Γ steps

(N to start from); entropy $S = T \log 3^{\Gamma} N$



change in free energy to get island ($F = E - TS$; assume $3^{\Gamma} \gg N$)

$$\Delta F \simeq \Gamma(2J - T \log 3)$$

hence transition temperature $T_c \simeq 2J/\log 3$.

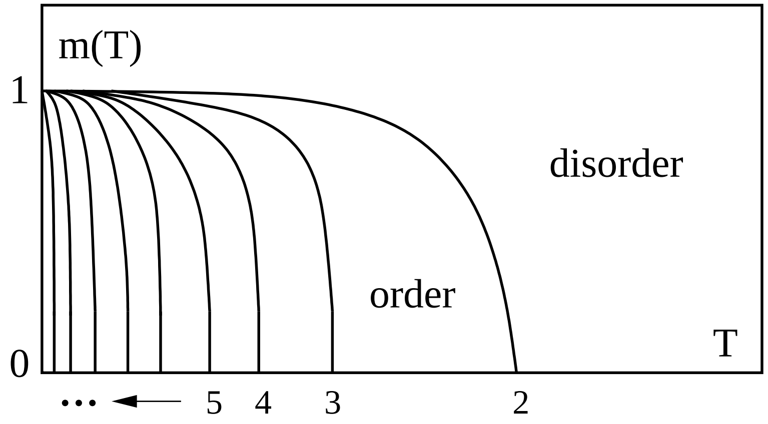
q-state Potts model: $\Delta E \simeq (1 - \cos[2\pi/q])J$

$\longrightarrow 0$ with large q: the larger q, the cheaper spin flip

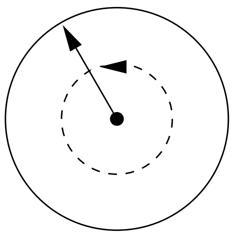
hence $T_c \simeq J(1 - \cos[2\pi/q])/\log 3$ vanishes for $q \to \infty$

For $q \to \infty$, $T_c(q) \to 0$, no long-range order at finite T,

and Potts model becomes



- **X−Y model**, continuous rotational symmetry:



$$\mathcal{H} = -J \sum_{\{i,j\}} \cos(\theta_i - \theta_j) - H \sum_i \cos(\theta_i)$$

but now $0 \le \theta_i \le 2\pi$.

For 2D: Mermin-Wagner theorem

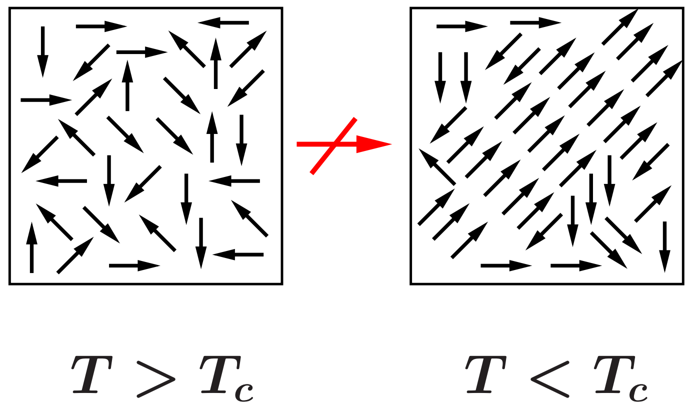
– no spontaneous symmetry breaking,

– no state of long range order for $T > 0$,

– state of long range order only for $T = 0$.

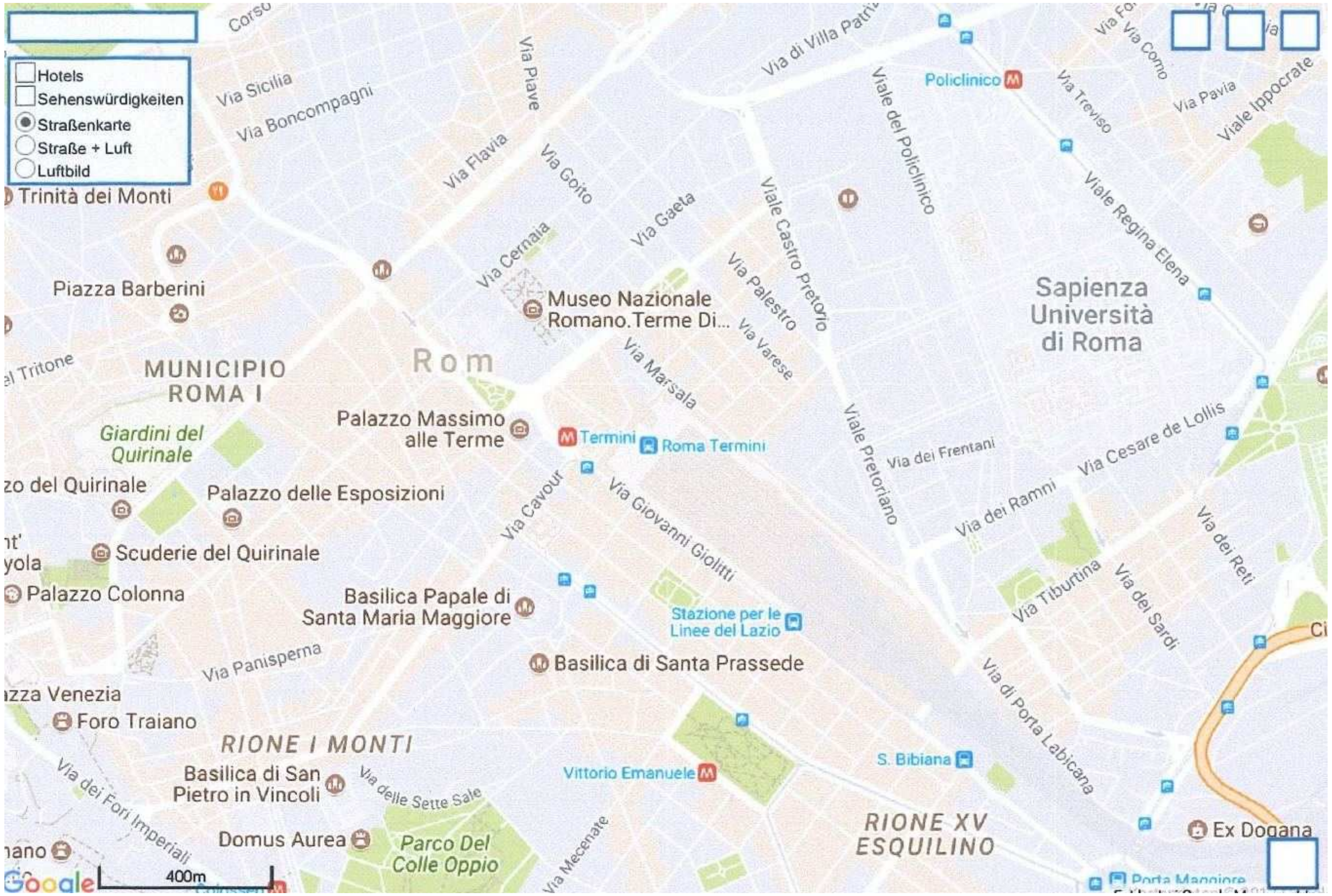
(*NB: finite temperature Kosterlitz-Thouless transition...)*

At finite $T > 0$, ferromagnetic transition not possible: there is no state with $m(T) \neq 0$ for $T > 0$



How can the birds fly away all in one direction?

Experiment: Rome 2005, The EU Starflag Project



European Starling
(*sturnus vulgaris*)

Palazzo Massimo alle Terme

ANIMAL BEHAVIOUR, 2008, **76**, 201–215
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Empirical investigation of starling flocks: a benchmark study in collective animal behaviour

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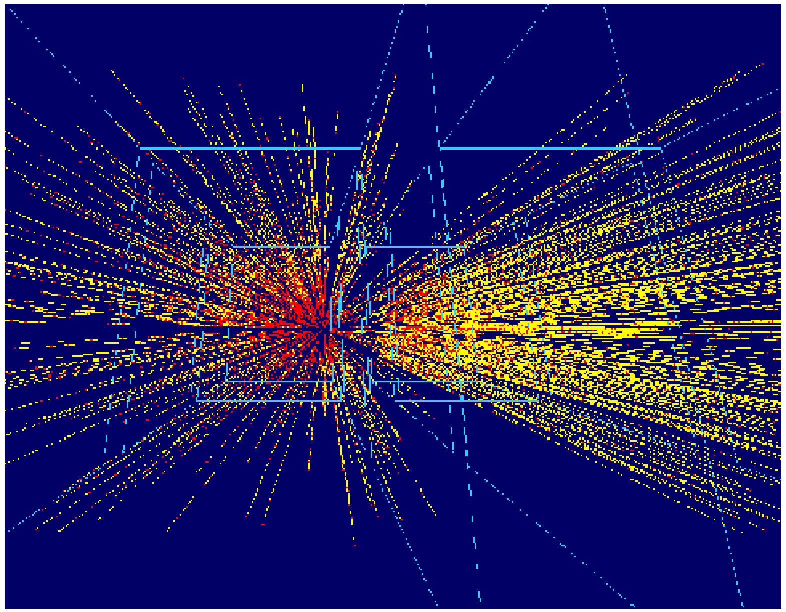
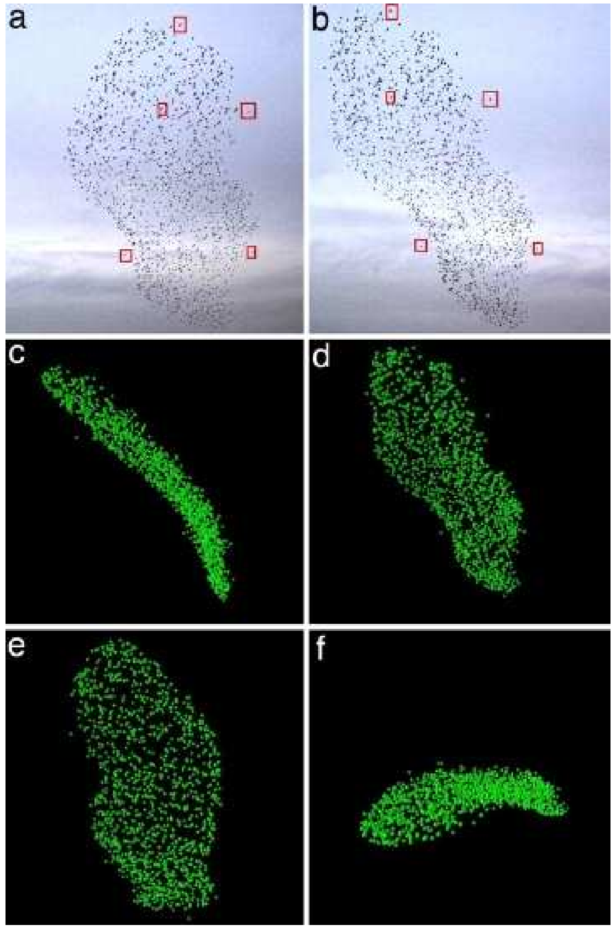
- **Two high speed cameras, 30m above ground, 25m apart, $\sim$ 100m from swarm, taking 10 frames/second for 8 seconds $\equiv$ event, 500 events, of 24 swarms from 100 up to 4000 birds.**

- Stereoscopic analysis (APE computers, CERN programs) identified each bird and allowed swarm structure determination.

CERN particle tracks
in AA collision





- Stereoscopic analysis (APE computers, CERN programs) identified each bird and allowed swarm structure determination.
- Swarm (N birds) is approximately 2D: flat re gravity, area L.
- Each bird interacts with six other birds = interaction range. Topological, not metric, independent of swarm size, density.
- Each bird is correlated to many other birds in the swarm (correlation length $\sim L \gg$ interaction range $\sim$ "stille Post").
- Some obserables:

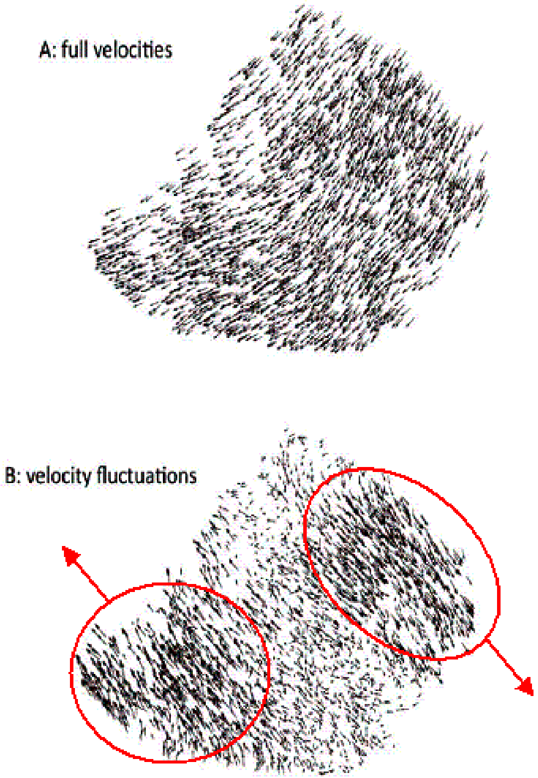
  - order parameter, polarization $\phi = || \frac{1}{N} \sum_1^N [\vec{v}_i / |v_i|] ||$

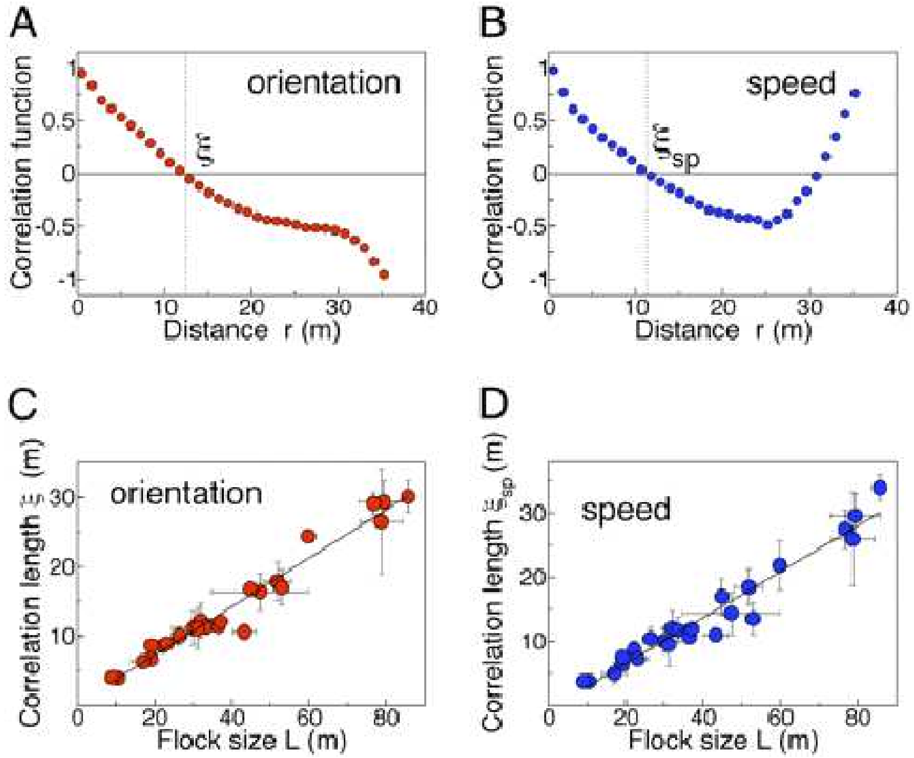
    $\phi = 0$ for rotational invariance;

  - cms velocity of bird i, velocity fluctuation $\vec{u}_i = \vec{v}_i - \frac{1}{N} \sum_1^N \vec{v}_i$

  - correlation function, $C(r) = \frac{\sum_{ij} \vec{u}_i \cdot \vec{u}_j \delta(r - r_{ij})}{\sum \delta(r - r_{ij})}$

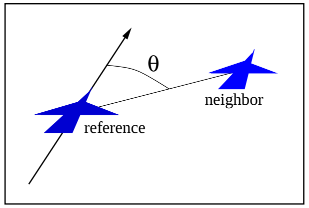
$C(r) \gg 0$: i, j distance r apart fly parallel, $\ll 0$ anti-parallel, 0 uncorrelated, cross-over defines correlation length ξ

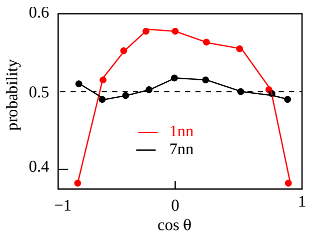


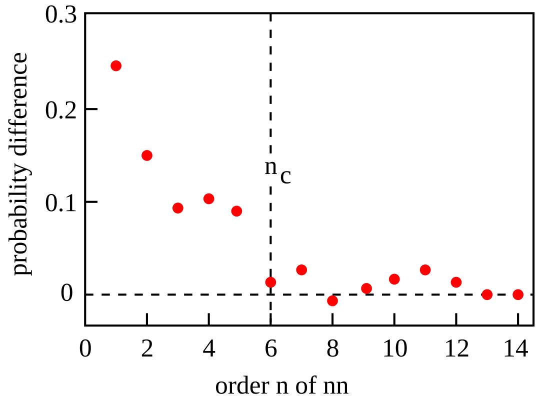


correlation length $\xi \sim L$

interaction range determined by neighbor anisotropy:







interaction range $\simeq$ six birds $\ll$ correlation range

correlation range ξ determined by $C(r = \xi) = 0$, grows with swarm size $\xi(bL) = b\xi(L)$; result for $L \to \infty$

$$C(r) = \frac{1}{r^\gamma} \exp -(r/\xi) \quad \to \quad \frac{1}{r^\gamma} \quad \text{with} \quad \gamma \simeq 0.2$$

non-critical behavior: exponential decay of $C(r)$

critical behavior: power-law decay.

Conclude: Bird swarm correlation is based on critical behavior with very weak decay, extremely long-range effect.

STARFLAG: "how starlings achieve such a strong correlation remains a mystery to us".

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$\longrightarrow$ mathematical models of swarm behavior.

Vicsek Model

basic idea: velocity vectors of birds $\sim$ spins in spin models

- **2D surface of area $A = L^2$, periodic boundary conditions**
- **N birds move in time, at constant speed, over surface**
- **each bird tries to follow its neigbors = birds within fixed distance R around it; R is independent of L**
- **following is subject to stochastic noise $\sim$ temperature for spins**
- **$\exists$ rotational symmetry; no Galilei-invariance, momentum not conserved**

Rules for bird motion:

- **initial state: random space distribution of N birds, random orientation of velocity vectors of unit length**
- **start time evolution**

$$\vec{x}_i(t+1) = \vec{x}_i(t) + \vec{v}_i(t)\Delta t$$

- **velocity $\vec{v}_i(t+1)$ is average over all bird vectors within a radius R around i, with noise $\Delta\theta_i$:**

$$\theta_i(t+1) = < \theta_i(t) >_{i \subset R_i} + \Delta\theta_i$$

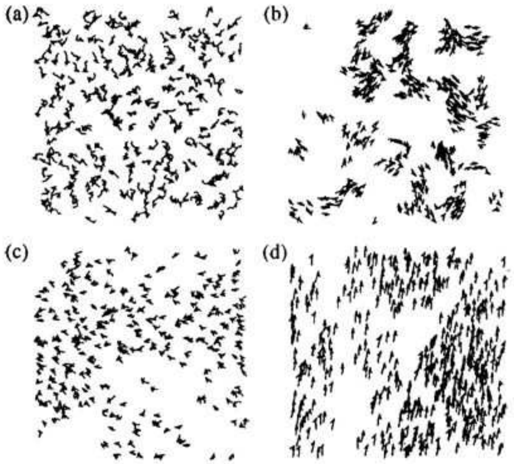
with $0 \leq \theta \leq 2\pi$ measured re some arbitrary reference direction

- **noise is random value subject to $-\eta \leq \Delta\theta_i \leq \eta \;\; \forall i$**
- **two parameters: density $\rho = N/A$ and noise $\eta \sim$ temperature**
- **iterate in unit time steps $\Delta t = 1$ until equilibration**

Vicsek Model

⇒ Self-Propelled Interacting Particles [Vicsek *et al.*, PRL **75**, 1226 (1995)]

- N locally aligning particles with noise and constant velocity $\boldsymbol{v}$
- Periodic boundary conditions
- Parameters: density of particles and amplitude of noise



(a) initial random setting

(b) low density, low noise

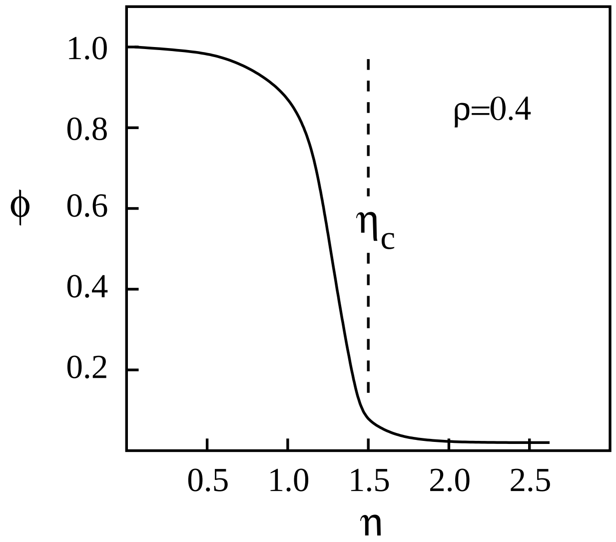
(c) high density, high noise

(d) high density, low noise

Order parameter: average momentum

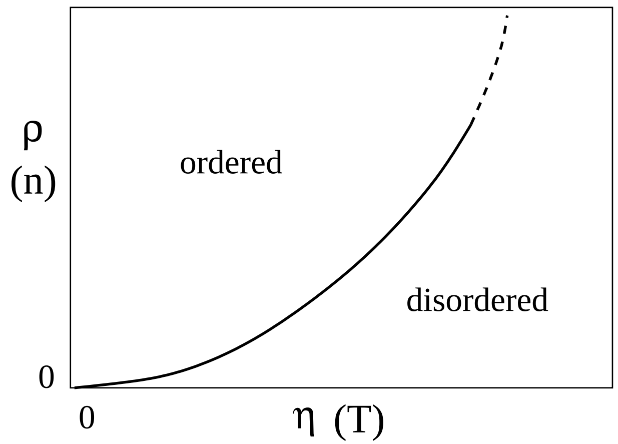
$$\varphi = \frac{1}{N}\left|\sum_i \boldsymbol{v}_i\right|$$

- order parameter $\phi = |\frac{1}{N}\sum_i \vec{v}_i|$



magnetization (average velocity ϕ) vs. temperature (orientational noise η) of a bird swarm; get η_c

similar ϕ vs. ρ to get ρ_c



first interpretation:
continuous transition, ρ vs. η like n vs. T phase diagram in condensed matter

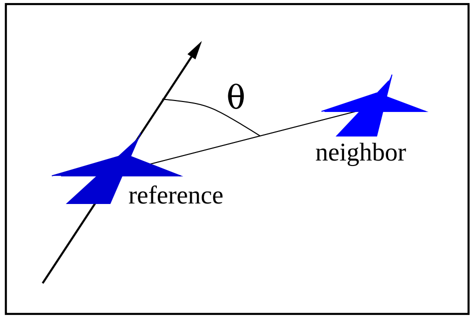
Lessons for phase structure from Vicsek model:

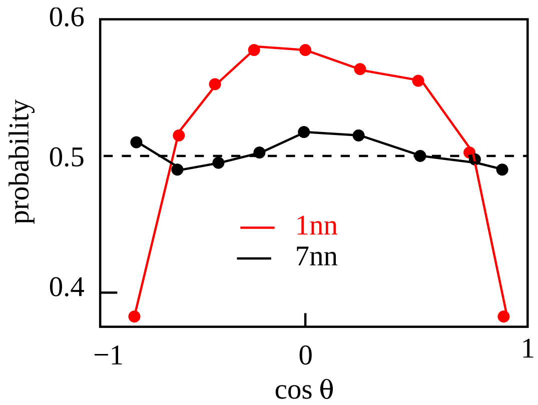
- **2d Vicsek model - spontaneous breaking of rotational symmetry, long range order, in contrast to**
- **2d $x - y$ model - no spontaneous symmetry breaking, no long range order**
- **non-equilibrium (motion, change of momentum): ordering effect**

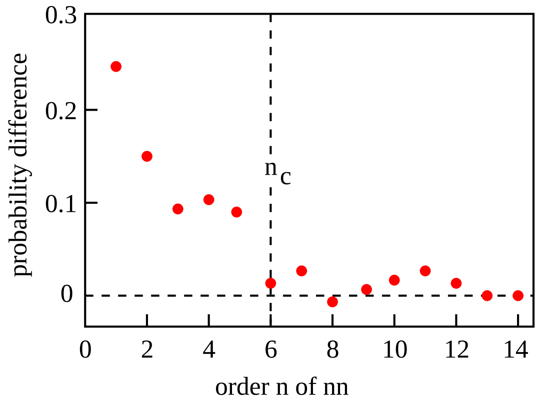
But: subsequent studies showed

- **theory: transition is discontinuous, intermediate ("coexistence") stage of band structures**
- **experiment (STARFLAG results): neighbor definition is topological, not metric**

return to experimental (bird) interaction range



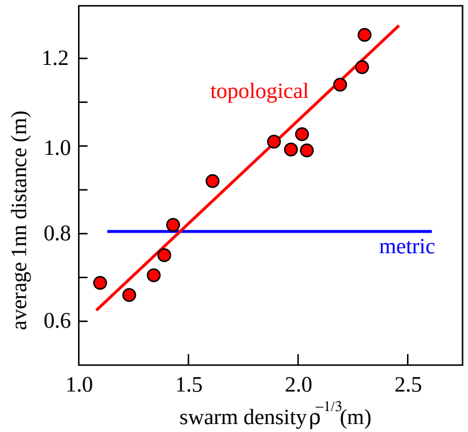




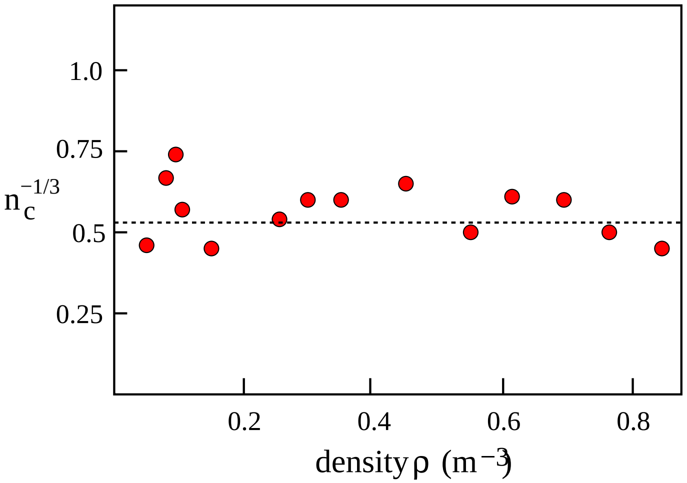
interaction range $\simeq$ six birds, $\neq x\,m$, nn can be $1\,m$ or $10\,m$ away, range size $\sim$ swarm density: topological, not metric

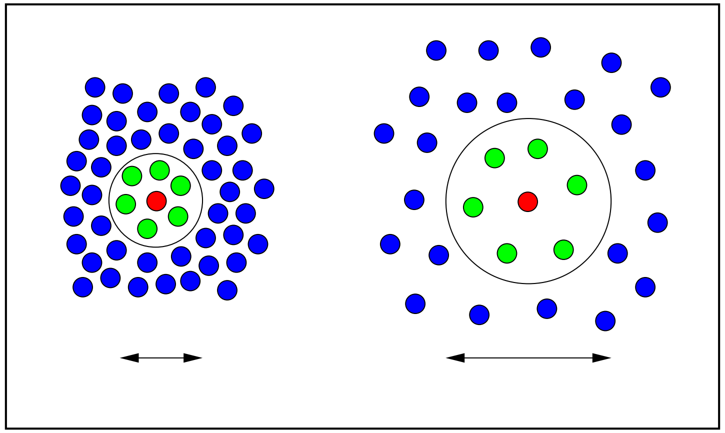
consider average next nearest neighbor distance r_1 as function of swarm density:

$$r_1 = a\rho^{-1/3}$$



how does interaction range depend on density?





high ρ   low ρ

Topological range (6 nearest birds) remains constant under density change, interaction structure unchanged as flock expands or contracts.

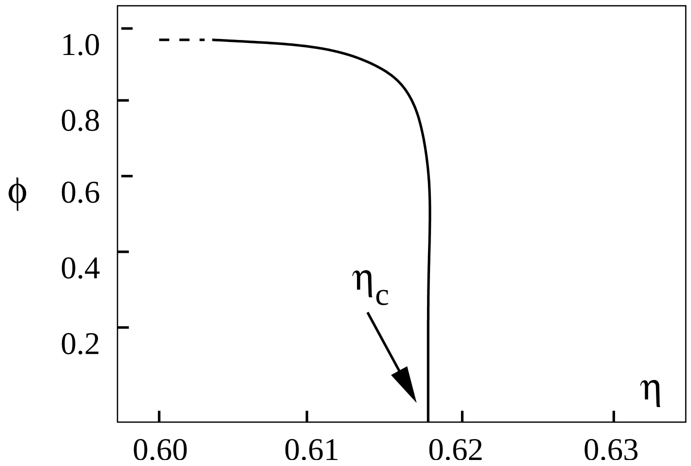
Need topological Vicsek model, with only noise η as parameter; density is irrelevant:

F. Ginelli and H. Chaté, PRL 105 (2010) 168103

Relevance of Metric-Free Interactions in Flocking Phenomena

Original Vicsek model: transition line ρ—η, discontinuous transition

topological flock model:
critical transition point η_c, continuous transition, no density dependence



result of numerical simulation (Ginelli & Chaté); why $\eta_c \simeq 0.615$?

determine critical exponents;
not mean field, not any known universality class

Topological definition: what are neighbors and why six?

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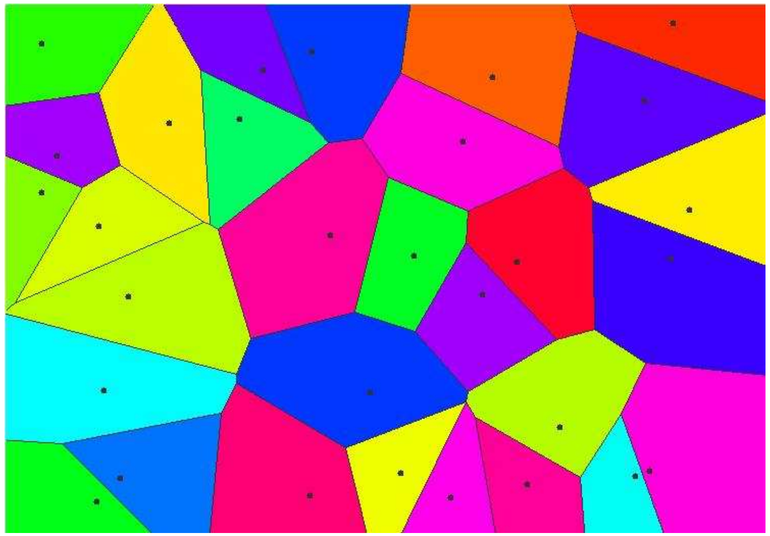
<u>Voronoi tesselation</u>

- **consider random distribution of N particles in a plane of area A**
- **define as neighborhood of any particle all points closer to it than to any other particle**
- **result: coverage of the plane ("tesselation") by areas**



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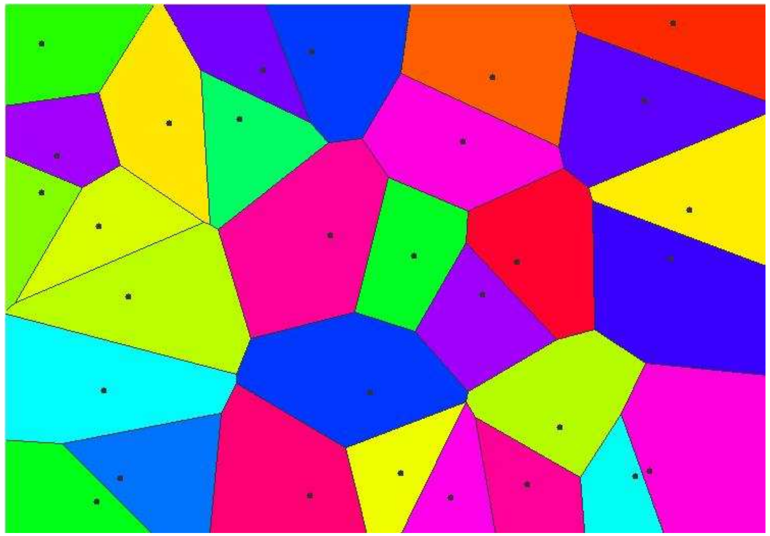
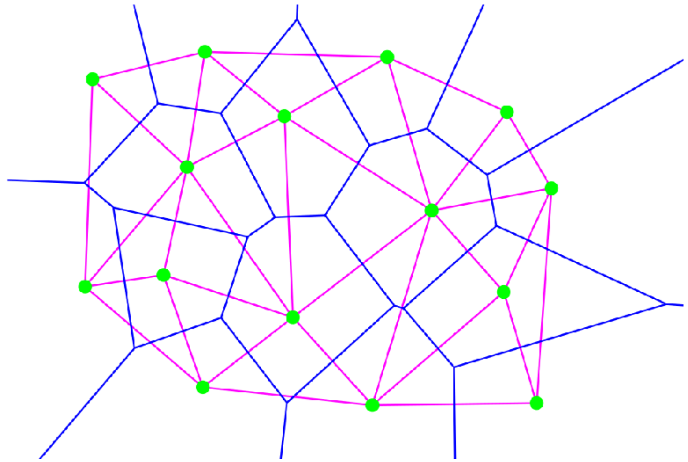
for $N \to \infty$, $N/A = \text{const.}$:

what is the average number of neighbors for each particle?



Delaunay Triangulation



triangles have
number of faces F,
of vertices/particles V, of edges E
Euler characteristic formula gives

$$\chi = V - E + F = 2$$

neglect borders:
face has three edges, edge touches two faces: 3F=2E

$$V = (1/3)E + 2$$

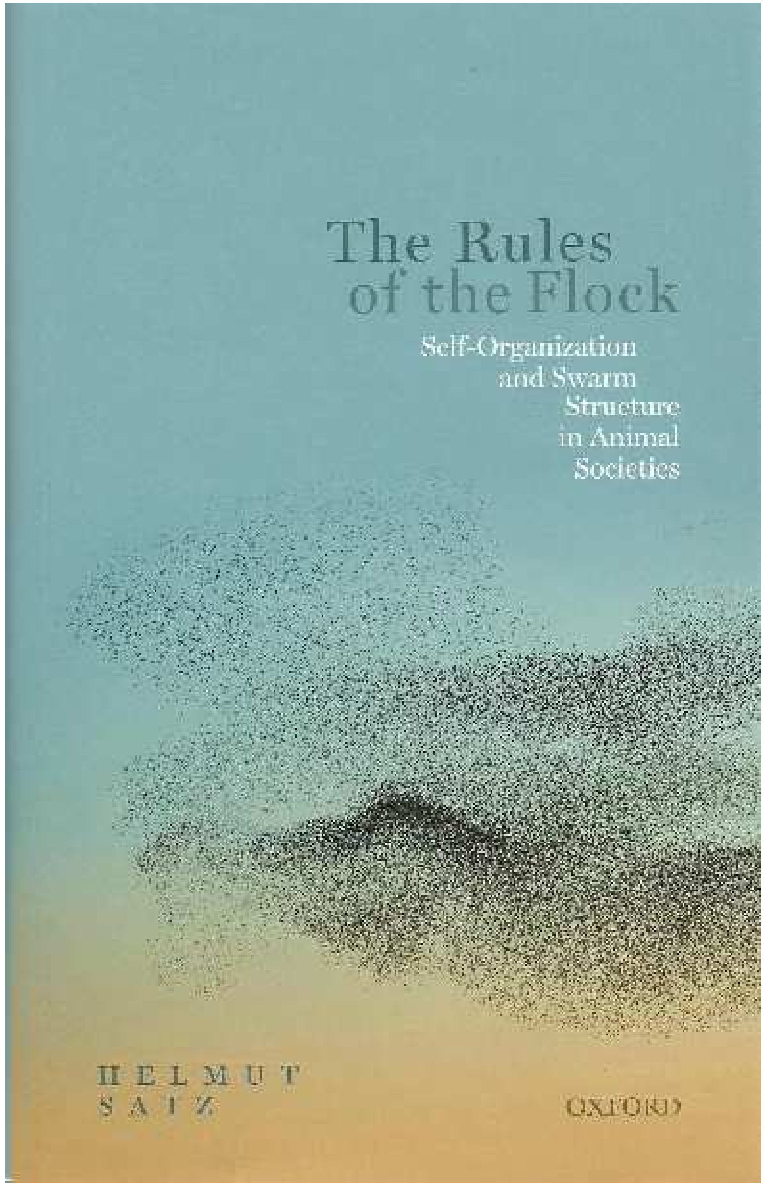
since each edge ∼ two vertices, from each vertex emerge ∼ six lines.

⇒ each Voronoi area has on the average six neighbors

NB: actual bird flocks are flat, but not 2d!

Conclusions

- **behavior of natural flocks of birds can be accounted for in terms of non-equilibrium statistical mechanics;**
- **it arises through self-organization based on local interactions of nearest neigbors only;**
- **observed global behavior is emergent.**


The Rules
of the Flock
Self-Organization
and Swarm
Structure
in Animal
Societies
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OXFORD