Aspects of scattering amplitudes

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Lecture 4: Differential equations, II

Differential equations, II

- Let's recall our guiding principle:

  The differential of a pure integral is a total derivative,

  & let's warm up with 1D integrals!

- Recall the simplest rational integral:

$$\int \frac{dx(a-b)}{(x-a)(x-b)} = \int (d\log\frac{x-a}{x-b})$$

- Let's raise the stakes a bit. Consider:

$$I_{a,b;F} = \int_0^\infty (d \log \frac{x-a}{x-b}) F(x, u_i)$$

- F is a pure transcendental function, u_i are parameters (See Claude Duhr's lectures)

- We will now compute the symbol of $I_{a,b;F}$ from the symbol of F

- Recall that

$$S[f] = a_1 \otimes \ldots \otimes a_n \leftrightarrow S[df] = (a_1 \otimes \ldots \otimes a_{n-1}) d \log a_n$$

- A good strategy is to shove the data about the rational part of the integrand, into the boundary: $x \to y$,

$$y = \frac{x-a}{x-b}, \quad x = \frac{a-by}{1-y}$$

- Then $I_{a,b;F} = \int_{a/b}^{1} (d_y \log y) F(x(y,a,b), u_i)$

  and the differential is simple to take:

$$dI_{a,b;F} = -(d \log \frac{a}{b}) F(x=0, u_i) + \int_{a/b}^{1} (d_y \log y) d_{\{a,b,u_i\}} F(x(y,a,b), u_i)$$

- By assumption $dF(x,u_i)$ is a sum of terms

$$G_j(x, u_i) d\log(x - x_j(u_i)), \quad G_0(x, u_i) d\log f(u_i)$$

where G_i, G_0 are pure transcendental functions

- It is not hard to compute each case, and re-express everything in terms of the original integral

$$I_{a,b;F} = \int_0^\infty (d\log\frac{x-a}{x-b}) F(x, u_i)$$

- We thus find 3 terms in $dI_{a,b;F}$:

1. $-F(x{=}0, u_i) d\log\frac{a}{b}$

2. For each zero $(x\text{-}x_j)$ in the last entry of F,

$$+(d\log(a-x_j))\int_0^\infty (d\log\frac{x-a}{x-x_j})G_j(x,u_i)$$
$$-(d\log(b-x_j))\int_0^\infty (d\log\frac{x-b}{x-x_j})G_j(x,u_i)$$

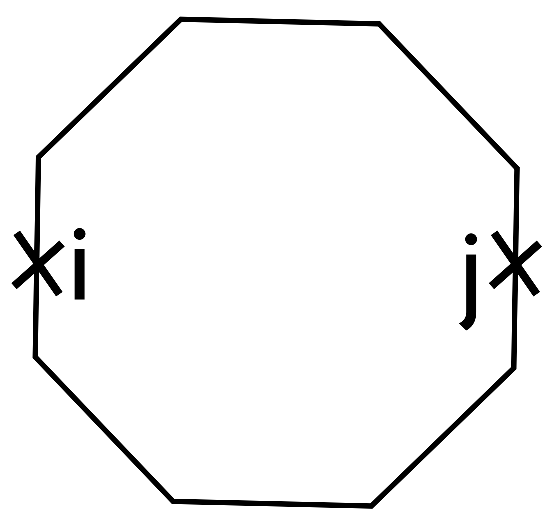
3. For each x-independent factor *f* in the last entry of F,

$$+(d\log f)\int_0^\infty (d\log\frac{x-a}{x-b})G_0(x,u_i)$$

- Application: from supersymmetry applied on Wilson loops in N=4,

$$dR_n|_{\bar{\chi},\chi=0} = \sum_{i,j} C_{i,j}\, d\log\langle i{-}1\, i\, i{+}1\, j\rangle.$$

(SCH, 1105.5606)

$C_{i,j} = \langle$  $\rangle$

Empirical observation: $C_{i,j}$ is a pure transcendental function (here, for R at two-loops, of degree 3), given as a 1-fold integral over dilogs (see Henn's talk)

Using essentially the technique just described, I obtained:

The differential of the n-point function is expressed as

$$dR_n = \sum_{i,j} C_{i,j} d\log\langle i{-}1ii{+}1j\rangle \tag{A.1}$$

where $C_{2,i}$ is the sum of the four contributions

$$\begin{aligned}
C_{2,i}^{(1)} &= \log u_{2,i-1,i,1} \times \sum_{j=2}^{i-1}\sum_{k=i}^{n+1}\left[\mathrm{Li}_2(1-u_{j,k,k-1,j+1}) + \log\frac{x_{j,k}^2}{x_{j+1,k}^2}\log\frac{x_{j,k}^2}{x_{j,k-1}^2}\right],\\
C_{2,i}^{(2)} &= \sum_{j=4}^{i-2}\Delta(1,2;j{-}1,j;i{-}1,i),\\
C_{2,i}^{(3)} &= \sum_{j=i+2}^{n}\Delta(2,1;j,j{-}1;i,i{-}1),\\
C_{2,i}^{(4)} &= -2\mathrm{Li}_3(1-\frac{1}{u}) - \mathrm{Li}_2(1-\frac{1}{u})\log u - \frac{1}{6}\log^3 u + \frac{\pi^2}{6}\log u,
\end{aligned} \tag{A.2}$$

and other $C_{i,j}$ are obtained by cyclic symmetry. In the first line, $x_{j+1} \equiv x_2$ when $j = i{-}1$, and $x_{k-1} \equiv x_1$ when $k = i$, and in the last line, $u = u_{2,i-1,i,1}$. The symbol of Δ is

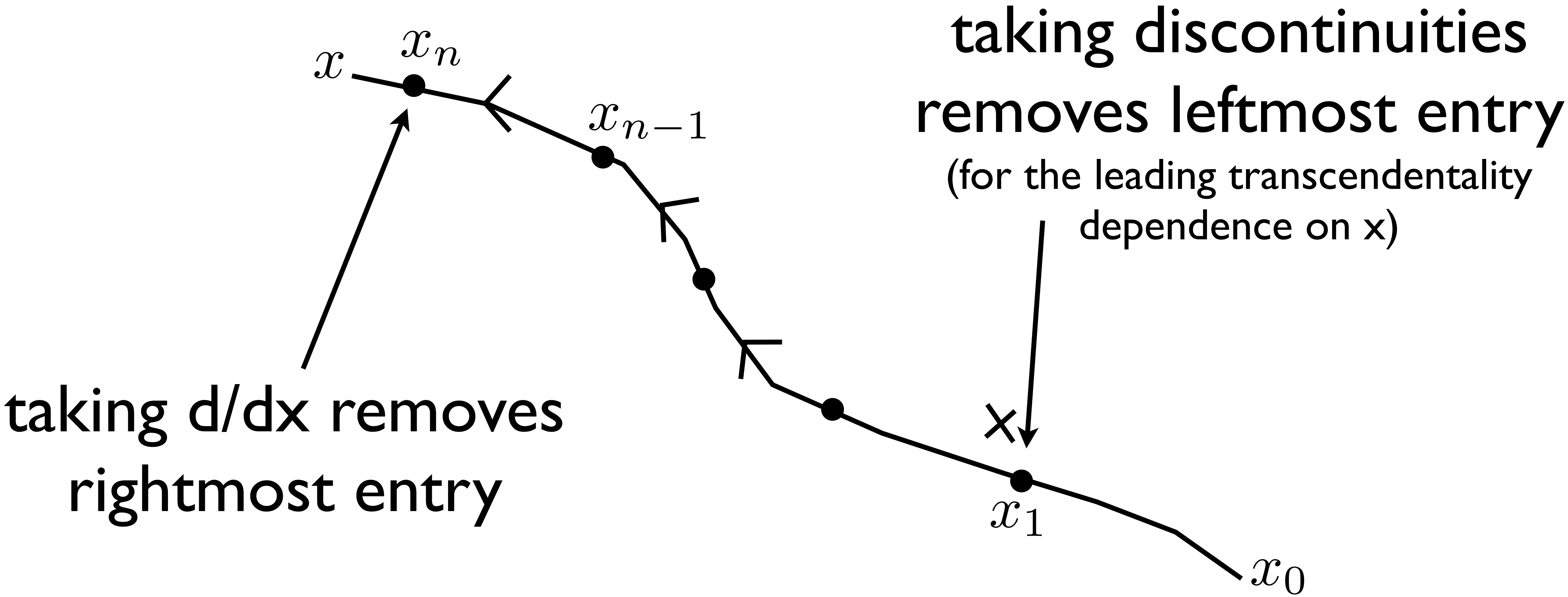
$$\begin{aligned}
&\mathcal{S}\Delta(1,2;j{-}1,j;i{-}1,i)\\
&= \left(\ \mathcal{S}[I_5(i;1,2;j{-}1,j)] \otimes \frac{\langle ii{+}1(\bar{2})\cap(\bar{j})\rangle\langle 23ij\rangle}{\langle j{-}1jj{+}1i\rangle\langle 123j\rangle\langle 23ii{+}1\rangle} - ((ii{+}1)\to(i{-}1i))\right)\\
&\left(\begin{array}{l}
\frac{1}{2}\mathcal{S}[\mathrm{Li}_2(1-u_{j,2,1,i-1}) - \mathrm{Li}_2(1-u_{j,2,1,i})] \otimes \left(\frac{\langle 123i\rangle\langle j{-}1jj{+}12\rangle\langle 23ij\rangle}{\langle 123j\rangle\langle j{-}1jj{+}1i\rangle\langle 23ii{+}1\rangle}\right)^2 \frac{\langle jj{+}1(\bar{2})\cap(\bar{i})\rangle\langle ii{+}1jj{+}1\rangle}{\langle 2ijj{+}1\rangle\langle 13(2i{-}1i)\cap(2jj{+}1)\rangle}\\
+\frac{1}{2}\mathcal{S}[\mathrm{Li}_2(1-u_{j,i-1,i,2}) - \mathrm{Li}_2(1-u_{j,i-1,i,1})] \otimes \left(\frac{\langle 12i{-}1i\rangle\langle 23ij\rangle}{\langle 123i\rangle\langle i{-}1ii{+}1j\rangle\langle 23i{-}1i\rangle}\right)^2 \frac{\langle jj{+}1(\bar{2})\cap(\bar{i})\rangle\langle i{-}1i{+}1(i23)\cap(ijj{+}1)\rangle}{\langle 2ijj{+}1\rangle\langle 12jj{+}1\rangle}\\
+\frac{1}{2}\mathcal{S}[\mathrm{Li}_2(1-u_{2,i-1,i,1})] \otimes \frac{\langle jj{+}1(\bar{2})\cap(\bar{i})\rangle\langle i{-}1i{+}1(i23)\cap(ijj{+}1)\rangle}{\langle 2ijj{+}1\rangle\langle 13(2i{-}1i)\cap(2jj{+}1)\rangle}\\
+\frac{1}{2}\mathcal{S}[\log u_{j,i-1,i,2}\log u_{j,2,1,i-1}] \otimes \left(\frac{\langle 23ij\rangle}{\langle 123j\rangle}\right)^2 \frac{\langle jj{+}1(\bar{2})\cap(\bar{i})\rangle\langle 13(2i{-}1i)\cap(2jj{+}1)\rangle}{\langle 2ijj{+}1\rangle\langle 23i{-}1i\rangle\langle i{-}1i{+}1(i23)\cap(ijj{+}1)\rangle}\\
\quad -((jj{+}1)\to(j{-}1j))
\end{array}\right)\\
&+\mathcal{S}[I_5(1;i{-}1,i;j{-}1,j)] \otimes \frac{\langle 12ij\rangle\langle 23i{-}1i\rangle}{\langle 12i{-}1i\rangle\langle 23ij\rangle}\\
&+\mathcal{S}[\log u_{i,j-1,j,1}\log u_{2,i-1,i,1}] \otimes \frac{\langle j{-}1j{+}1(j12)\cap(jii{+}1)\rangle\langle 123i\rangle\langle 23i{-}1i\rangle}{\langle 123j\rangle\langle j{-}1jj{+}1i\rangle\langle 12i{-}1i\rangle\langle 23ii{+}1\rangle}.
\end{aligned} \tag{A.3}$$

The factors of $\frac{1}{2}$ cancel telescopically in the sum over j, and there are no $\frac{1}{2}$ in front of anything in the full symbol of the amplitude (e.g., inside the big parenthesis, only the squared factors do not telescope away). The symbol could be written more succinctly by exploiting these telescopic cancellations; this particular presentation makes the individual term Δ integrable and parity covariant. I_5 is the "pentagon integral"

$$\begin{aligned}
I_5(X;1,2;i{-}1,i) =\ & \mathrm{Li}_2(1-u_{X,1,2,i}) - \mathrm{Li}_2(1-u_{X,1,2,i-1}) + \mathrm{Li}_2(1-u_{2,i-1,i,1})\\
&+\mathrm{Li}_2(1-u_{X,i,i-1,1}) - \mathrm{Li}_2(1-u_{X,i,i-1,2}) + \log u_{X,1,2,i}\log u_{X,i,i-1,1}.
\end{aligned} \tag{A.4}$$

- Actually, the technique I did the 1D integral then was based on monodromies
- Recall that symbols represent iterated integrals

$$\int^{x} d\log a_n(x_n) \int^{x_n} d\log a_n(x_{n-1} \ldots \int_{x_0}^{x_2} d\log a_1(x_1)$$



- # Monodromy technique

We hope to elaborate elsewhere about the algorithm we have used to compute the discontinuities. Let us just try outline the method for a one-dimensional integral such as Eq. (4.21). Basically, there are exactly three phenomena to keep track of:

- A pole of the integrand makes a loop around an integration endpoint.
- A branch cut endpoint of the integrand makes a loop around an integration endpoint.
- The value of the integrand at an endpoint undergoes monodromy.

- Monodromy technique

derivatives

We hope to elaborate elsewhere about the algorithm we have used to compute ~~the discontinuities~~. Let us just try outline the method for a one-dimensional integral such as Eq. (4.21). Basically, there are exactly three phenomena to keep track of:

- A pole of the integrand makes a loop around an integration endpoint.
- A branch cut endpoint of the integrand makes a loop around an ~~integration endpoint.~~ pole
- The value of the integrand at an ~~endpoint~~ pole undergoes monodromy.

The two methods are literally the same!!!

- Two ways to compute symbols: differential equations and monodromies

- *Poincaré duality*:

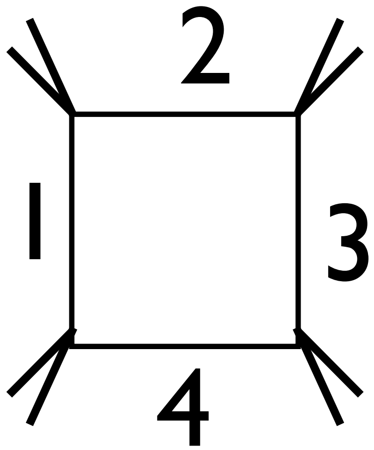
$$S\int_0^\infty (d\log\frac{x-a}{x-b})F(x,u_i) = S^T\int_a^b (d\log x)F^T(x,u_i),$$

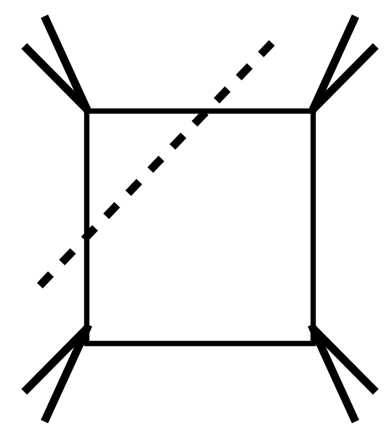
where ^{T} reverses the entries of the symbol!

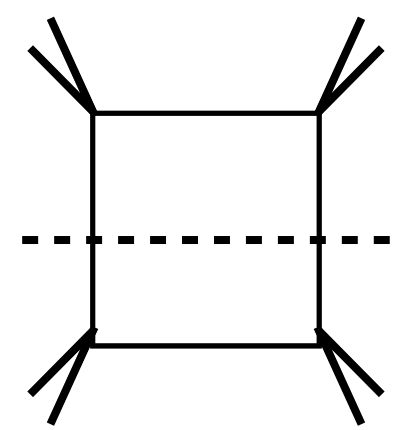
- This can be generalized to higher-dimensional integrals, such as appears in two-loop computations! (Arkani-Hamed & SCH, to appear)

- Poincaré duality in spacetime

Monodromy viewpoint:

S 

= $(x_1 - x_2)^2 \otimes$ 

$+(x_1 - x_3)^2 \otimes$ 

+ 4 terms

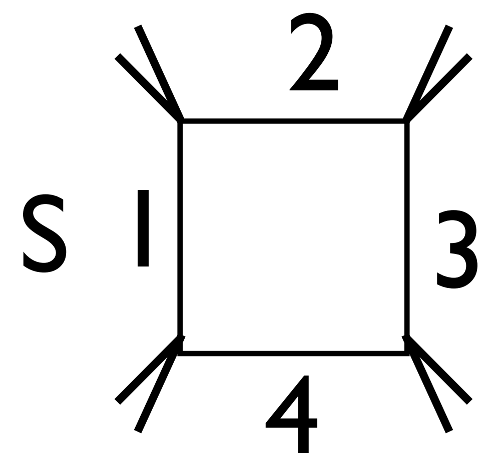
“Cutkowski, 1960”

- Poincaré duality in spacetime

Differential equation viewpoint:

$$X_1\cdot\frac{d}{dX_4}I_4^{4m} \quad = \quad \int_X K\cdot\frac{d}{dX}\frac{X\cdot V\sqrt{\text{Det }G}}{X\cdot X_2 X\cdot X_3 X\cdot X_4 X\cdot K}$$

Integral localizes to the real S^2 where X.K vanishes!



$$= \left(\int_X \delta(X\cdot K)\delta(X\cdot\tilde{K})[\ldots]\right)\otimes(1-\alpha_+)$$

+ ...

Formally very similar to a unitarity cut!

- It seems that a new operation on loops, "dual" to unitarity cuts, computes derivatives
- Works cleanly on *pure integrals*
- I have no doubt that this operation will be defined at all loop orders.
  Though I have no clue what its physical meaning is yet

Conclusions

- The scattering amplitude world is ripe with important and interesting computations, waiting to be done

- New, "motivic" (?) ideas may be trying to tell us something new about quantum field theory