# N=4 SYM and N=8 supergravity amplitudes

Lecture 3

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Lecture 1:

- Unitarity method for loop amplitudes
- N=4 super-Yang Mills
- Example of amplitude construction
- Quadruple cut, hepta cut

Lecture 2:

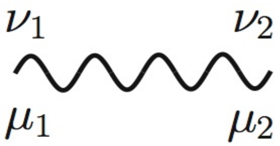
- Non-planar amplitudes
- N=8 supergravity
- Kawai-Lewellen-Tye relations
- Calculation of UV divergences

Lecture 3:

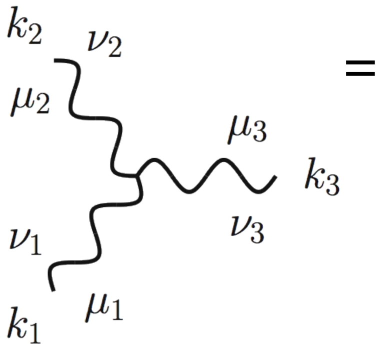
- Color/Kinematics duality
- Open problems

# Einstein Gravity Feynman rules

de Donder gauge: $\mathcal{L} = \frac{2}{\kappa^2}\sqrt{g}R, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$



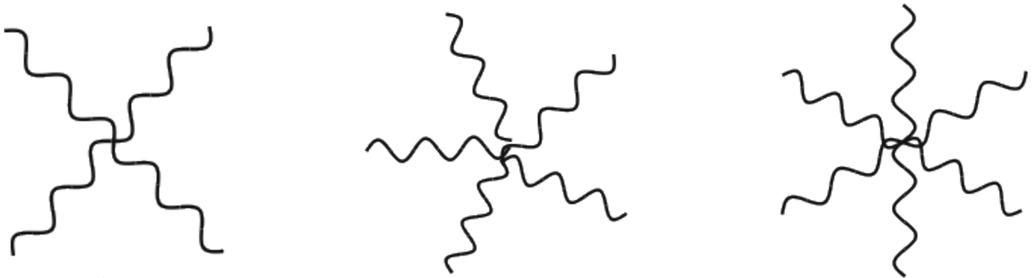
$$= \frac{1}{2}\left[\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2} + \eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} - \frac{2}{D-2}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\right]\frac{i}{p^2 + i\epsilon}$$



$$\begin{aligned}= \mathrm{sym}[&-\frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}) - \frac{1}{2}P_6(k_{1\mu_1}k_{1\nu_2}\eta_{\mu_1\nu_1}\eta_{\mu_3\nu_3}) + \frac{1}{2}P_3(k_1\cdot k_2\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\eta_{\mu_3\nu_3})\\ &+P_6(k_1\cdot k_2\eta_{\mu_1\nu_1}\eta_{\mu_2\mu_3}\eta_{\nu_2\nu_3}) + 2P_3(k_{1\mu_2}k_{1\nu_3}\eta_{\mu_1\nu_1}\eta_{\nu_2\mu_3}) - P_3(k_{1\nu_2}k_{2\mu_1}\eta_{\nu_1\mu_1}\eta_{\mu_3\nu_3})\\ &+P_3(k_{1\mu_3}k_{2\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + P_6(k_{1\mu_3}k_{1\nu_3}\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}) + 2P_6(k_{1\mu_2}k_{2\nu_3}\eta_{\nu_2\mu_1}\eta_{\nu_1\mu_3})\\ &+2P_3(k_{1\mu_2}k_{2\mu_1}\eta_{\nu_2\mu_3}\eta_{\nu_3\nu_1}) - 2P_3(k_1\cdot k_2\eta_{\nu_1\mu_2}\eta_{\nu_2\mu_3}\eta_{\nu_3\mu_1})]\end{aligned}$$

After symmetrization
~ 100 terms !

higher order
vertices...



~$10^3$ terms

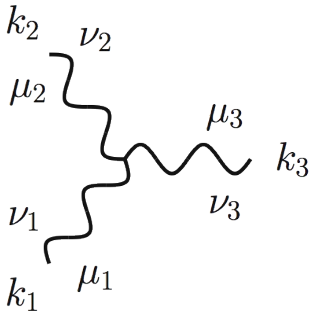
. . .

# On-shell simplifications

Graviton plane wave: $\varepsilon^{\mu}(p)\varepsilon^{\nu}(p)\, e^{ip\cdot x}$

Yang-Mills polarization

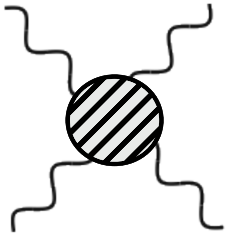
On-shell 3-graviton vertex:



$$= i\kappa\Big(\eta_{\mu_1\mu_2}(k_1-k_2)_{\mu_3} + \text{cyclic}\Big)\Big(\eta_{\nu_1\nu_2}(k_1-k_2)_{\nu_3} + \text{cyclic}\Big)$$

Yang-Mills vertex

Gravity scattering amplitude:



$$M_4^{\text{tree}}(1,2,3,4) = -i\frac{st}{u}A_4^{\text{tree}}(1,2,3,4)\tilde{A}_4^{\text{tree}}(1,2,3,4)$$

Yang-Mills amplitude

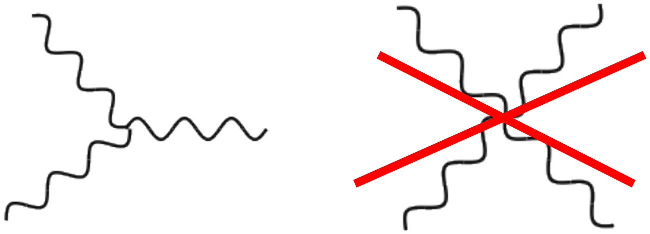
On-shell gravity objects are "squares" of Yang-Mills objects !

# Gravity should be cubic

Yang-Mills → cubic

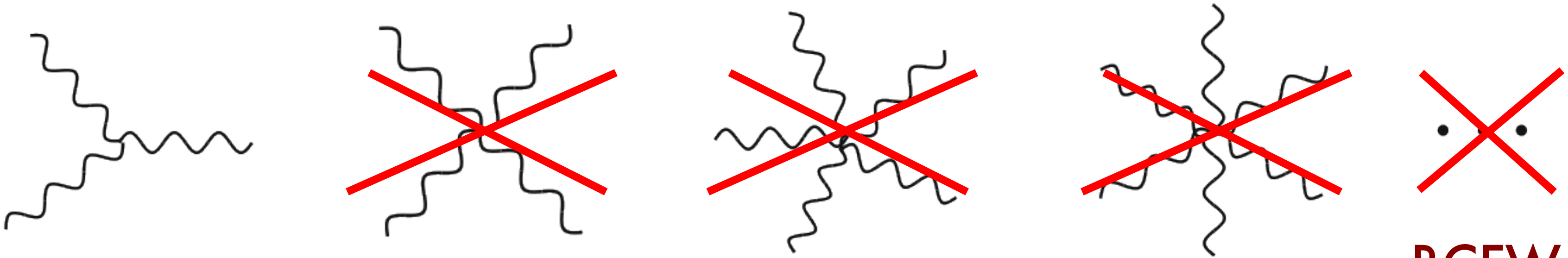
schematically: $\mathcal{L}_{\rm YM} \sim A\Box A + \partial A^3$

schematic derivative



Einstein gravity → cubic

schematically: $\mathcal{L}_{\rm G} \sim h\Box h + \partial^2 h^3$



e.g. BCFW

And gravity should be a double copy of a YM theory:

$$h^{\mu\nu} \sim A^\mu A^\nu$$

$$V_{\rm G}(k_1, k_2, k_3) = V_{\rm YM}(k_1, k_2, k_3) V_{\rm YM}(k_1, k_2, k_3)$$

# Gauge theory is the key

The simplicity of gravity stems from a novel structure in Yang-Mills

- represent amplitudes using cubic graphs only:

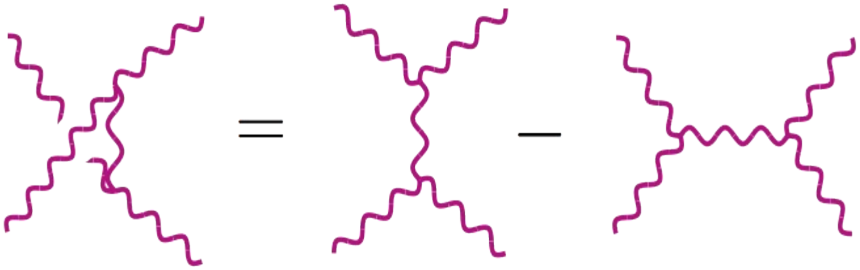
numerators

color factors

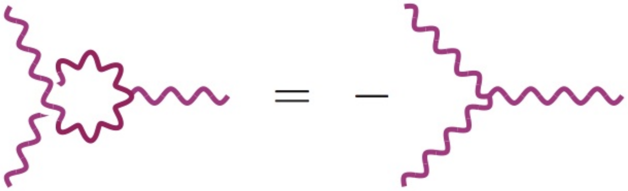
$$\mathcal{A}_m^{(L)} = \sum_{i\in\Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

propagators

Both $c_i$ and $n_i$ satisfy the algebra:



Jacobi identity



antisymmetry

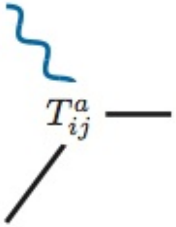
These are the relations that should define a Lie Algebra

Duality: color ↔ kinematics

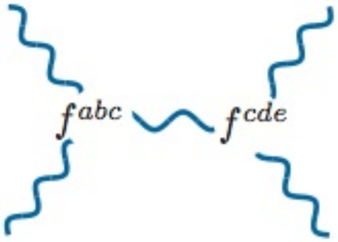
Bern, Carrasco, HJ
[BCJ]

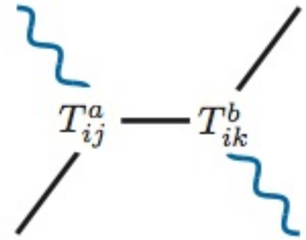
# Color diagramatics

Gauge group algebra:

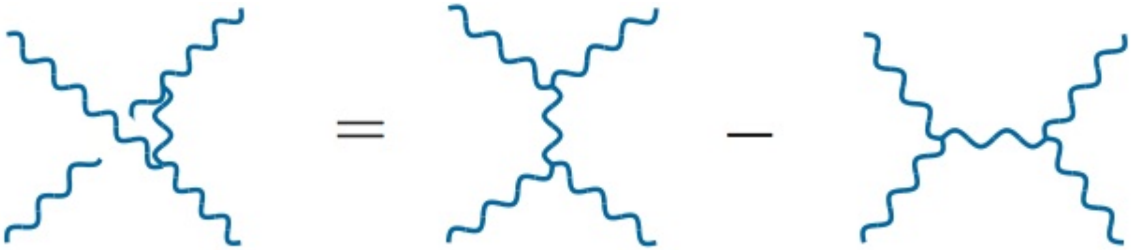


Build color structures





Relations



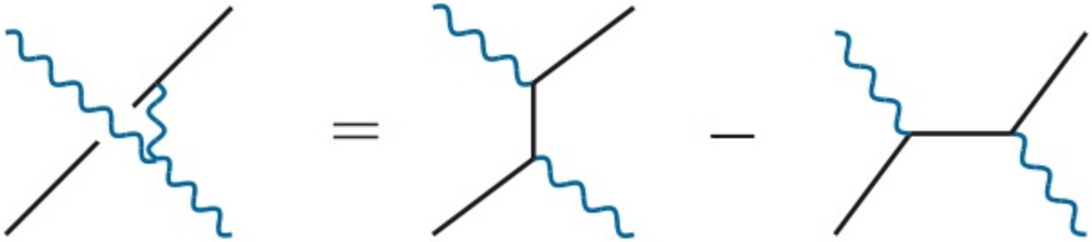
$$f^{adc} f^{ceb} = f^{eac} f^{cbd} - f^{abc} f^{cde}$$

# Color diagramatics

Gauge group algebra:

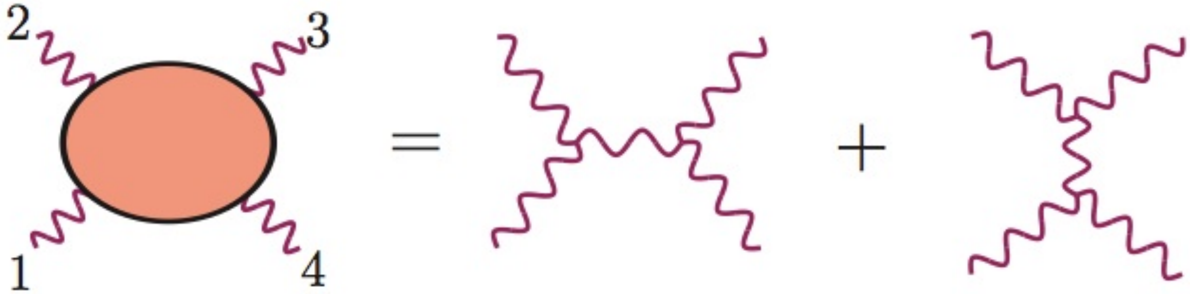
Build color structures

Relations



$$f^{cba}T^c_{ik} = T^b_{ij}T^a_{jk} - T^a_{ij}T^b_{jk}$$

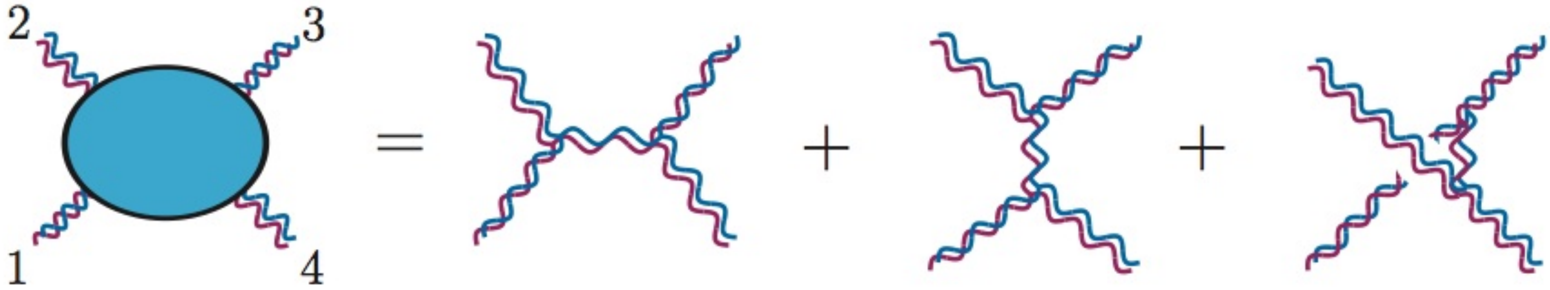
# Kinematic diagrams



$$A^{\text{tree}}(1,2,3,4) = \frac{n_s}{s} + \frac{n_t}{t}$$

color-stripped,
color-ordered,
partial ampl.

(absorb contact
terms using $1 = s/s$)



$$\mathcal{A}_4^{\text{tree}} = \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}$$

color
dressed

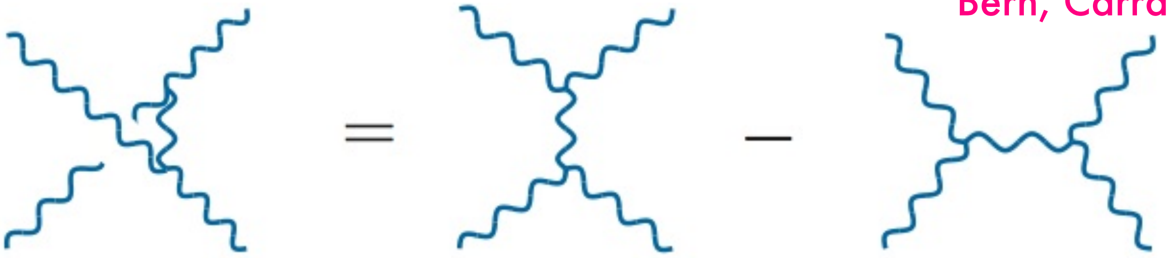
color factors: $c_s = f^{abc} f^{cde}$

kinematic factors: Feynman rules,
BCFW etc.

# kinematics is dual to color

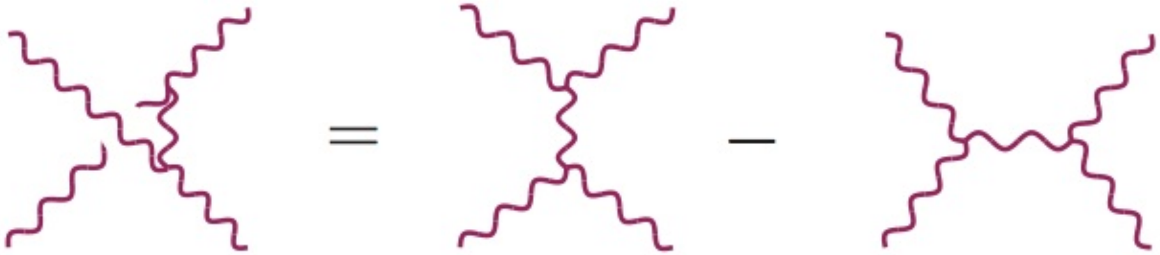
Bern, Carrasco, HJ

color Jacobi



$$c_u = c_t - c_s$$

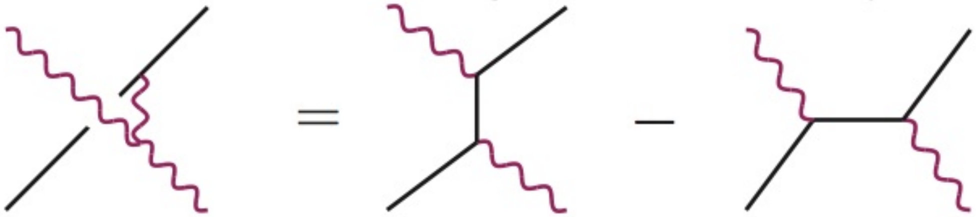
kinematic Jacobi



$$n_u = n_t - n_s$$

can be checked for 4pt on-shell ampl. using Feynman rules Halzen, Zhu

e.g.



$$\varepsilon_2 \cdot (\bar{u}_1 \not{V} u_3) \cdot \varepsilon_4 = \bar{u}_1 \not{\varepsilon}_4 \not{p}_t \not{\varepsilon}_2 u_3 - \bar{u}_1 \not{\varepsilon}_2 \not{p}_s \not{\varepsilon}_4 u_3$$

**Homework:** Check this! Also for 4 gluons!

# Gravity is a double copy

- Gravity amplitudes are obtained after replacing color by kinematics

$$\mathcal{A}_m^{(L)} = \sum_{i\in\Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

[BCJ]

$$\mathcal{M}_m^{(L)} = \sum_{i\in\Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_l}^2}$$

- The two numerators can belong to different theories:

| $n_i$ | | $\tilde{n}_i$ | | |
|---|---|---|---|---|
| ($\mathcal{N}$=4) | × | ($\mathcal{N}$=4) | → | $\mathcal{N}$=8 sugra |
| ($\mathcal{N}$=4) | × | ($\mathcal{N}$=2) | → | $\mathcal{N}$=6 sugra |
| ($\mathcal{N}$=4) | × | ($\mathcal{N}$=0) | → | $\mathcal{N}$=4 sugra |
| ($\mathcal{N}$=0) | × | ($\mathcal{N}$=0) | → | Einstein gravity + axion+ dillaton |

similar to Kawai-Lewellen-Tye but works at loop level

# Five-point example

- Decomposing 5pt amplitude in terms of 15 cubic diagrams

kinematic numerator

color factor

propagators

$$\mathcal{A}_5^{\rm tree} = g^3\Big(\frac{n_1c_1}{s_{12}s_{45}} + \frac{n_2c_2}{s_{23}s_{51}} + \frac{n_3c_3}{s_{34}s_{12}} + \frac{n_4c_4}{s_{45}s_{23}} + \frac{n_5c_5}{s_{51}s_{34}} + \frac{n_6c_6}{s_{14}s_{25}}$$
$$+ \frac{n_7c_7}{s_{32}s_{14}} + \frac{n_8c_8}{s_{25}s_{43}} + \frac{n_9c_9}{s_{13}s_{25}} + \frac{n_{10}c_{10}}{s_{42}s_{13}} + \frac{n_{11}c_{11}}{s_{51}s_{42}} + \frac{n_{12}c_{12}}{s_{12}s_{35}}$$
$$+ \frac{n_{13}c_{13}}{s_{35}s_{24}} + \frac{n_{14}c_{14}}{s_{14}s_{35}} + \frac{n_{15}c_{15}}{s_{13}s_{45}}\Big)\,,$$

$$s_{ij} = (k_i + k_j)^2$$

- Equivalent to partial amplitudes

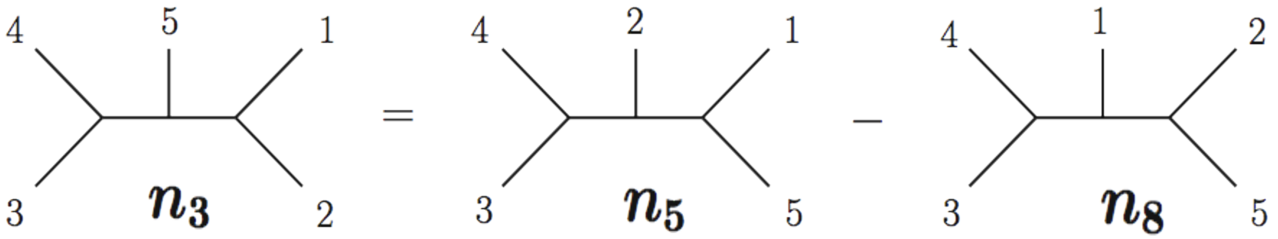
$$A_5^{\rm tree}(1,2,3,4,5) \equiv \frac{n_1}{s_{12}s_{45}} + \frac{n_2}{s_{23}s_{51}} + \frac{n_3}{s_{34}s_{12}} + \frac{n_4}{s_{45}s_{23}} + \frac{n_5}{s_{51}s_{34}}$$

etc…

- Duality between color and kinematics can be imposed, but not automatic

$$n_3 - n_5 + n_8 = 0 \qquad \Leftrightarrow \qquad c_3 - c_5 + c_8 = 0$$

$$c_3 \equiv \tilde f^{a_3a_4b}\tilde f^{ba_5c}\tilde f^{ca_1a_2}$$
$$c_5 \equiv \tilde f^{a_3a_4b}\tilde f^{ba_2c}\tilde f^{ca_1a_5}$$
$$c_8 \equiv \tilde f^{a_3a_4b}\tilde f^{ba_1c}\tilde f^{ca_2a_5}$$



checked through 8pts. All multiplicity solution known: Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard Vanhove

# Gauge theory amplitude properties

- Tree level, adjoint representation

$$\mathcal{A}_n^{\text{tree}}(1,2,\ldots,n) = g^{n-2} \sum_{\mathcal{P}(2,\ldots,n)} \text{Tr}[T^{a_1}T^{a_2}\cdots T^{a_n}]\, A_n^{\text{tree}}(1,2,\ldots,n)$$

gauge invariant

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- Well-known partial amplitude properties

$$\left.\begin{array}{ll} A_n^{\text{tree}}(1,2,\ldots,n) = A_n^{\text{tree}}(2,\ldots,n,1) & \text{cyclic symmetry} \\ A_n^{\text{tree}}(1,2,\ldots,n) = (-1)^n A_n^{\text{tree}}(n,\ldots,2,1) & \text{reflection symmetry} \end{array}\right\} (n-1)!/2$$

$$\left.\begin{array}{ll} \displaystyle\sum_{\sigma\in\text{cyclic}} A_n^{\text{tree}}(1,\sigma(2,3,\ldots,n)) = 0 & \text{“photon”-decoupling identity} \\ A_n^{\text{tree}}(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \displaystyle\sum_{\{\sigma\}_i\in\text{OP}(\{\alpha\},\{\beta^T\})} A_n^{\text{tree}}(1,\{\sigma\}_i,n) & \text{Kleiss-Kuijf relations} \end{array}\right\} (n-2)!$$

---

- *New BCJ relations reduce independent basis to* **$(n - 3)!$**

# Duality gives new amplitude relations

In color-ordered tree amplitudes 3 legs can be fixed: $(n$-$3)!$ basis

$$A_4^{\rm tree}(1,2,\{4\},3) = \frac{A_4^{\rm tree}(1,2,3,4)s_{14}}{s_{24}}$$

$s_{ij..} = (k_i+k_j+\ldots)^2$

$$A_5^{\rm tree}(1,2,\{4\},3,\{5\}) = \frac{A_5^{\rm tree}(1,2,3,4,5)(s_{14}+s_{45}) + A_5^{\rm tree}(1,2,3,5,4)s_{14}}{s_{24}}\,,$$

$$A_5^{\rm tree}(1,2,\{4,5\},3) = \frac{-A_5^{\rm tree}(1,2,3,4,5)s_{34}s_{15} - A_5^{\rm tree}(1,2,3,5,4)s_{14}(s_{245}+s_{35})}{s_{24}s_{245}}$$

...relations obtained for any multiplicity

These were later found to be equivalent to monodromy relations on the open string worldsheet Bjerrum-Bohr, Damgaard, Vanhove; Stieberger

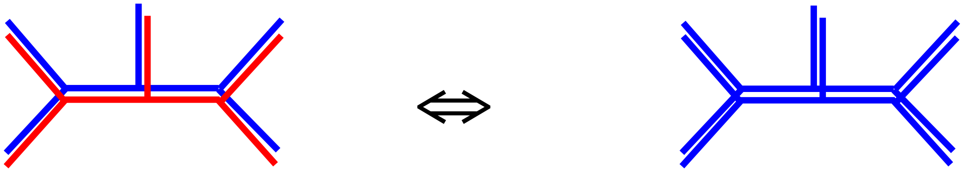
Also field theory proofs through BCFW: Feng, Huang, Jia; Chen, Du, Feng

Used in the solution of all open string disk amplitudes Mafra, Schlotterer, Stieberger

# Tree-level gravity checks

- Original conjecture checked through 8 points

$$\mathcal{A}_n^{\text{tree}} = \sum_i \frac{n_i c_i}{\prod_\alpha p_\alpha^2} \quad \Leftrightarrow \quad \mathcal{M}_n^{\text{tree}} = \sum_i \frac{n_i \tilde{n}_i}{\prod_\alpha p_\alpha^2}$$



double copy of YM

- All-multiplicity proof assuming gauge theory duality: Bern, Dennen, Huang, Kiermaier

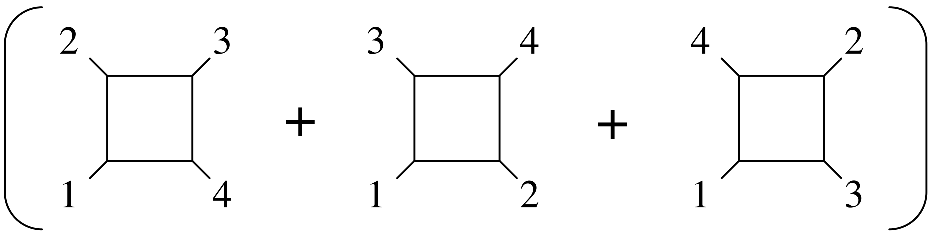
# Duality at loop level

- Duality is known to exist to all multiplicity at tree level
- Double copy relation to gravity is proven
- Open problem: Existence of duality at loop level ?
- Pedestrian approach: do case-by-case checks for various different amplitudes

# Manifest duality in $\mathcal{N}=4$ SYM 4-pt ampl.

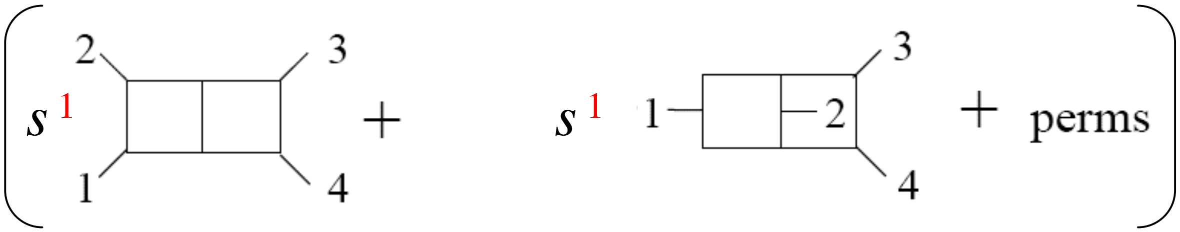
For some very simple loop amplitudes the loop-level duality follows from the tree-level one.

1-loop: $K^{1}$



Green, Schwarz, Brink (1982)

2-loop: $K^{1}$



Bern, Dixon, Dunbar, Perelstein and Rozowsky (1998)

prefactor contains helicity structure:

$$K = st A_4^{\text{tree}}$$

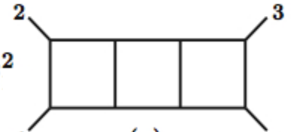
**Duality: $\mathcal{N}=8$ SG is obtained if $1 \rightarrow 2$ (numerator squaring)**

# Old form of 3-loop amplitude

Problem: no double copy in 0808.4112 [hep-th] (Bern, Carrasco, Dixon, HJ, Roiban)

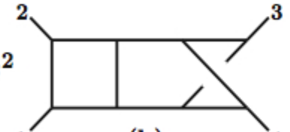
## N=4 SYM

$s^2$



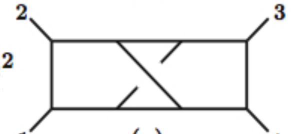
(a)

$s^2$



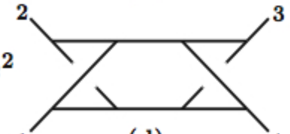
(b)

$s^2$



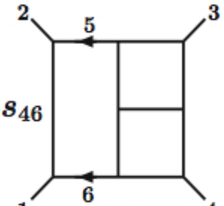
(c)

$s^2$



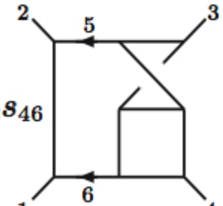
(d)

$s_{12}s_{46}$



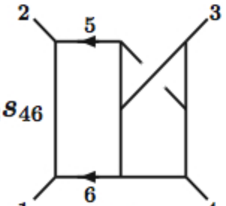
(e)

$s_{12}s_{46}$



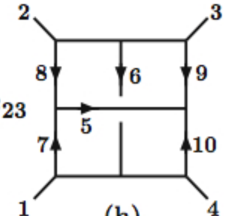
(f)

$s_{12}s_{46}$



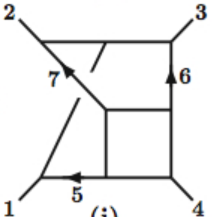
(g)

$s_{12}(\tau_{26}+\tau_{36})+s_{23}(\tau_{15}+\tau_{25})+s_{12}s_{23}$



(h)

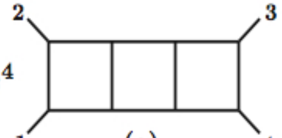
$s_{12}s_{45}-s_{23}s_{46}-\frac{1}{3}(s_{12}-s_{23})l_7^2$



(i)

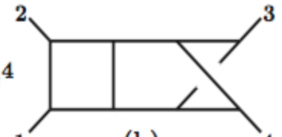
## N=8 SG

$s^4$



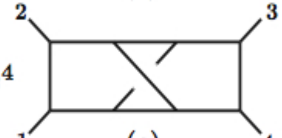
(a)

$s^4$



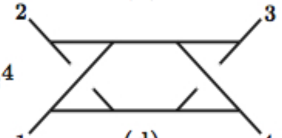
(b)

$s^4$



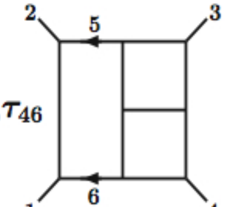
(c)

$s^4$



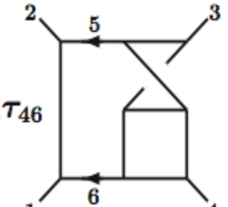
(d)

$s^2\tau_{35}\tau_{46}$



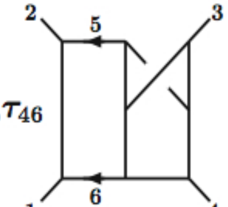
(e)

$s^2\tau_{35}\tau_{46}$



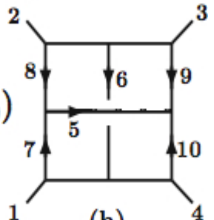
(f)

$s^2\tau_{35}\tau_{46}$



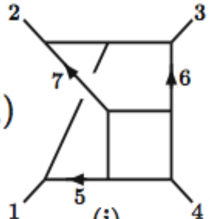
(g)

$$
\begin{aligned}
&(s(\tau_{26}+\tau_{36})+t(\tau_{15}+\tau_{25})+st)^2\\
&+(s^2(\tau_{26}+\tau_{36})-t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})\\
&+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10})+t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})\\
&+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})
\end{aligned}
$$



(h)

$$
\begin{aligned}
&(s\tau_{45}-t\tau_{46})^2-\tau_{27}(s^2\tau_{45}+t^2\tau_{46})\\
&-\tau_{15}(s^2\tau_{47}+u^2\tau_{46})-\tau_{36}(t^2\tau_{47}+u^2\tau_{45})\\
&+l_5^2s^2t+l_6^2st^2-\tfrac{1}{3}l_7^2stu
\end{aligned}
$$



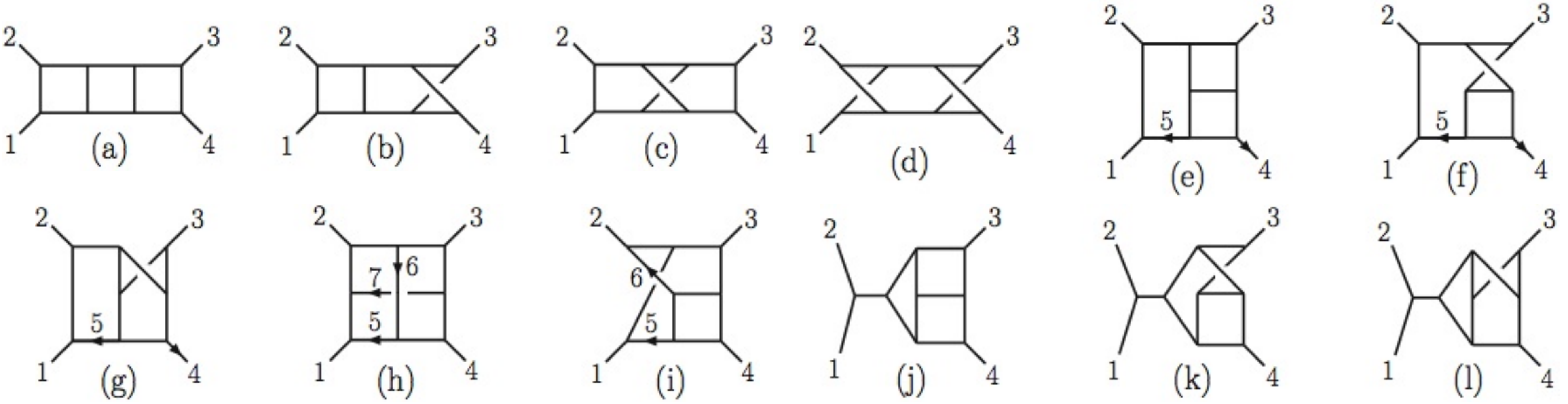
(i)

$\tau_{ij}=2k_i\cdot l_j$

# After nontrivial reshuffling

3-loop $\mathcal{N}$=4 SYM admits manifest realization of duality – and $\mathcal{N}$=8 SG is simply the square

1004.0476 [hep-th]
Bern, Carrasco, HJ

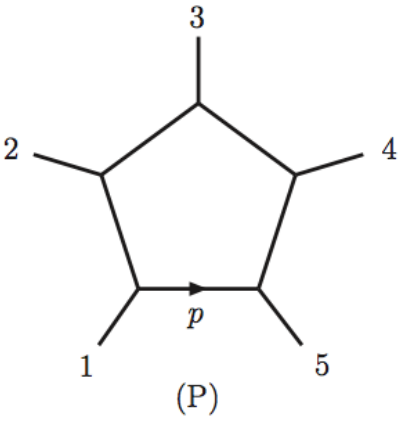


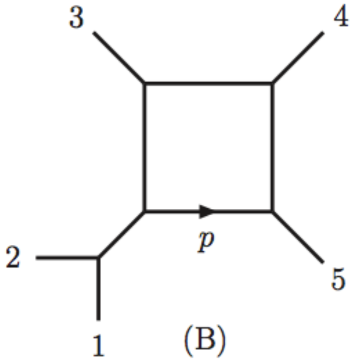
| Integral $I^{(x)}$ | $\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8\text{ supergravity}}$) numerator |
|---|---|
| (a)–(d) | $s^2$ |
| (e)–(g) | $\left(s\left(-\tau_{35}+\tau_{45}+t\right)-t\left(\tau_{25}+\tau_{45}\right)+u\left(\tau_{25}+\tau_{35}\right)-s^2\right)/3$ |
| (h) | $(s(2\tau_{15}-\tau_{16}+2\tau_{26}-\tau_{27}+2\tau_{35}+\tau_{36}+\tau_{37}-u)$<br>$+t(\tau_{16}+\tau_{26}-\tau_{37}+2\tau_{36}-2\tau_{15}-2\tau_{27}-2\tau_{35}-3\tau_{17})+s^2)/3$ |
| (i) | $(s(-\tau_{25}-\tau_{26}-\tau_{35}+\tau_{36}+\tau_{45}+2t)$<br>$+t(\tau_{26}+\tau_{35}+2\tau_{36}+2\tau_{45}+3\tau_{46})+u\,\tau_{25}+s^2)/3$ |
| (j)-(l) | $s(t-u)/3$ |

$\tau_{ij} = 2k_i \cdot l_j$

# 1-loop 5-pts $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 SG

1106.4711 [hep-th]



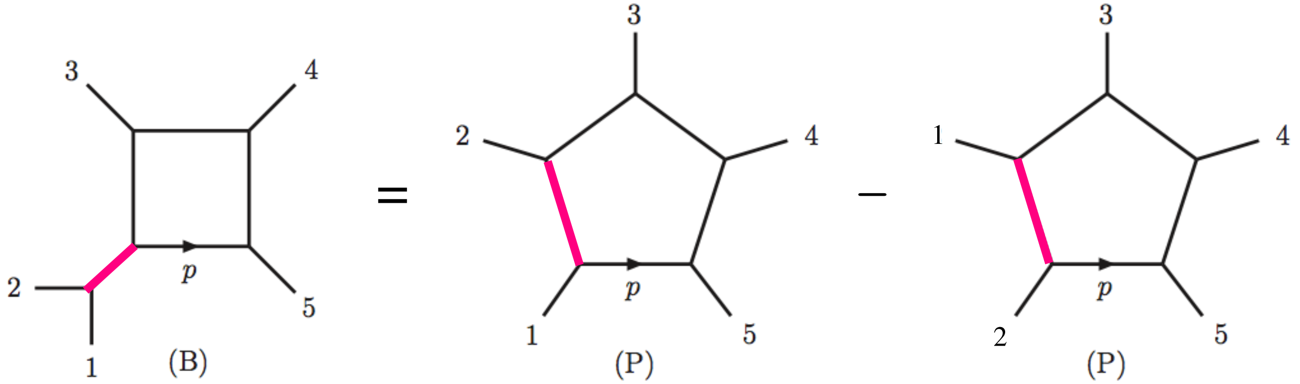


$$\beta_{12345} \equiv N^{(\mathrm{P})} = \delta^{(8)}(Q)\frac{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]}{4i\varepsilon(1,2,3,4)}$$

$$\gamma_{12} \equiv N^{(\mathrm{B})} = \delta^{(8)}(Q)\frac{[1\,2]^2\,[3\,4]\,[4\,5]\,[3\,5]}{4i\varepsilon(1,2,3,4)}$$

- The five-point amplitude makes the duality manifest !
- $\mathcal{N}$=8 SG is obtained through the numerator double copy

e.g. Jacobi relation:



$$N^{(\mathrm{B})}(1,2,3,4,p) = N^{(\mathrm{P})}(1,2,3,4,p) - N^{(\mathrm{P})}(2,1,3,4,p)$$

# All-loop 5pt $\mathcal{N}=4$ ansatz

Extrapolating the one-loop solution we can predict the all-loop structure

All cubic Feynman-like diagrams

$$\mathcal{A}_5^{(L)} = i g^{2L+3} \sum_{i \in \Gamma_3} \int \frac{d^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{N_i C_i}{p_{i_1}^2 p_{i_2}^2 p_{i_3}^2 \cdots p_{i_m}^2}$$

with numerators

$$N_i = \sum_{j,k,n} a_{i;jk;n} \, \gamma_{jk} \, M_n^{(L)}$$

constants

non-local state/helicity factors

local momentum invariants, dimension: $2L$-2

gammas and betas interchangeable

$$\gamma_{12} = \beta_{12345} - \beta_{21345}$$

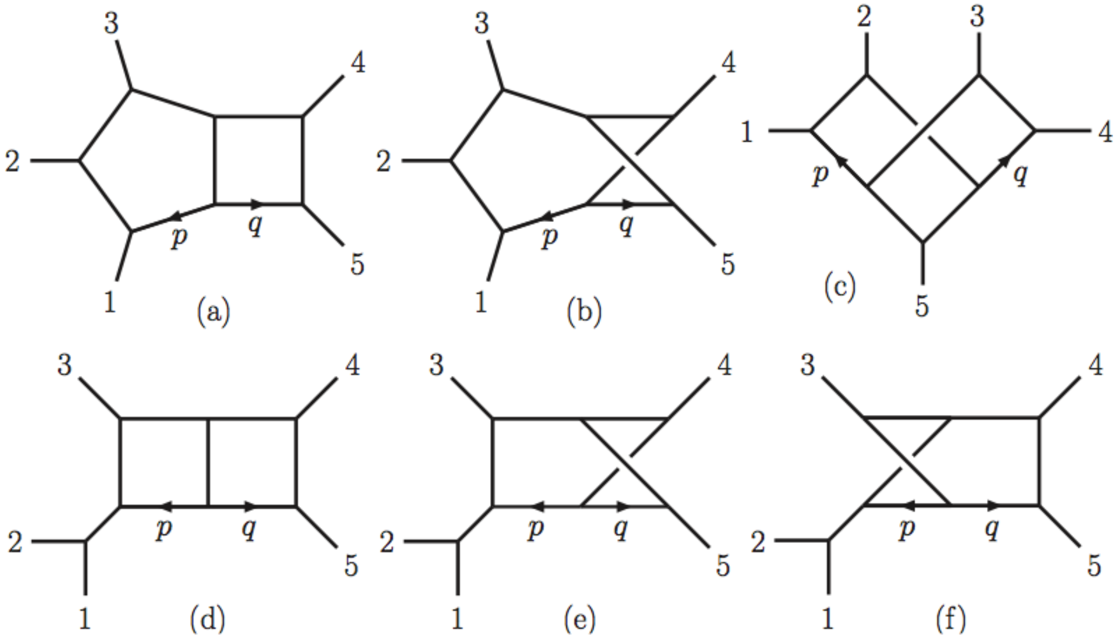
$$\beta_{12345} = \frac{1}{2}(\gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} + \gamma_{34})$$

and satisfies relations

$$\sum_{i=1}^{5} \gamma_{ij} = 0 \qquad \gamma_{ij} = -\gamma_{ji}$$

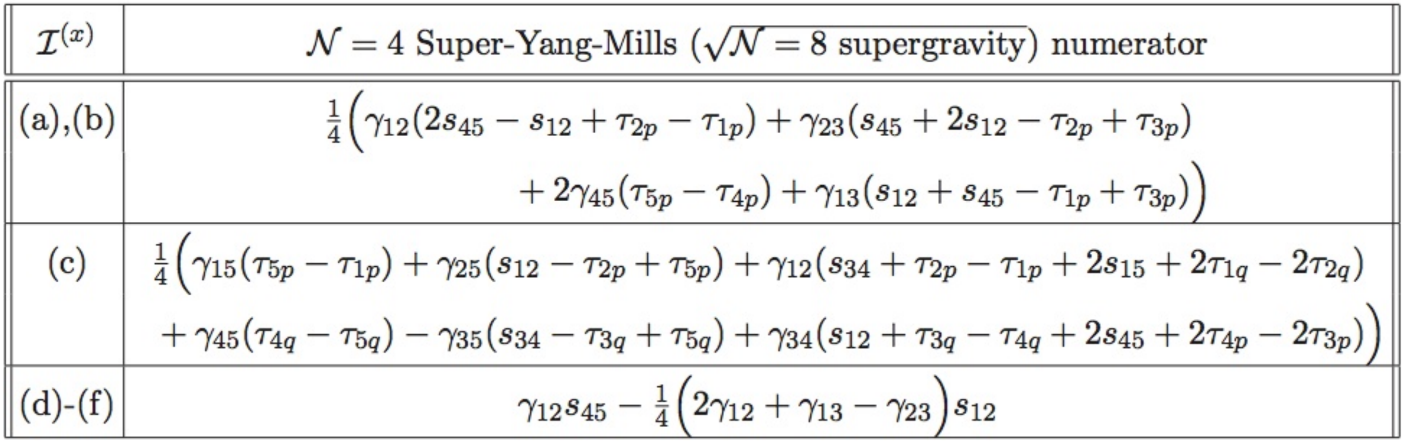
only 6 independent gammas!

# 2-loop 5-pts $\mathcal{N}$=4 SYM and $\mathcal{N}$=8 SG



1106.4711 [hep-th]

The 2-loop 5-point amplitude with duality exposed



| $\mathcal{I}^{(x)}$ | $\mathcal{N}=4$ Super-Yang-Mills ($\sqrt{\mathcal{N}=8 \text{ supergravity}}$) numerator |
|---|---|
| (a),(b) | $\frac{1}{4}\Big(\gamma_{12}(2s_{45}-s_{12}+\tau_{2p}-\tau_{1p})+\gamma_{23}(s_{45}+2s_{12}-\tau_{2p}+\tau_{3p})$ <br> $+2\gamma_{45}(\tau_{5p}-\tau_{4p})+\gamma_{13}(s_{12}+s_{45}-\tau_{1p}+\tau_{3p})\Big)$ |
| (c) | $\frac{1}{4}\Big(\gamma_{15}(\tau_{5p}-\tau_{1p})+\gamma_{25}(s_{12}-\tau_{2p}+\tau_{5p})+\gamma_{12}(s_{34}+\tau_{2p}-\tau_{1p}+2s_{15}+2\tau_{1q}-2\tau_{2q})$ <br> $+\gamma_{45}(\tau_{4q}-\tau_{5q})-\gamma_{35}(s_{34}-\tau_{3q}+\tau_{5q})+\gamma_{34}(s_{12}+\tau_{3q}-\tau_{4q}+2s_{45}+2\tau_{4p}-2\tau_{3p})\Big)$ |
| (d)-(f) | $\gamma_{12}s_{45}-\frac{1}{4}\Big(2\gamma_{12}+\gamma_{13}-\gamma_{23}\Big)s_{12}$ |

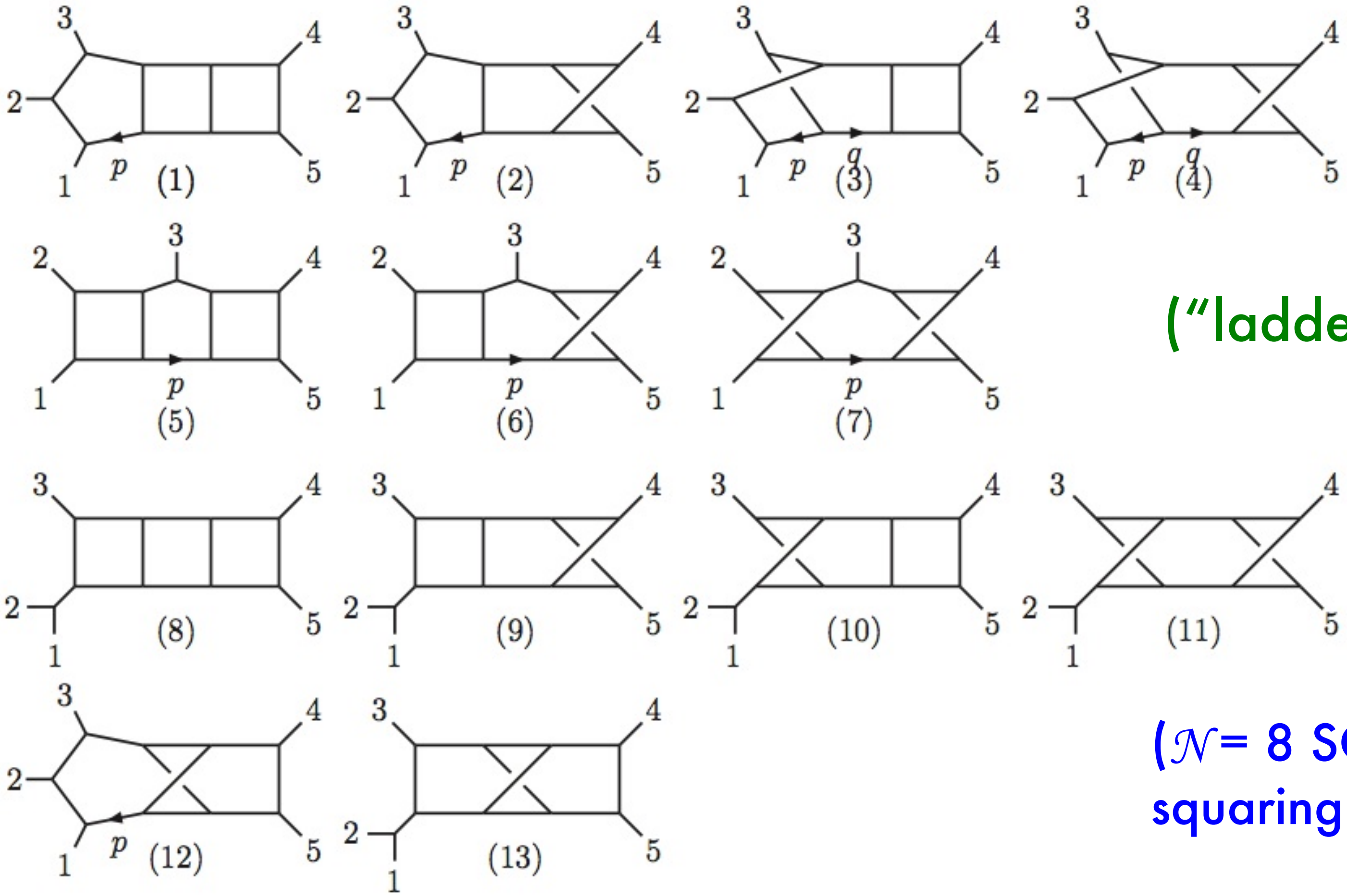
$\mathcal{N}$= 8 SG obtained from numerator double copies

$$\tau_{ip}=2k_i\cdot p$$

# 3-loop 5-point SYM and $\mathcal{N}$=8 SG

Again the color-dressed D-dimensional amplitude admits a representation with manifest duality

(to be published)



("ladder-like" diagrams)

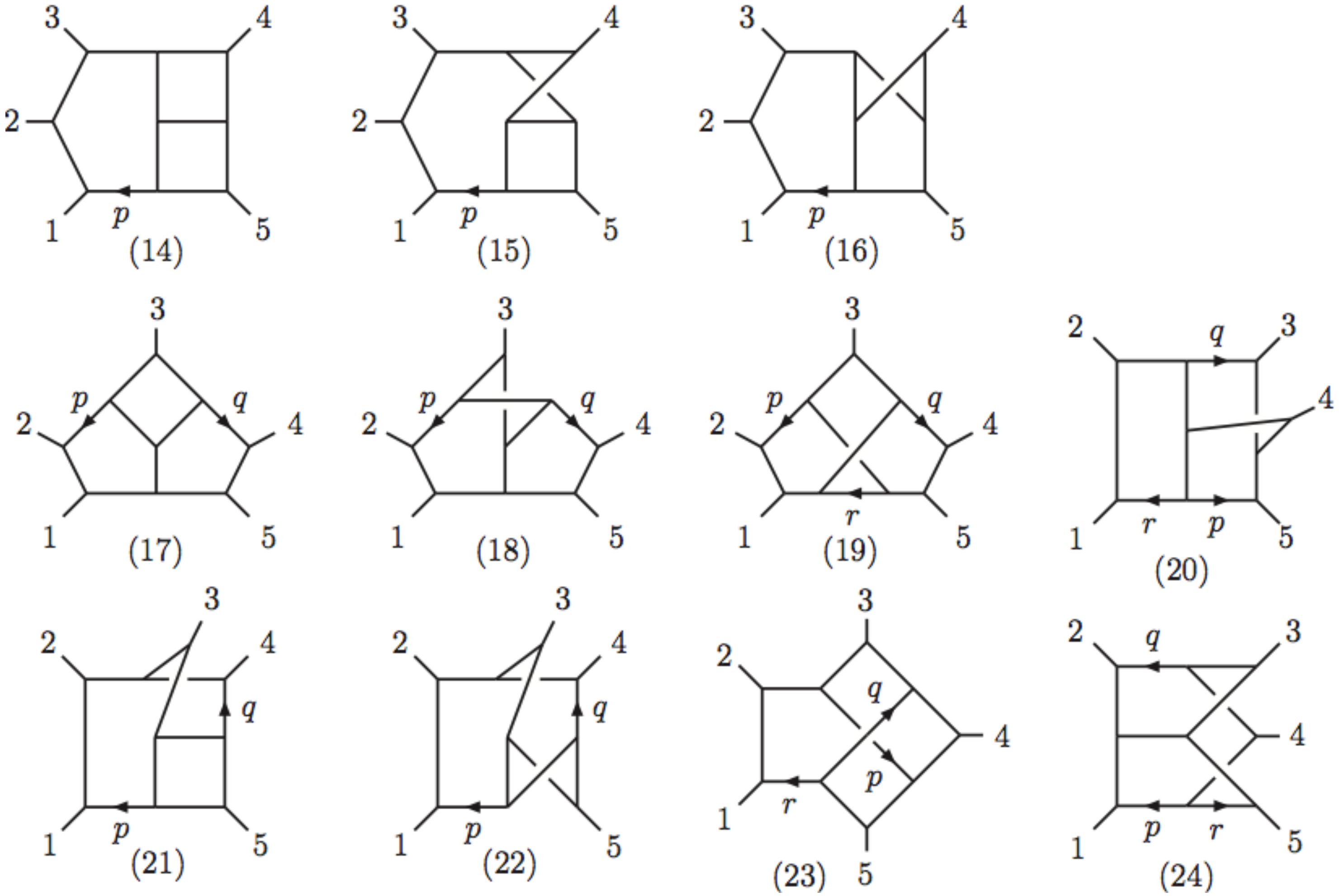
($\mathcal{N}$= 8 SG obtained from squaring the numerators)

$$N_8 = \gamma_{12} s_{45}^2 - \frac{1}{12} s_{12} \Big( \gamma_{13}(2s_{13}+12s_{23}-s_{12}) - \gamma_{23}(2s_{23}+12s_{13}-s_{12}) - \gamma_{12}(7s_{12}-11s_{45}) \Big)$$

# 3-loop 5-point SYM and $\mathcal{N}$=8 SG

some “Mercedes-like” diagrams…

(to be published)



$$N_{14} = \frac{1}{2}\gamma_{45}(\tau_{1p}^2 + \tau_{2p}^2 + \tau_{3p}^2 + \tau_{4p}^2 + \tau_{5p}^2) + \text{ subleading in } p$$

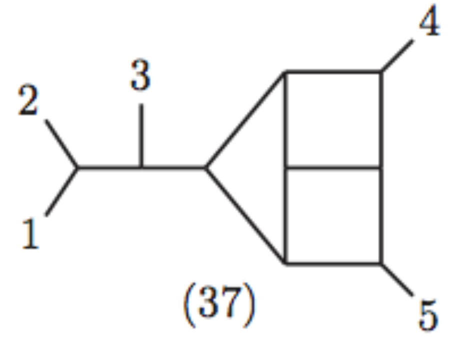
$$\tau_{ip}^2 = 2k_i \cdot p$$

# 3-loop 5-point SYM and $\mathcal{N}$=8 SG

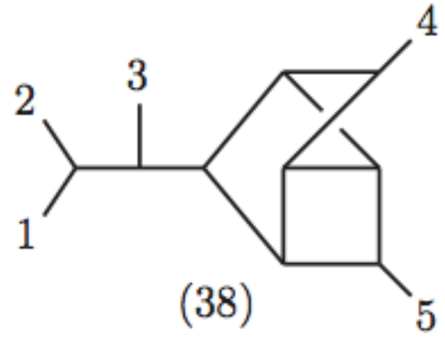
...in total 42 diagrams.

(to be published)

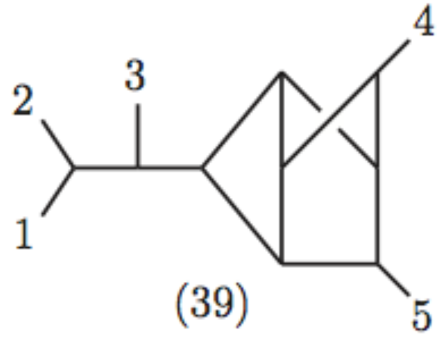
Conveniently the UV divergent diagrams (in D=6) are very simple:



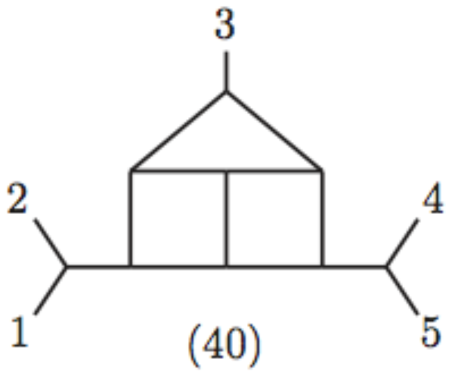
(37)



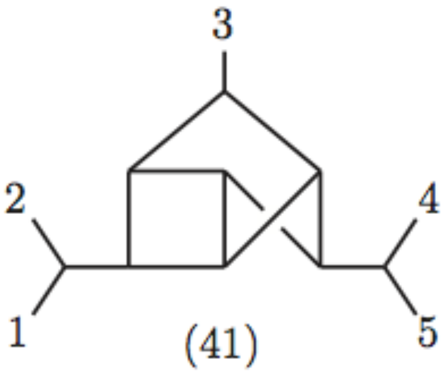
(38)



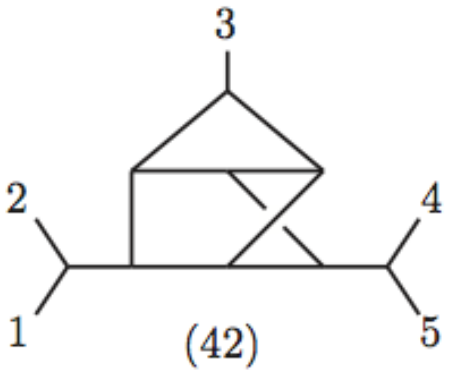
(39)



(40)



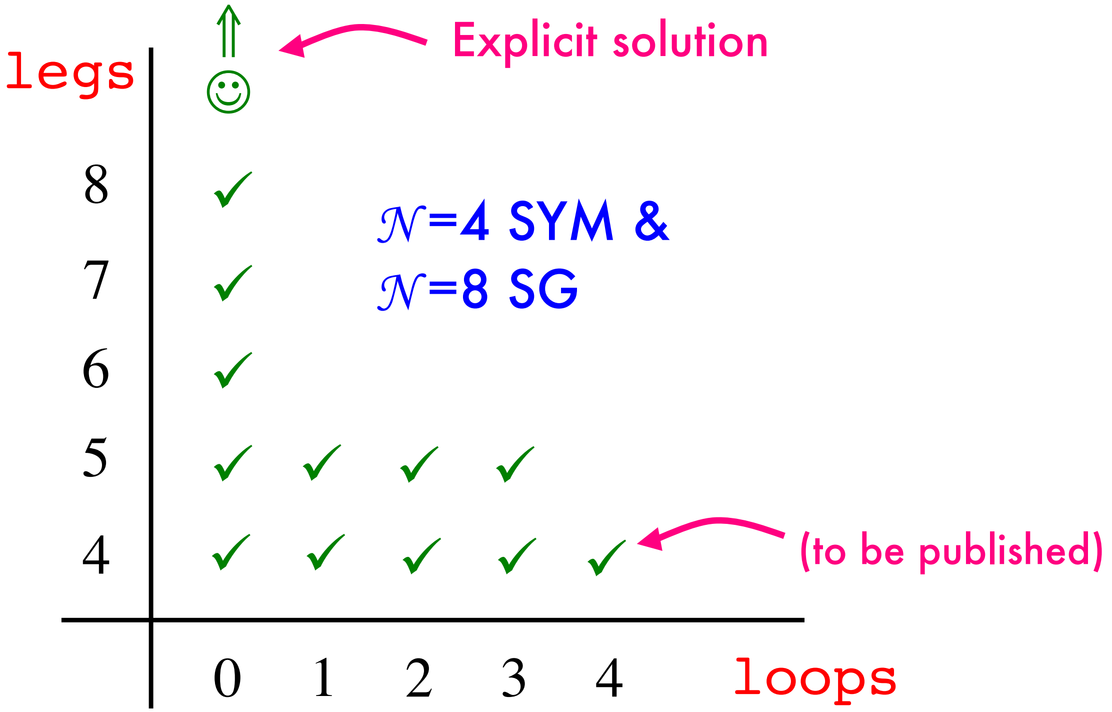
(41)



(42)

(for SG the UV div. comes from the other diagrams as well)

# Summary of checks of duality



| | | | | | | |
|---|---|---|---|---|---|---|
| legs | ⇑<br>☺ | | | | | |
| 8 | ✓ | | | | | |
| 7 | ✓ | | | | | |
| 6 | ✓ | | | | | |
| 5 | ✓ | ✓ | ✓ | ✓ | | |
| 4 | ✓ | ✓ | ✓ | ✓ | ✓ | |
| | 0 | 1 | 2 | 3 | 4 | loops |

Explicit solution

$\mathcal{N}$=4 SYM &
$\mathcal{N}$=8 SG

(to be published)

Less-SUSY theories:

Tree level: all pure gauge theories have the same
tree amplitudes as $\mathcal{N}$=4 SYM ✓

One-loop: $\mathcal{N}$=4,5,6 SG, 4p ✓

Two-loop: $\mathcal{N}$=0 YM, 4p all-plus helicity ✓
$\mathcal{N}$=4,5,6 SG, 4p ✓ Dixon, Boucher-Veronneau

# Lagrangian and Lie Algebra

- First attempt at Lagrangian with manifest duality

1004.0693 [hep-th]
Bern, Dennen, Huang, Kiermaier

YM Lagrangian receives corrections at 5 points and higher

$$\mathcal{L}_{YM} = \mathcal{L} + \mathcal{L}'_5 + \mathcal{L}'_6 + \dots$$

corrections proportional to the Jacobi identity (thus equal to zero)

$$\mathcal{L}'_5 \sim \mathrm{Tr}\,[A^\nu, A^\rho]\frac{1}{\Box}([[\partial_\mu A_\nu, A_\rho], A^\mu] + [[A_\rho, A^\mu], \partial_\mu A_\nu] + [[A^\mu, \partial_\mu A_\nu], A_\rho])$$

Introduction of auxiliary “dynamical” fields gives local cubic Lagrangian

$$\mathcal{L}_{YM} = \tfrac{1}{2}A^{a\mu}\Box A^a_\mu - B^{a\mu\nu\rho}\Box B^a_{\mu\nu\rho} - g f^{abc}\underbrace{(\partial_\mu A^a_\nu + \partial^\rho B^a_{\rho\mu\nu})A^{b\mu}A^{c\nu} \;+ \dots}_{\text{kinematical structure constants}}$$

- Monteiro and O'Connell (1105.2565 [hep-th]) identifies a Lie algebra in the self-dual YM sector ⇒ kin. structure constants for MHV tree amplitudes.

# Open Problems

- Open problems

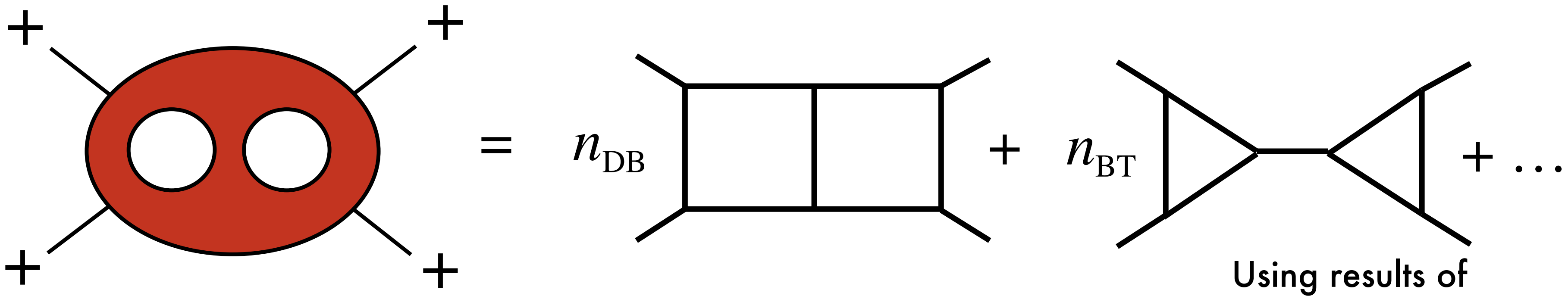
1. More one-loop calculations needed.
    - 6-pt and higher N=4 SYM
    - N<4 SYM and QCD
2. What is the kinematic Lie algebra ?
    - Find structure constants
3. Find Lagrangian with manifest duality
4. Implications: Planar <-> Non-planar
5. Proof or more evidence

# Extra slides

# Works for non-susy theories

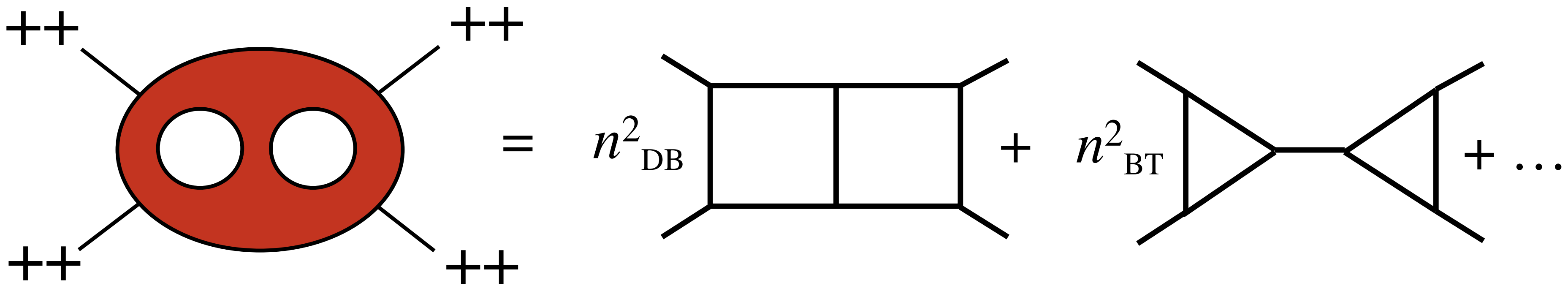
1004.0476 [hep-th]
Bern, Carrasco, HJ

All-plus-helicity QCD amplitude:



Using results of
hep-ph/0001001
(Bern, Dixon, Kosower)

All-plus-helicity Einstein gravity amplitude:

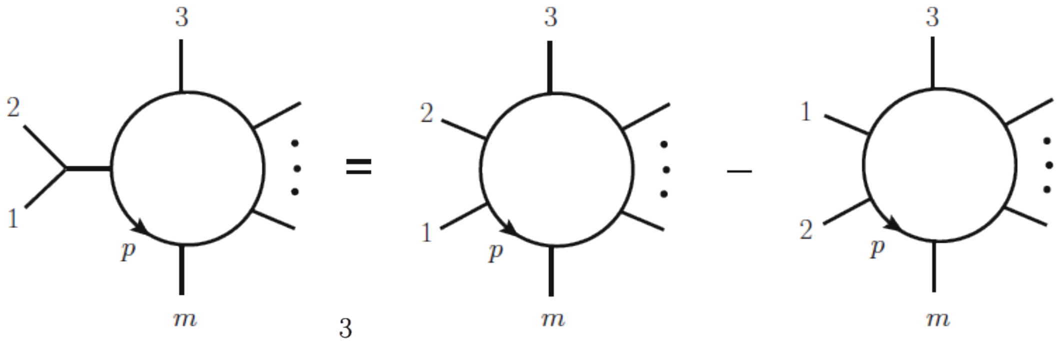


(with dilation and axion in loops)

# $\mathcal{N} < 8$ supergravity at one loop

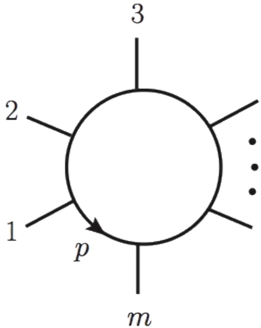
Bern, Boucher-Veronneau, HJ

Eliminate 1PR diagrams:



only ring diagrams remain

color basis of Del Duca, Dixon and Maltoni



$\mathrm{Tr}_A[f^{a_1} f^{a_2} f^{a_3} \cdots f^{a_m}]$

Yang-Mills:

$$\mathcal{A}_m^{\text{1-loop}} = g^m \sum_{S_m/(Z_m\times Z_2)} \int \frac{d^D p}{(2\pi)^D}\, c_{123\ldots m}\, \underbrace{\mathscr{A}_m(1,2,\ldots,m;p)}_{\text{“integrand”}}$$

need one copy of duality satisfying numerators:

Gravity:

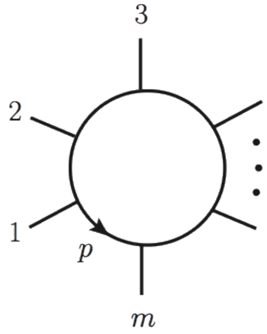
$$\mathcal{M}_m^{\text{1-loop}} = \left(\frac{\kappa}{2}\right)^m \sum_{S_m/(Z_m\times Z_2)} \int \frac{d^D p}{(2\pi)^D}\, \boxed{\tilde{n}_{123\ldots m}(p)}\, \boxed{\mathscr{A}_m(1,2,\ldots,m;p)}$$

$(\mathcal{N}=4) \times (\mathcal{N}= 0,1,2,4)$

# Relations between integrated ampls.

$$\mathcal{M}_m^{\text{1-loop}} = \left(\frac{\kappa}{2}\right)^m \sum_{S_m/(Z_m\times Z_2)} \int \frac{d^Dp}{(2\pi)^D}\, \tilde{n}_{123\ldots m}(p)\, \mathscr{A}_m(1,2,\ldots,m;p)$$

($\mathcal{N}$=4 SYM) × ($\mathcal{N}$= $p$ SYM)



Assume $\tilde{n}_{123\ldots m}(p) = \tilde{n}_{123\ldots m}$ (true for $m$ = 4,5)

Relations between integrated amplitudes:

$$\mathcal{M}_m^{\text{1-loop}} = \left(\frac{\kappa}{2}\right)^m \sum_{S_m/(Z_m\times Z_2)} \tilde{n}_{123\ldots m}\, A^{\text{1-loop}}_{\mathcal{N}\ \text{susy}}(1,2,\ldots,m)$$

$\mathcal{N}$=4 SYM numerators

4pt: Green, Schwarz, Brink

5pt: Carrasco, HJ

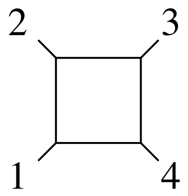
$\mathcal{N}$=0,1,2 amplitudes

4pt: Bern and Morgan

5pt in hep-ph/9302280 (Bern, Dixon, Kosower)

⇒ It works! reproduce $\mathcal{N}$=4,5,6 supergravity ampl. Dunbar, Ettle, Perkins, Dunbar and Norridge

# Four-point check details



Green, Schwarz, Brink: $n_{1234} = n_{1243} = n_{1423} = istA^{\text{tree}}(1,2,3,4)$

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}+4\text{ susy}}(1,2,3,4) \;=\; \left(\frac{\kappa}{2}\right)^4 istA^{\text{tree}}(1,2,3,4)\Big(A^{\text{1-loop}}_{\mathcal{N}\text{ susy}}(1,2,3,4)+A^{\text{1-loop}}_{\mathcal{N}\text{ susy}}(1,2,4,3) + A^{\text{1-loop}}_{\mathcal{N}\text{ susy}}(1,4,2,3)\Big)\,,$$

Bern, Morgan in D dim.

Can work with the simpler matter multiplet contributions, we get

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=6,\text{mat.}}(1^-,2^-,3^+,4^+) \;=\; -\frac{ic_\Gamma}{2}\left(\frac{\kappa}{2}\right)^4\frac{\langle 1\,2\rangle^4\,[3\,4]^4}{s^2}\left[\ln^2\left(\frac{-t}{-u}\right)+\pi^2\right]+\mathcal{O}(\epsilon)$$

$$\mathcal{M}^{\text{1-loop}}_{\mathcal{N}=4,\text{mat.}}(1^-,2^-,3^+,4^+) \;=\; \frac{1}{2}\left(\frac{\kappa}{2}\right)^4\frac{\langle 1\,2\rangle^2\,[3\,4]^2}{[1\,2]^2\,\langle 3\,4\rangle^2}\Big[ic_\Gamma s^2 + s(u-t)\Big(I_2(t)-I_2(u)\Big) - 2I_4^{D=6-2\epsilon}(t,u)stu\Big]+\mathcal{O}(\epsilon)\,,$$

From this one gets any $\mathcal{N} \geq 4$ supergravity theory ampl.

Agrees with Dunbar and Norridge