N=4 SYM and N=8 supergravity amplitudes

Lecture 2

Henrik Johansson

School of analytic computing

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Plan

Lecture 1:
- Unitarity method for loop amplitudes
- N=4 super-Yang Mills
- Example of amplitude construction
- Quadruple cut, hepta cut

Lecture 2:
- Non-planar amplitudes
- N=8 supergravity
- Kawai-Lewellen-Tye relations
- Calculation of UV divergences

Lecture 3:
- Color/Kinematics duality
- Open problems

Non-planar & color

- For non-planar sector of gauge theory it is important to keep track of the amplitudes' color dependence

- There are two main ways to encode color (adjoint rep.)

    - SU(N_c): Trace basis (Chan-Paton factors)

    - Color tensors built out of structure constants f^{abc}

Trace basis

Trace-basis gives full color structure of SU(N_c) gauge theory amplitude:

$$\mathcal{A} = g^{n-2}(g^2 N_c)^L \sum_{\substack{\text{non-cyclic}\\ \text{perms}}} A^{\text{single-trace}}(1,2,\ldots,n)\,\mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]$$

$$+g^n(g^2 N_c)^{L-1} \sum_{\substack{\text{non-cyclic}\\ \text{perms}}} A^{\text{double}}_{\text{trace}}(1,\ldots,m;m+1,\ldots,n)\,\mathrm{Tr}[T^{a_1}\cdots T^{a_m}]\mathrm{Tr}[T^{a_{m+1}}\cdots T^{a_n}]$$

$$\ldots$$

$$+g^{n+2p-2}(g^2 N_c)^{L-p+1} \sum_{\substack{\text{non-cyclic}\\ \text{perms}}} A^{p\text{-trace}}(1,\ldots,n)\prod_p \mathrm{Tr}[TTT\cdots T]$$

$$A^{\text{single-trace}} = A^{\text{planar}} + \frac{1}{N_c^2}A^{(2)} + \frac{1}{N_c^4}A^{(4)} + \cdots + \frac{1}{N_c^p}A^{(p)}$$

Examples

At tree level there are only planar contributions:

$$\mathcal{A}^{tree} = g^{n-2} \sum_{\substack{non-cyclic \\ perms}} A^{planar}(1,2,\ldots,n)\, \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]$$

At one-loop level there are planar single trace, and leading double-trace terms

$$\mathcal{A}^{1-loop} = g^{n} N_c \sum_{\substack{non-cyclic \\ perms}} A^{planar}(1,2,\ldots,n)\, \mathrm{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]$$

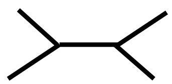
$$+ g^{n} \sum_{\substack{non-cyclic \\ perms}} A^{\mathrm{double}}_{\mathrm{trace}}(1,\ldots,m;m+1,\ldots,n)\, \mathrm{Tr}[T^{a_1}\cdots T^{a_m}]\mathrm{Tr}[T^{a_{m+1}}\cdots T^{a_n}]$$

At two loops:	single-trace	double-trace	triple-trace
	$\mathcal{O}(N_c^2)$	$\mathcal{O}(N_c)$	$\mathcal{O}(1)$
	$\mathcal{O}(1)$		

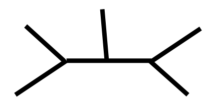
Color tensors

The set of all color tensors can be built by contracting f^{abc} in all possible way. Examples:

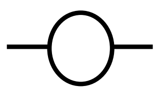
$$c^{abcd} = f^{abe} f^{ecd}$$



$$c^{abcde} = f^{abg} f^{gch} f^{hde}$$

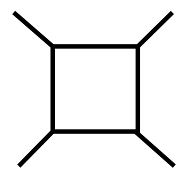


$$c^{ab} = f^{acd} f^{dcb}$$



$$= C_A \, \delta^{ab}$$ (quadratic Casimir)

$$c^{abcd} = f^{e_1 a e_2} f^{e_2 b e_3} f^{e_3 c e_4} f^{e_4 d e_1}$$



Feynman rules in an adjoint theory only have f^{abc} color dependence

$\Rightarrow$ Amplitude is linear combination of color tensors

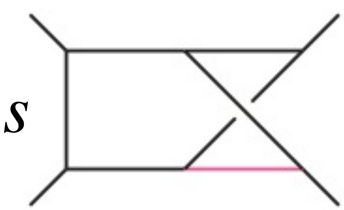
Planar vs. non-planar

- All tree amplitudes are planar
- One-loop amplitudes can be planar or non-planar
- However, all one-loop integrals are planar!
- Non-planar part are linear comb. of planar

$$A^{\rm double}_{\rm trace}(1,2,\ldots,m;m+1,\ldots,n) = \sum_{\sigma\in \mathrm{shuffle}\{1,2,\ldots,m\}\{m+1,\ldots,n\}} A^{\rm planar}(\sigma)$$

- First non-planar integrals show up at two loops

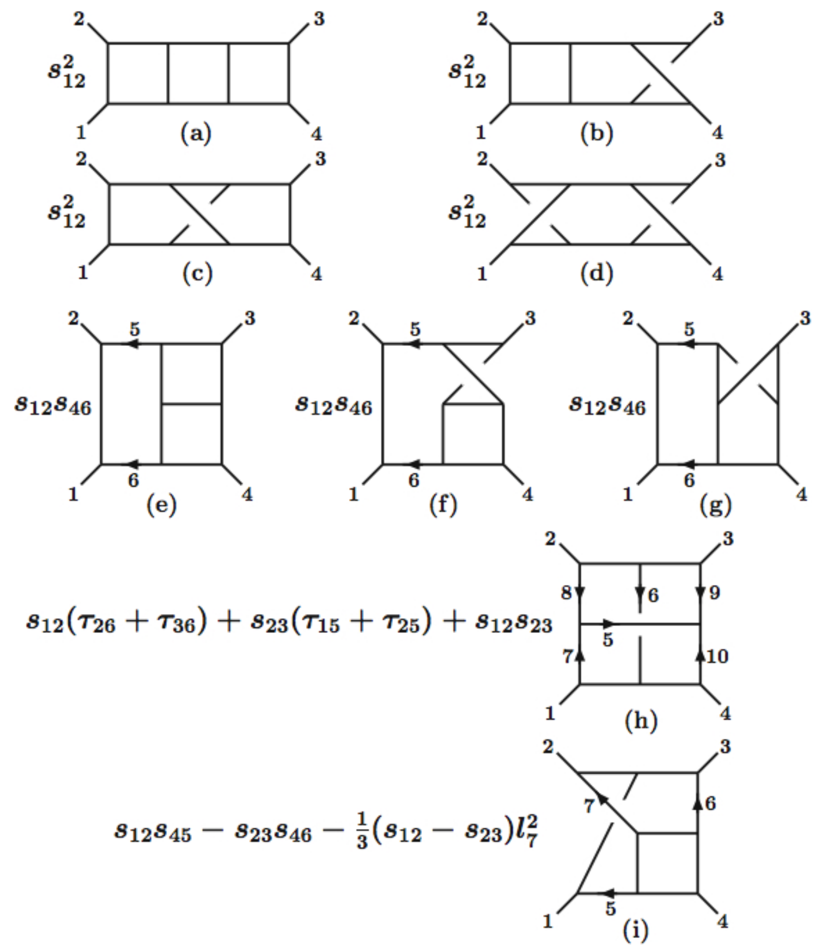
$$\mathcal{A}^{2-loop} = stA^{tree}\times \Big(s$$

$$+\, s$$

$$+\, perms \Big)$$

Example: non-planar 3loops

- Best way to understand nonplanar? Look at example:
- 3-loop 4pt N=4 SYM
- constructed 2007 using
  unitarity method

- Integration: open problem



$$s_{ij} = (k_i + k_j)^2$$
$$\tau_{ij} = 2k_i \cdot l_j$$

Gravity amplitudes

• In particle physics gravity (spin-2) theories are considered to be strange cousins of Yang-Mills (spin-1) theories

• Gravity has no color, thus no notion of planar amplitudes (non-planar integrals show up starting at two loops)

• The simplest 3-particle interaction in the Lagrangian involves two derivatives, unlike Yang-Mills one derivative

• Feynman diagrams thus have more momentum in the numerators $\Rightarrow$ in general, gives bad UV divergences

•(Alternatively, the coupling constant is dimensionful $\Rightarrow$ gravity is non-renormlizable)

• Tree-level amplitudes are always finite,

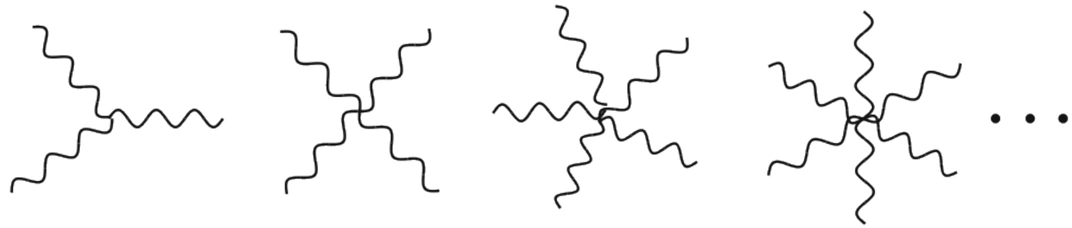
• Loop amplitudes may diverge $\Rightarrow$ needs to be checked!

Perturbative gravity is messy!

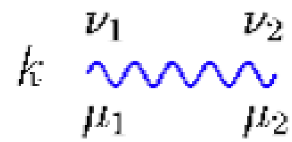
- Einstein gravity

$$\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

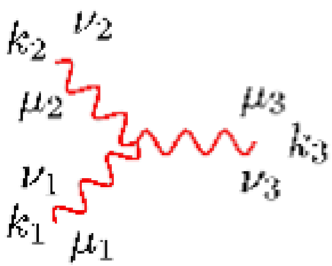


propagator: de Donder gauge



$$P_{\mu_1\nu_1\,\mu_2\nu_2}(k) = \frac{1}{2}\left[\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + \eta_{\mu_1\nu_2}\eta_{\mu_2\nu_1} - \frac{2}{D-2}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\right]\frac{i}{k^2+i\epsilon}$$

cubic vertex:



$$G_{3\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}(k_1,k_2,k_3) =$$
$$\mathrm{sym}[-\frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}) - \frac{1}{2}P_6(k_{1\mu_1}k_{1\nu_2}\eta_{\mu_1\nu_1}\eta_{\mu_3\nu_3}) + \frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\eta_{\mu_3\nu_3})$$
$$+P_6(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3}) + 2P_3(k_{1\mu_2}k_{1\nu_3}\eta_{\mu_1\nu_1}\eta_{\nu_2\mu_3}) - P_3(k_{1\nu_2}k_{2\mu_1}\eta_{\nu_1\mu_1}\eta_{\mu_3\nu_3})$$
$$+P_3(k_{1\mu_3}k_{2\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + P_6(k_{1\mu_3}k_{1\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + 2P_6(k_{1\mu_2}k_{2\nu_3}\eta_{\nu_2\mu_1}\eta_{\nu_1\mu_3})$$
$$+2P_3(k_{1\mu_2}k_{2\mu_1}\eta_{\nu_2\mu_3}\eta_{\nu_3\nu_1}) - 2P_3(k_1\cdot k_2\eta_{\nu_1\mu_2}\eta_{\nu_2\mu_3}\eta_{\nu_3\mu_1})]$$

After symmetrization ~ 100 terms !

(note: ampls are much prettier!)

N=8 supergravity

- N = 8 supergravity (*D* = 4) by Cremmer, Julia and Scherk (1978, 1979)
- 8 susys -- maximum number of susys for a spin-2 theory
- One supermultiplet of 256 massless states ($=2^8$)

	h^-	ψ^-_i	v^-_{ij}	χ^-_{ijk}	ϕ_{ijkl}	χ^+_{ijk}	v^+_{ij}	ψ^+_i	h^+
helicity	-2	-3/2	-1	-1/2	0	1/2	1	3/2	2
# states	1	8	28	56	70	56	28	8	1

Note: binomial coefficients of $(x + y)^8$

- Reasons work on this theory:
  - With more susy expect better UV properties.
  - High symmetry implies ampls will be simple.
  - Contains a hidden $E_{7(7)}$ symmetry

Where is the first UV divergence?

3 loops	Conventional superspace power counting	Green, Schwarz, Brink (1982) Howe and Stelle (1989) Marcus and Sagnotti (1985)
5 loops	Partial analysis of unitarity cuts; *If* $\mathcal{N}$ = 6 harmonic superspace exists; algebraic renormalisation	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998) Howe and Stelle (2003,2009)
6 loops	*If* $\mathcal{N}$ = 7 harmonic superspace exists	Howe and Stelle (2003)
7 loops	*If* $\mathcal{N}$ = 8 harmonic superspace exists; lightcone gauge locality arguments; Hints from string U-duality analysis. $E_{7(7)}$ analysis.	Grisaru and Siegel (1982); Kallosh (2009); Green, Russo, Vanhove; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger (2010)
8 loops	Explicit identification of potential susy invariant counterterm with full non-linear susy	Kallosh; Howe and Lindström (1981)
9 loops	Assume Berkovits' superstring non-renormalization theorems can be carried over to $\mathcal{N}$ = 8 supergravity	Green, Russo, Vanhove (2006)
Finite	Identified cancellations in multiloop amplitudes; lightcone gauge locality and $E_{7(7)}$	Bern, Dixon,Roiban (2006), Kallosh (2009–11)

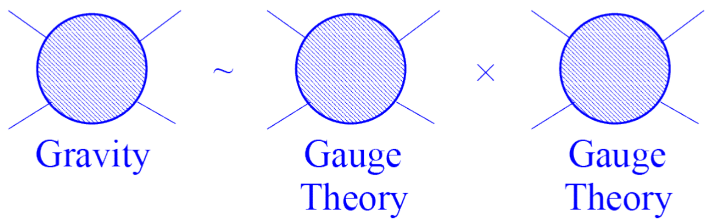
Note: thus far, no divergence has been found!

Kawai-Lewellen-Tye relations

- In string theory: closed strings (gravity) can be written as two open strings (gauge theory)

closed string ~ (left open string) × (right open string)

- This is 'trivial' for free strings, but nontrivial for interacting ones
- Kawai, Lewellen and Tye showed how this is true for tree amplitudes
- Consider the limit where the strings become point-like: field theory



gravity states are products of gauge theory states:

$$|1\rangle_{\rm grav} = |1\rangle_{\rm gauge} \otimes |1\rangle_{\rm gauge}$$

$$M_3^{\rm tree}(1,2,3) = A_3^{\rm tree}(1,2,3)\tilde{A}_3^{\rm tree}(1,2,3)$$

$$M_4^{\rm tree}(1,2,3,4) = -is_{12}A_4^{\rm tree}(1,2,3,4)\,\widetilde{A}_4^{\rm tree}(1,2,4,3)$$

$$M_5^{\rm tree}(1,2,3,4,5) = is_{12}s_{34}A_5^{\rm tree}(1,2,3,4,5)\,\widetilde{A}_5^{\rm tree}(2,1,4,3,5) + is_{13}s_{24}A_5^{\rm tree}(1,3,2,4,5)\,\widetilde{A}_5^{\rm tree}(3,1,4,2,5)$$

Example: 4pt tree

- Let's compute the 4pt tree-level scattering of gravitons

$$M^{\rm tree}(1^-,2^-,3^+,4^+) = s\,A^{\rm tree}(1^-,2^-,3^+,4^+)\,A^{\rm tree}(1^-,2^-,4^+,3^+)$$

Use:

$$A^{\rm tree}(1^-2^-3^+4^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = -i\frac{\langle 12\rangle^2[34]^2}{st}$$

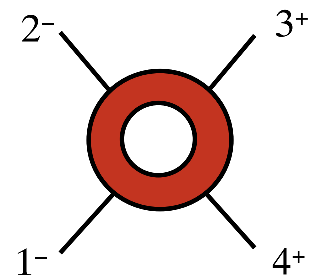
$$M^{\rm tree}(1^-,2^-,3^+,4^+) = s\times\frac{\langle 12\rangle^2[34]^2}{st}\times\frac{\langle 12\rangle^2[43]^2}{su}$$

$$= \frac{\langle 12\rangle^4[34]^4}{stu}$$

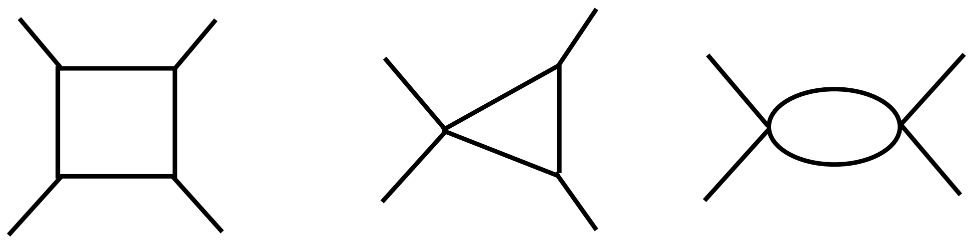
Homework from Hell: show that the Feynman rules gives the same answer!

Example: 4pt one-loop N=8 SG

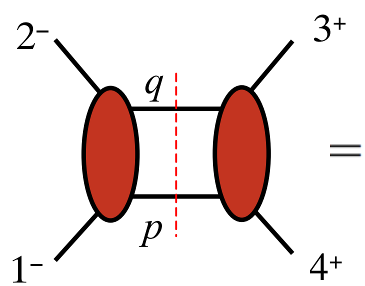
$$M^{\text{1-loop}}\,(1^-,2^-,3^+,4^+) =$$



What kind of integrals are expected?

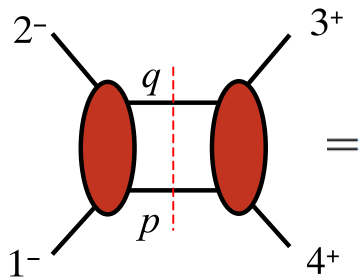


Compute the s-channel cut



$$= \int dLIPS \sum_{states} M^{\text{tree}}(1^-,2^-,q,p)M^{\text{tree}}(-p,-q,3^+,4^+)$$

s-channel cut



$$= \int dLIPS \sum_{states} M^{\text{tree}}(1^-, 2^-, q, p) M^{\text{tree}}(-p, -q, 3^+, 4^+)$$

- What are the states in the sum? $h^+, \lambda^+, g^+, f^+, s, f^-, g^-, \lambda^-, h^-$

⇒Naively 256^2 combinations

- SWI make all but one of them vanish!

$$\int dLIPS \sum_{states} M^{\text{tree}}(1^-, 2^-, q, p) M^{\text{tree}}(-p, -q, 3^+, 4^+)$$

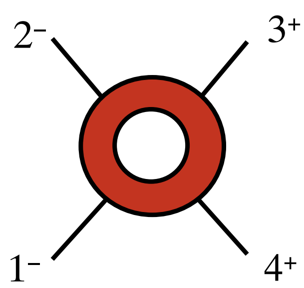
$$= \frac{\langle 12\rangle^4 [pq]^4}{s(k_1+p)^2(k_2+p)^2} \times \frac{\langle pq\rangle^4 [34]^4}{s(k_4-p)^2(k_3-p)^2} = \langle 12\rangle^4[34]^4 \frac{s^2}{(k_1+p)^2(k_2+p)^2(k_4-p)^2(k_3-p)^2}$$

$$= \langle 12\rangle^4[34]^4 \frac{((k_1+p)^2+(k_2+p)^2)((k_4-p)^2+(k_3-p)^2)}{(k_1+p)^2(k_2+p)^2(k_4-p)^2(k_3-p)^2}$$

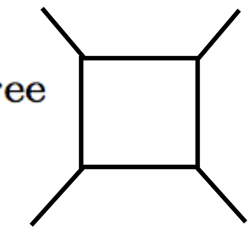
$$= \langle 12\rangle^4[34]^4 \left[\frac{1}{(k_2+p)^2(k_3-p)^2} + \frac{1}{(k_2+p)^2(k_4-p)^2} + \frac{1}{(k_1+p)^2(k_3-p)^2} + \frac{1}{(k_1+p)^2(k_4-p)^2}\right]$$

The result

- The 1-loop 4pt N=8 supergravity amplitude:



$$= stu M_4^{\text{tree}}$$

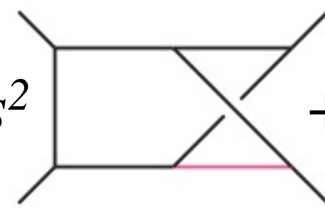


$$+ \ perms$$

- The 2-loop 4pt N=8 supergravity amplitude:

$$M_4^{2-loop} = stu M_4^{\text{tree}} \Bigg(s^2$$

$$+ \ s^2$$

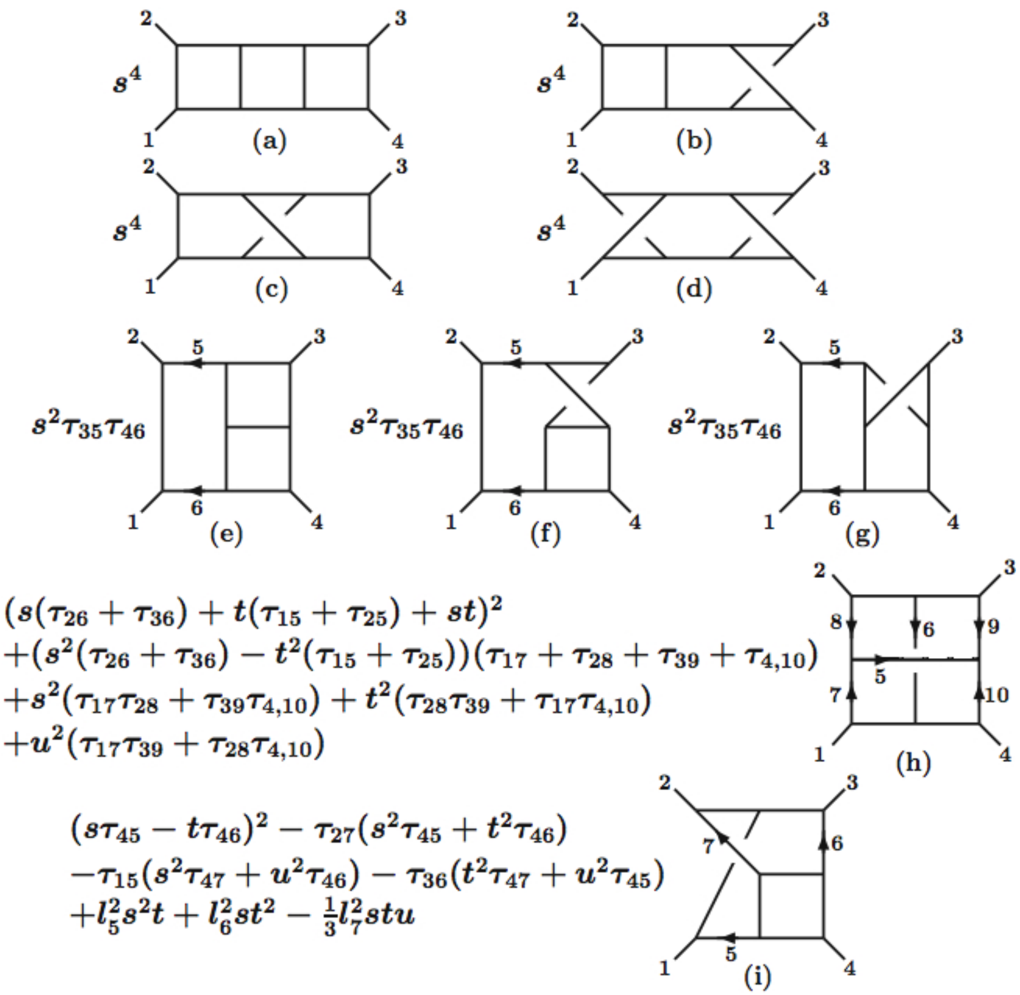


$$+ \ perms \Bigg)$$

- Homework: Show that it has the correct two-particle cuts!

3-loop N=8 SG

- The 3-loop 4pt N=8 supergravity amplitude:



$$
\begin{aligned}
&(s(\tau_{26}+\tau_{36})+t(\tau_{15}+\tau_{25})+st)^2\\
&+(s^2(\tau_{26}+\tau_{36})-t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})\\
&+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10})+t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})\\
&+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})
\end{aligned}
$$

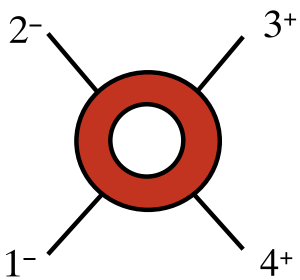
$$
\begin{aligned}
&(s\tau_{45}-t\tau_{46})^2-\tau_{27}(s^2\tau_{45}+t^2\tau_{46})\\
&-\tau_{15}(s^2\tau_{47}+u^2\tau_{46})-\tau_{36}(t^2\tau_{47}+u^2\tau_{45})\\
&+l_5^2 s^2 t+l_6^2 s t^2-\tfrac{1}{3}l_7^2 stu
\end{aligned}
$$

Bern, Carrasco, Dixon, HJ, Kosower, Roiban, (2007)
Bern, Carrasco, Dixon, HJ, Roiban (2008)

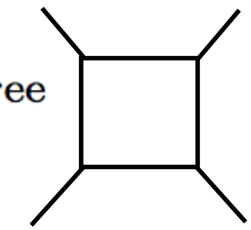
$$\tau_{ij}=2k_i\cdot l_j$$

UV divergences: 1-loop

- Let's analyze the UV behavior!
- Question: Is the one-loop amplitude UV finite in D=4?



$$= stu M_4^{\text{tree}}$$



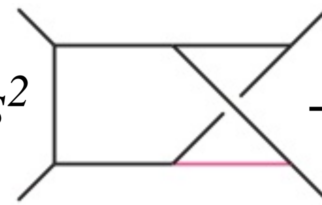
$$+ \; perms$$

- Question: In what dimension does it diverge?

- Answer: D≥8

UV divergences: 2-loop

- Let's analyze the UV behavior!
- Question: Is the two-loop amplitude UV finite in D=4?

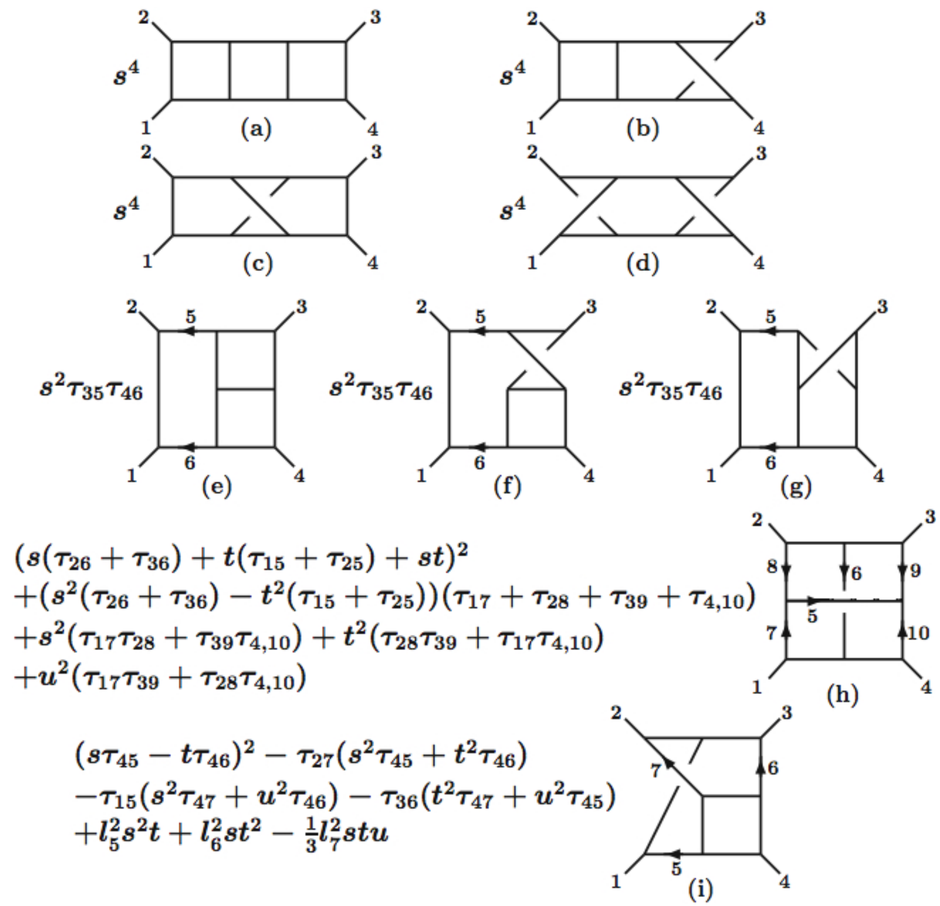
$M_4^{2-loop} = stu M_4^{\text{tree}} \Big(s^2$  $+ s^2$  $+ \, perms \Big)$

- Question: In what dimension does it diverge?

- Answer: D≥7

UV divergences: 3-loop

- Let's analyze the UV behavior!
- Question: Is the 3-loop amplitude UV finite in D=4?

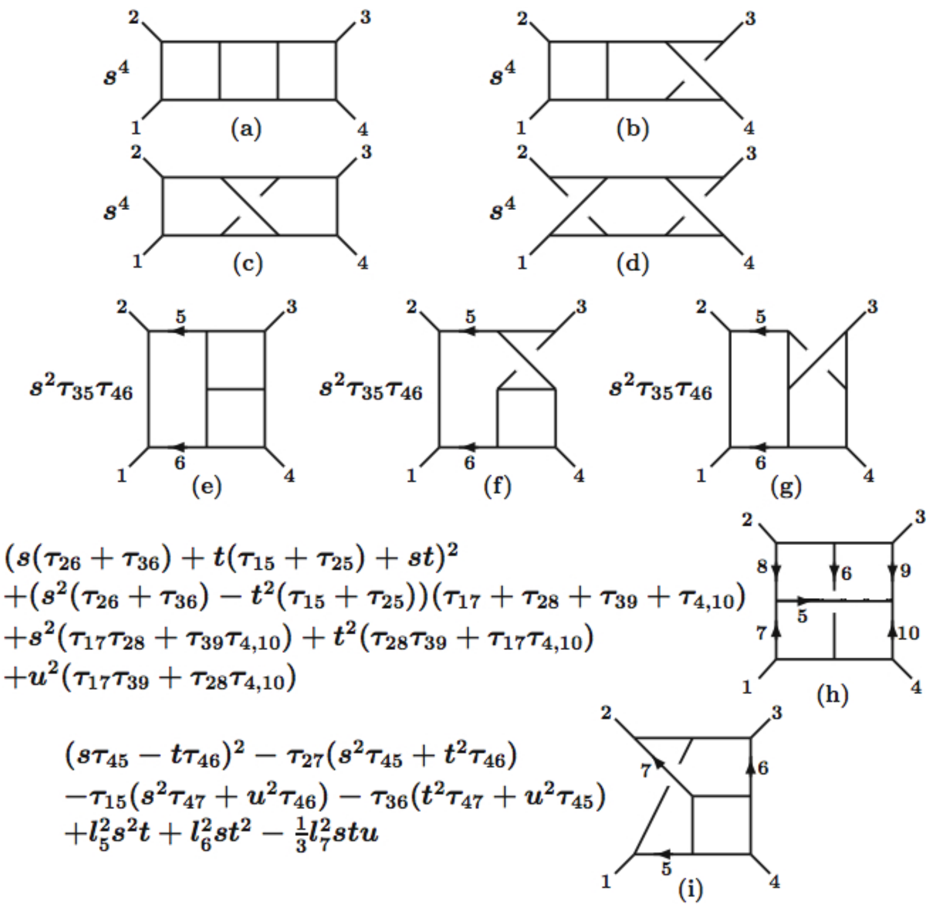


- Question: In what dimension does it diverge?

- Answer: D≥6

UV divergences: 3-loop

- Let's analyze the UV behavior!
- Question: Is the 3-loop amplitude UV finite in D=4?



- Question: In what dimension does it diverge?

- Answer: D≥6