Momentum twistors, special functions and symbols

Lecture 1

Claude Duhr

School of analytic computing
Atrani, 06/10 - 11/10 2011

Aim of this lecture

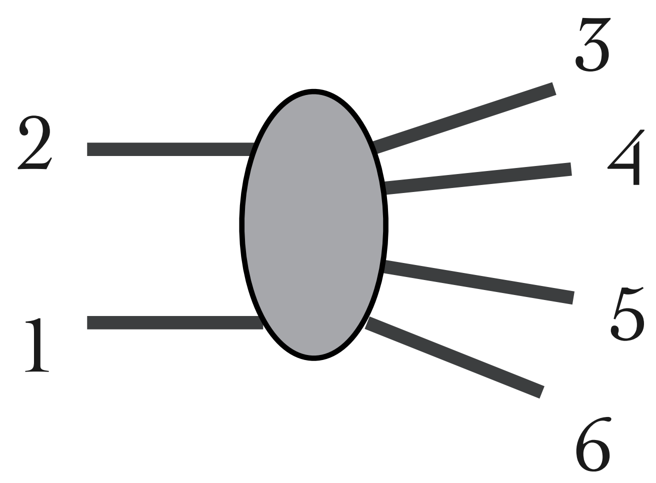
- Present mathematical tools that are useful for loop computations.
- Plan:
    - ➡ Topic 1: Kinematics
    - ➡ Topic 2: Multiple Polylogarithms
    - ➡ Topic 3: Some more formal theorems about the special numbers and functions that appear in loops.
    - ➡ Topic 4: Symbols
- There will likely be connections to other lectures, where some of these concepts will show up.

Kinematics

General considerations

Kinematics of a scattering

- We consider a $2 \longrightarrow n$ scattering.



- A priori: Function of n external momenta, i.e., of $4n$ real degrees of freedom.
- This set of variables is of course highly overconstrained.
- Question: What is a 'good' set of variables?

Kinematics of a scattering

- Assume we have expressed all our tensor integrals as scalar integrals.
  - ➡ Integrals can only depend on scalar products $s_{ij} = (p_i + p_j)^2$
- Counting of two-particle invariants ($i \neq j$):
  - ➡ A priori: $\binom{n}{2} = \frac{n(n-1)}{2}$

Kinematics of a scattering

- Assume we have expressed all our tensor integrals as scalar integrals.
  - ➡ Integrals can only depend on scalar products $s_{ij} = (p_i + p_j)^2$
- Counting of two-particle invariants ($i \neq j$):
  - ➡ A priori: $\binom{n}{2} = \frac{n(n-1)}{2}$
  - ➡ Momentum conservation: $\sum_{i=1}^{n} p_i = 0 \Rightarrow \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$

Kinematics of a scattering

- Assume we have expressed all our tensor integrals as scalar integrals.
  - ➡ Integrals can only depend on scalar products $s_{ij} = (p_i + p_j)^2$
- Counting of two-particle invariants ($i \neq j$):
  - ➡ A priori: $\binom{n}{2} = \frac{n(n-1)}{2}$
  - ➡ Momentum conservation: $\sum_{i=1}^{n} p_i = 0 \Rightarrow \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$
  - ➡ All momenta must be on-shell, $p_i^2 = m_i^2$

Kinematics of a scattering

- Assume we have expressed all our tensor integrals as scalar integrals.
  - ➡ Integrals can only depend on scalar products $s_{ij} = (p_i + p_j)^2$
- Counting of two-particle invariants ($i \neq j$):
  - ➡ A priori: $\binom{n}{2} = \frac{n(n-1)}{2}$
  - ➡ Momentum conservation: $\sum_{i=1}^{n} p_i = 0 \Rightarrow \binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$
  - ➡ All momenta must be on-shell, $p_i^2 = m_i^2$
  - ➡ A sum of (n-1) on-shell momenta does not necessarily satisfy the on-shellness constraint for p_n

$$m_n^2 = p_n^2 = (p_1 + \ldots + p_{n-1})^2 = \text{polynomial in } s_{ij}$$

$$\binom{n-1}{2} - 1 = \frac{n(n-3)}{2}$$

Example

- A four-point function depends on 4 momenta satisfying

$$p_1 + p_2 + p_3 + p_4 = 0 \qquad\qquad p_i^2 = m_i^2$$

➡ Need only to consider invariants that depend on p_1, p_2, p_3

$$s_{12} = s \qquad\qquad s_{23} = t \qquad\qquad s_{13} = u$$

- On-shellness constraint:

$$m_4^2 = p_4^2 = (p_1 + p_2 + p_3)^2 = s + t + u - m_1^2 - m_2^2 - m_3^2$$

- Counting:

$$\binom{4-1}{2} - 1 = \frac{4(4-3)}{2} = 2$$

- Exercise: Show that for *n*=5, the kinematics is described by 5 external masses, and by the 5 invariants $s_{i,i+1}$

Gram determinants

- Starting from 6 points, momentum conservation and on-shellness are no longer enough in 4 dimensions:
  - ➡ Momentum conservation implies 5 independent momenta (subject to the onshellness constraint).
  - ➡ But only 4 momenta can be linearly independent in 4 dimensions!

$$\mathrm{Gram}(p_1, p_2, p_3, p_4, p_5) = 0$$

  - ➡ We obtain a complicated polynomial relation among the invariants.

Gram determinants

- Starting from 6 points, momentum conservation and on-shellness are no longer enough in 4 dimensions:
  - ➡ Momentum conservation implies 5 independent momenta (subject to the onshellness constraint).
  - ➡ But only 4 momenta can be linearly independent in 4 dimensions!

$$\mathrm{Gram}(p_1, p_2, p_3, p_4, p_5) = 0$$

$$\begin{aligned}
&s_2\, t_2^2\, s^2 + s_2^2\, t_2\, s^2 - s_2\, t_1\, t_2\, s^2 + s_2\, t_1\, t_3\, s^2 - s_2\, t_2\, t_3\, s^2 + s_2^2\, s_3^2\, s + s_3^2\, t_1^2\, s - s_3\, s_{456} t_1^2\, s + s_2^2\, t_2^2\, s \\
&- s_2\, s_{345}\, t_2^2\, s - s_2 s_{456} t_2^2\, s - s_{345}\, s_{456}\, t_2^2\, s + s_3^2\, t_3^2\, s - s_3\, s_{345}\, t_3^2\, s - 2\, s_2\, s_3^2\, t_1\, s + s_2\, s_3\, s_{456}\, t_1\, s \\
&- 2 s_2^2\, s_3\, t_2\, s + s_2\, s_3\, s_{345}\, t_2\, s + s_2\, s_3\, s_{456}\, t_2\, s - 2 s_2\, s_{345}\, s_{456}\, t_2\, s + 2 s_2\, s_3\, t_1\, t_2\, s - s_3\, s_{345}\, t_1\, t_2\, s \\
&+ s_2\, s_{456}\, t_1\, t_2\, s + s_3\, s_{456}\, t_1\, t_2\, s + s_{345}\, s_{456}\, t_1\, t_2\, s - 2 s_2\, s_3^2\, t_3\, s + s_2\, s_3\, s_{345}\, t_3\, s - 2 s_3^2\, t_1\, t_3\, s \\
&- 4 s_2\, s_3\, t_1\, t_3\, s + s_3\, s_{345}\, t_1\, t_3\, s + s_3\, s_{456}\, t_1\, t_3\, s - s_{345}\, s_{456}\, t_1\, t_3\, s + 2 s_2\, s_3\, t_2\, t_3\, s + s_2\, s_{345}\, t_2\, t_3\, s \\
&+ s_3\, s_{345}\, t_2\, t_3\, s - s_3\, s_{456}\, t_2\, t_3\, s + s_{345}\, s_{456}\, t_2\, t_3\, s + s_3\, s_{456}^2\, t_1^2 + s_{345}\, s_{456}^2\, t_2^2 + s_{345}^2\, s_{456}\, t_2^2 \\
&- s_2\, s_{345}\, s_{456}\, t_2^2 + s_3\, s_{345}^2\, {t_3}^2 - s_2\, {s_3}^2\, s_{345}\, s_{456} + s_3^2\, s_{456}^2\, t_1 - s_3\, s_{345}\, s_{456}^2\, t_1 + s_3^2\, s_{345}\, s_{456}\, t_1 \\
&+ s_{345}^2\, s_{456}^2\, t_2 - s_3\, s_{345}\, s_{456}^2\, t_2 - s_3\, s_{345}^2\, s_{456}\, t_2 + 2 s_2\, s_3\, s_{345}\, s_{456}\, t_2 - s_3\, s_{456}^2\, t_1\, t_2 \\
&- s_{345}\, s_{456}^2\, t_1\, t_2 - s_3\, s_{345}\, s_{456}\, t_1\, t_2 + s_3^2\, s_{345}^2\, t_3 - s_3\, s_{345}^2\, s_{456}\, t_3 + s_3^2\, s_{345}\, s_{456}\, t_3 \\
&+ 2 s_3\, s_{345}\, s_{456}\, t_1\, t_3 - s_3\, s_{345}^2\, t_2\, t_3 - s_{345}^2\, s_{456}\, t_2\, t_3 - s_3\, s_{345}\, s_{456}\, t_2\, t_3 = 0.
\end{aligned}$$

Gram determinants

- Starting from 6 points, momentum conservation and on-shellness are no longer enough in 4 dimensions:
  - ➡ Momentum conservation implies 5 independent momenta (subject to the onshellness constraint).
  - ➡ But only 4 momenta can be linearly independent in 4 dimensions!

$$\text{Gram}(p_1, p_2, p_3, p_4, p_5) = 0$$

  - ➡ We obtain a complicated polynomial relation among the invariants.
- Counting:

$$\frac{n(n-3)}{2} - \binom{n-4}{2} = 3n - 10$$

- N.B.: For n=4,5, we have

$$\frac{n(n-3)}{2} = 3n - 10$$

Summary

- Contraints:
  - ➡ Momentum conservation.
  - ➡ On-shellness.
  - ➡ Gram determinant.

  } $3n-10$ independent variables in 4 dimensions

- Can we do better than this?
- Can we find better variables where some of the constraints are trivial?
- Let's restrict ourselves to
  - ➡ massless external states
  - ➡ planar graphs

Kinematics

Massless particles: Spinor-helicity formalism

Spinor-Helicity formalism

- Real 4-vectors can be parametrized by hermitian 2x2 matrices:

$$P_i^{a\dot{a}} = p_i^\mu \, \sigma_\mu^{a\dot{a}} \qquad\qquad \det P_i = ||p_i||^2$$

- For null vectors, we can parametrize this matrix by

$$P_i^{a\dot{a}} = \lambda_i^a \, \bar{\lambda}_i^{\dot{a}}$$

where λ_i^a and $\bar{\lambda}_i^{\dot{a}}$ are two component (1/2,0) and (0,1/2) spinors.

- Mandelstam invariants are expressed via *spinor products*.

$$\langle i \; j \rangle = \epsilon_{ab} \, \lambda_i^a \, \lambda_j^b = \bar{u}_-(i) u_+(j) \qquad\qquad [i \; j] = \epsilon_{\dot{a}\dot{b}} \, \bar{\lambda}_i^{\dot{a}} \, \bar{\lambda}_j^{\dot{b}} = \bar{u}_+(i) u_-(j)$$

$$s_{ij} = \langle i \; j \rangle \, [i \; j]$$

Spinor-Helicity formalism

- Advantage: the spinor-helicity solves the on-shellness constraint!

$$\langle i\ j\rangle = \epsilon_{ab}\,\lambda_i^a\,\lambda_j^b = \bar{u}_-(i)u_+(j) \qquad [i\ j] = \epsilon_{\dot{a}\dot{b}}\,\bar{\lambda}_i^{\dot{a}}\,\bar{\lambda}_j^{\dot{b}} = \bar{u}_+(i)u_-(j)$$

$$s_{ij} = \langle i\ j\rangle\,[i\ j]$$

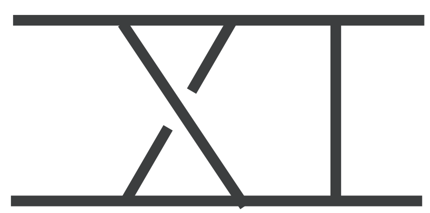
- In other words, choose n spinors λ_i^a (and their complex conjugates $\bar{\lambda}_i^{\dot{a}}$) that constraint by
  - ➡ Momentum conservation: $\sum_i \lambda_i^a\,\bar{\lambda}_i^{\dot{a}} = 0$
  - ➡ Satisfy the Gram determinant constraint.

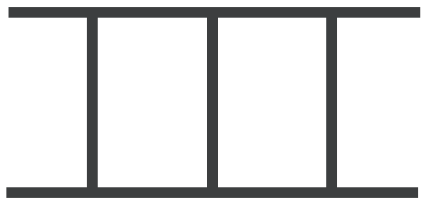
Kinematics

Planar graphs: Dual coordinates

Planar graphs

- **Definition:** A graph is said to be planar if it can be drawn in a plane without selfcrossings.
- **Examples:**

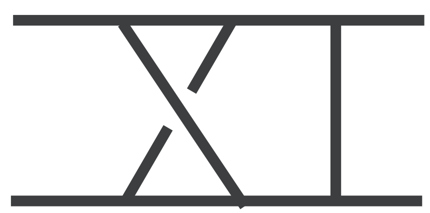




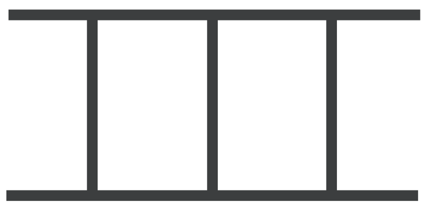
- **N.B.:** Tree and one-loop graphs are always planar! [Why?]

Planar graphs

- **Definition:** A graph is said to be planar if it can be drawn in a plane without selfcrossings.
- **Examples:**



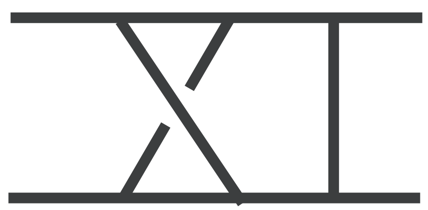
non planar



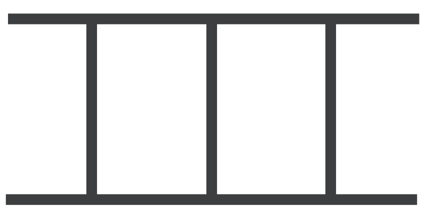
- **N.B.:** Tree and one-loop graphs are always planar! [Why?]

Planar graphs

- **Definition:** A graph is said to be planar if it can be drawn in a plane without selfcrossings.
- **Examples:**



non planar

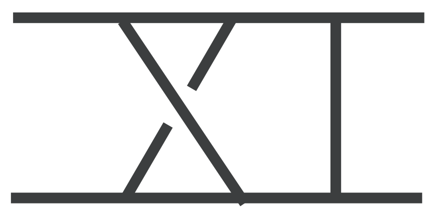


planar

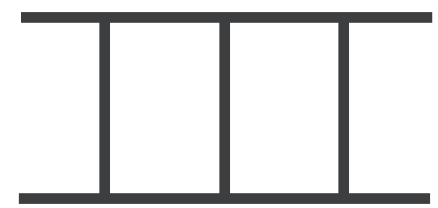
- **N.B.:** Tree and one-loop graphs are always planar! [Why?]

Planar graphs

- Definition: A graph is said to be planar if it can be drawn in a plane without selfcrossings.
- Examples:



non planar



planar

- N.B.: Tree and one-loop graphs are always planar! [Why?]
- Planar graphs appear for example in the limit of a large number of colors.

Planar graphs

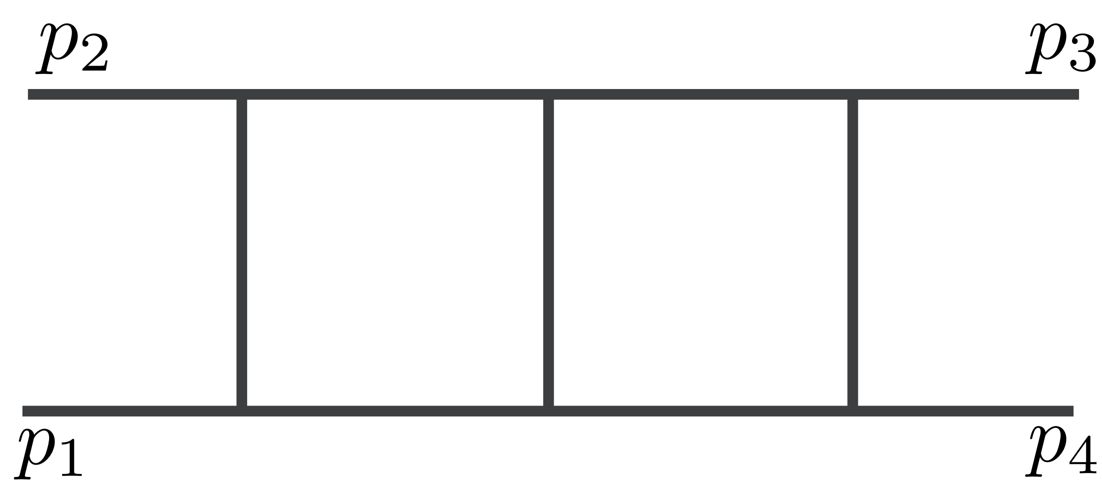
- Definition: A graph is said to be planar if it can be drawn in a plane without selfcrossings.
- Examples:

non planar planar

- N.B.: Tree and one-loop graphs are always planar! [Why?]
- Planar graphs appear for example in the limit of a large number of colors.
- Planar graphs can only depend on *consecutive* Mandelstam invariants.

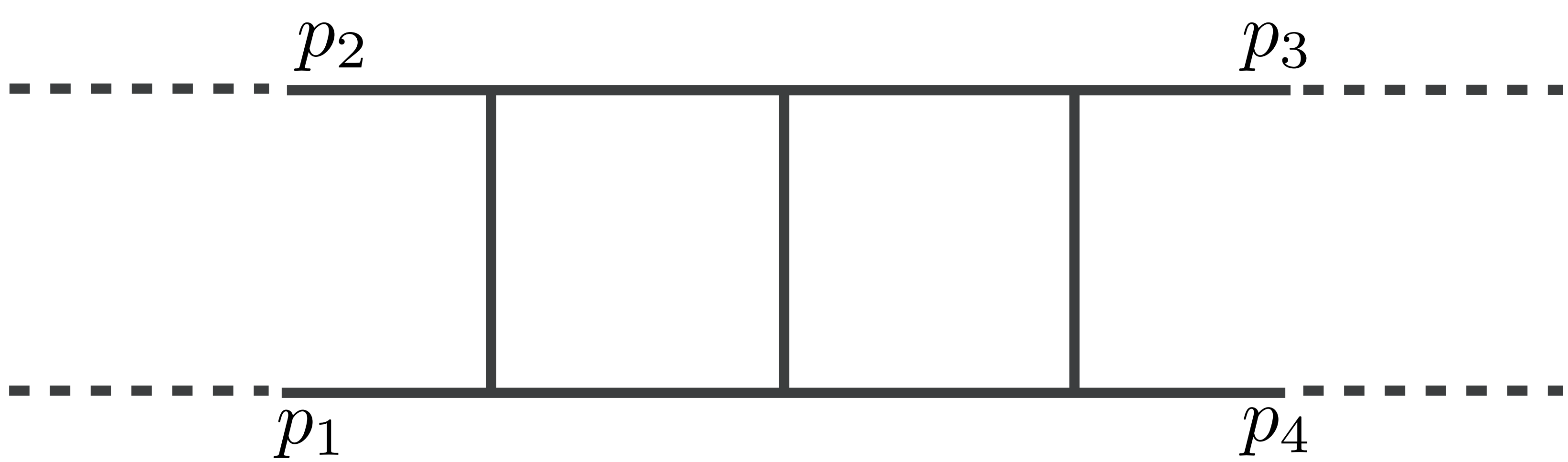
Dual coordinates

- In a planar graph, there is a natural way to define so-called *dual coordinates* (or *region momenta*).



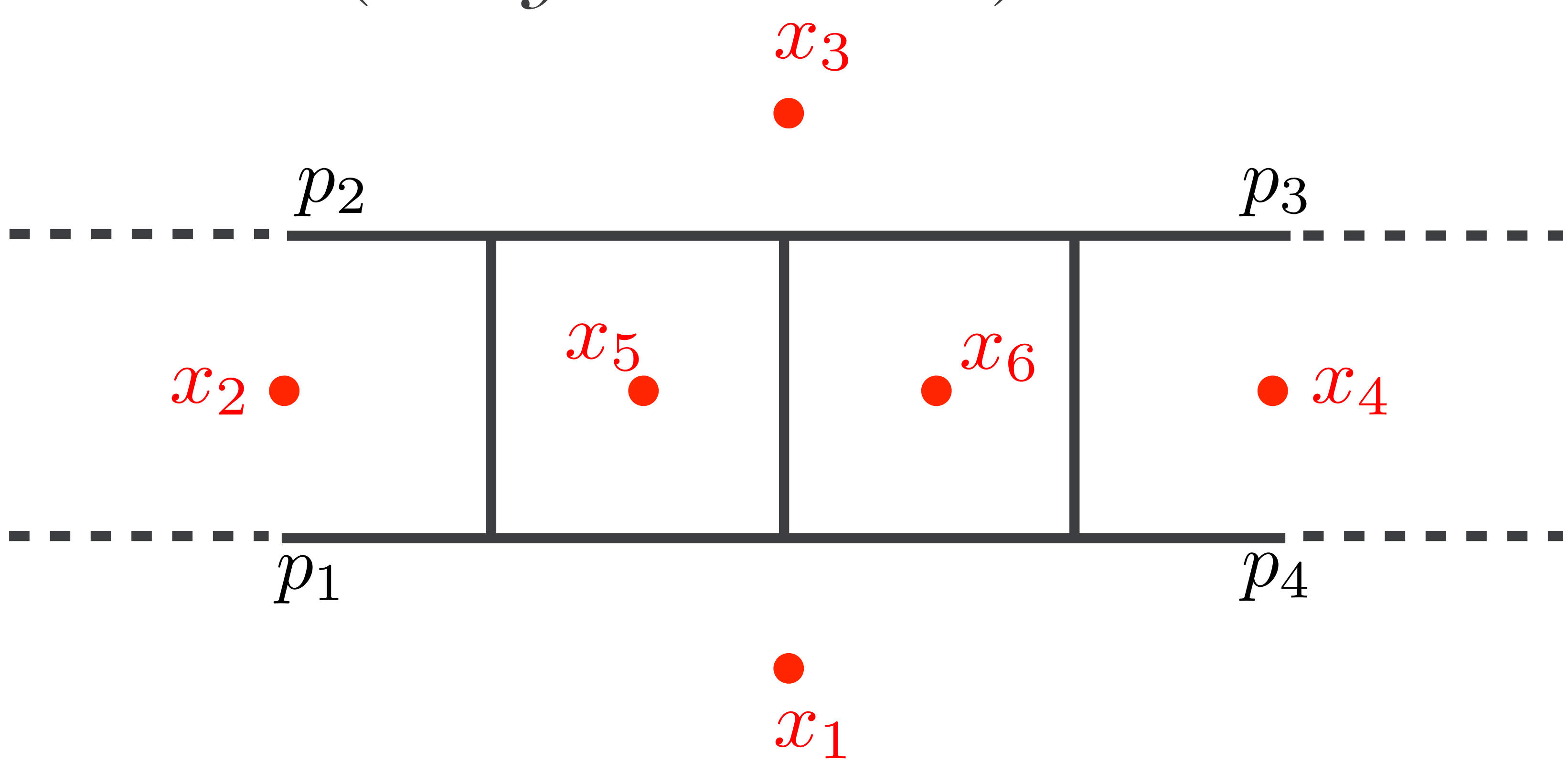
Dual coordinates

- In a planar graph, there is a natural way to define so-called *dual coordinates* (or *region momenta*).



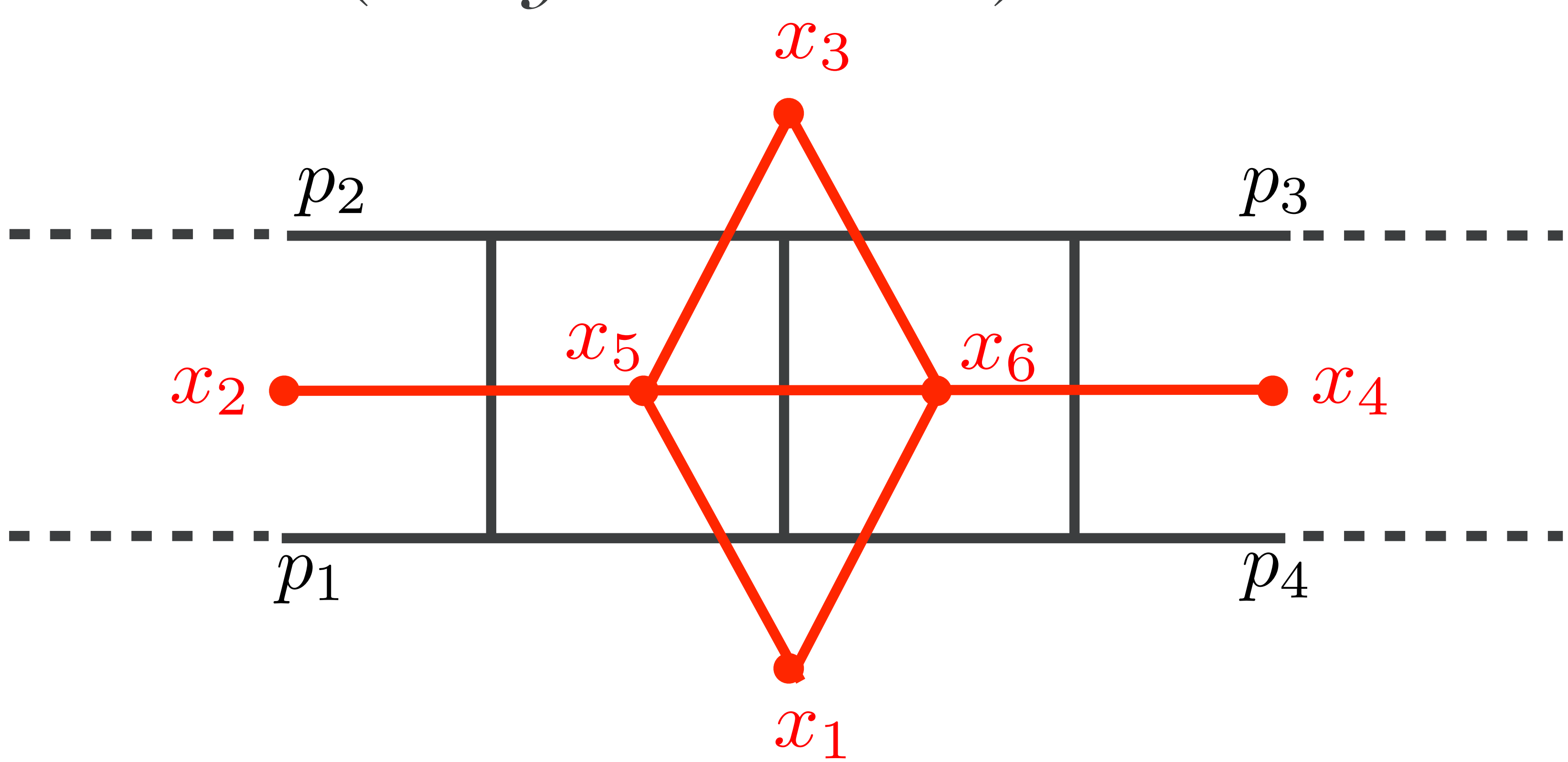
Dual coordinates

- In a planar graph, there is a natural way to define so-called *dual coordinates* (or *region momenta*).



Dual coordinates

- In a planar graph, there is a natural way to define so-called *dual coordinates* (or *region momenta*).

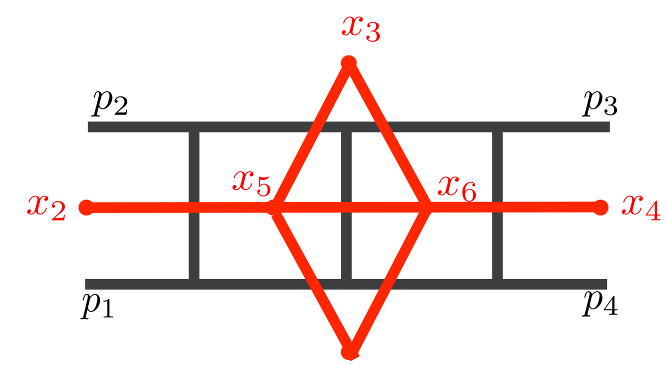


- External momenta take the form $p_i = x_i - x_{i+1}$.
- Consecutive Mandelstam invariants take the form

$$(p_i + p_{i+1} + \ldots + p_{j-1})^2 = (x_i - x_j)^2 \equiv x_{ij}^2$$

Dual coordinates

- The integral can be directly written in terms of dual coordinates:

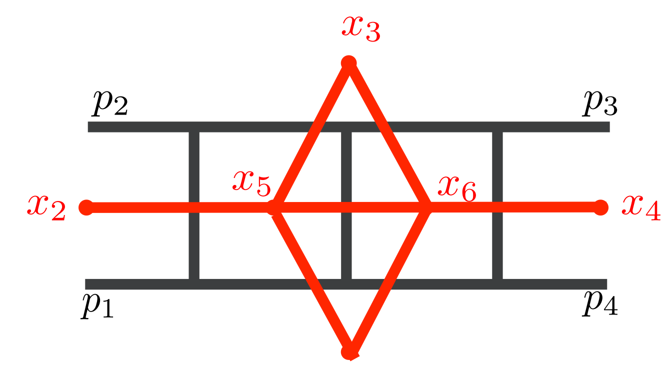


$$\int \frac{\mathrm{d}^D k\, \mathrm{d}^D l}{(i\pi^{D/2})^2} \frac{1}{k^2\,(k+p_1)^2\,(k+p_1+p_2)^2\,l^2\,(l-k)^2\,(l+p_1+p_2)^2\,(l-p_4)^2}$$

$$l = x_6 - x_1$$

Dual coordinates

- The integral can be directly written in terms of dual coordinates:



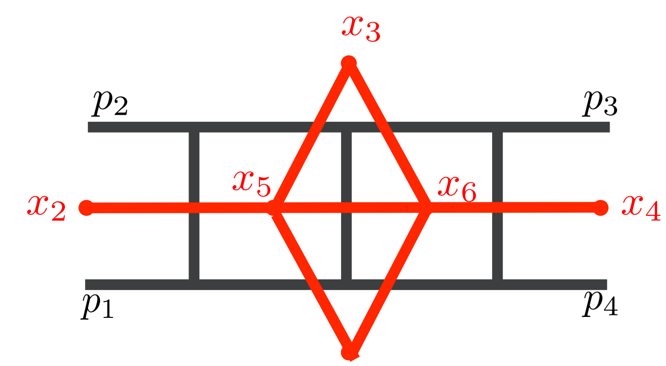
$$\int \frac{\mathrm{d}^D k \, \mathrm{d}^D l}{(i\pi^{D/2})^2} \frac{1}{k^2 \, (k+p_1)^2 \, (k+p_1+p_2)^2 \, l^2 \, (l-k)^2 \, (l+p_1+p_2)^2 \, (l-p_4)^2}$$

- We perform the change of variables:

$$p_i = x_i - x_{i+1} \qquad k = x_5 - x_1 \qquad l = x_6 - x_1$$

Dual coordinates

- The integral can be directly written in terms of dual coordinates:



$$\int \frac{d^D k \, d^D l}{(i\pi^{D/2})^2} \frac{1}{k^2 \, (k+p_1)^2 \, (k+p_1+p_2)^2 \, l^2 \, (l-k)^2 \, (l+p_1+p_2)^2 \, (l-p_4)^2}$$

- We perform the change of variables:

$$p_i = x_i - x_{i+1} \qquad k = x_5 - x_1 \qquad l = x_6 - x_1$$

$$\int \frac{d^D x_5 \, d^D x_6}{(i\pi^{D/2})^2} \frac{1}{x_{51}^2 \, x_{52}^2 \, x_{53}^2 \, x_{56}^2 \, x_{63}^2 \, x_{64}^2 \, x_{61}^2}$$

- Exercise: Proof this!

Dual coordinates

- Some properties:
  - ➡ The integral can only depend on distances

$$x_{ij}^2 = (x_i - x_j)^2 (= (p_i + p_{i+1} + \ldots + p_{j-1})^2)$$

  - ➡ Dual coordinates make momentum conservation manifest.

Dual coordinates

- Some properties:
  - ➡ The integral can only depend on distances

$$x_{ij}^2 = (x_i - x_j)^2 (= (p_i + p_{i+1} + \ldots + p_{j-1})^2)$$

  - ➡ Dual coordinates make momentum conservation manifest.

$$p_1 + p_2 + \ldots + p_n$$

Dual coordinates

- Some properties:
  - ➡ The integral can only depend on distances

$$x_{ij}^2 = (x_i - x_j)^2 (= (p_i + p_{i+1} + \ldots + p_{j-1})^2)$$

  - ➡ Dual coordinates make momentum conservation manifest.

$$p_1 + p_2 + \ldots + p_n$$

$$= (x_1 - x_2) + (x_2 - x_3) + \ldots + (x_n - x_1)$$

Dual coordinates

- Some properties:
  - ➡ The integral can only depend on distances

$$x_{ij}^2 = (x_i - x_j)^2 (= (p_i + p_{i+1} + \ldots + p_{j-1})^2)$$

  - ➡ Dual coordinates make momentum conservation manifest.

$$\begin{aligned} &p_1 + p_2 + \ldots + p_n \\ &\quad = (x_1 - x_2) + (x_2 - x_3) + \ldots + (x_n - x_1) \\ &\quad = 0 \end{aligned}$$

Dual coordinates

- Some properties:
  - ➡ The integral can only depend on distances

$$x_{ij}^2 = (x_i - x_j)^2 (= (p_i + p_{i+1} + \ldots + p_{j-1})^2)$$

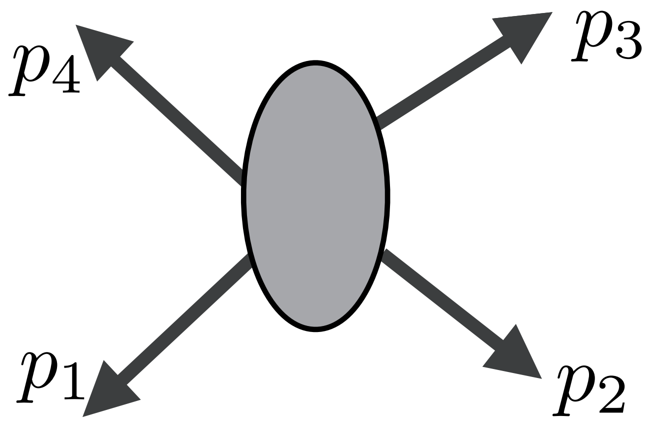
  - ➡ Dual coordinates make momentum conservation manifest.
  - ➡ The on-shellness constraint must however be imposed by hand.

Dual coordinates

- Some properties:
  - ➡ The integral can only depend on distances
  $$x_{ij}^2 = (x_i - x_j)^2 (= (p_i + p_{i+1} + \ldots + p_{j-1})^2)$$
  - ➡ Dual coordinates make momentum conservation manifest.
  - ➡ The on-shellness constraint must however be imposed by hand.
- A side remark: The kinematics is encoded in a polygon in dual space! $\longrightarrow$ Link to Wilson loops! [See Henn's lecture]

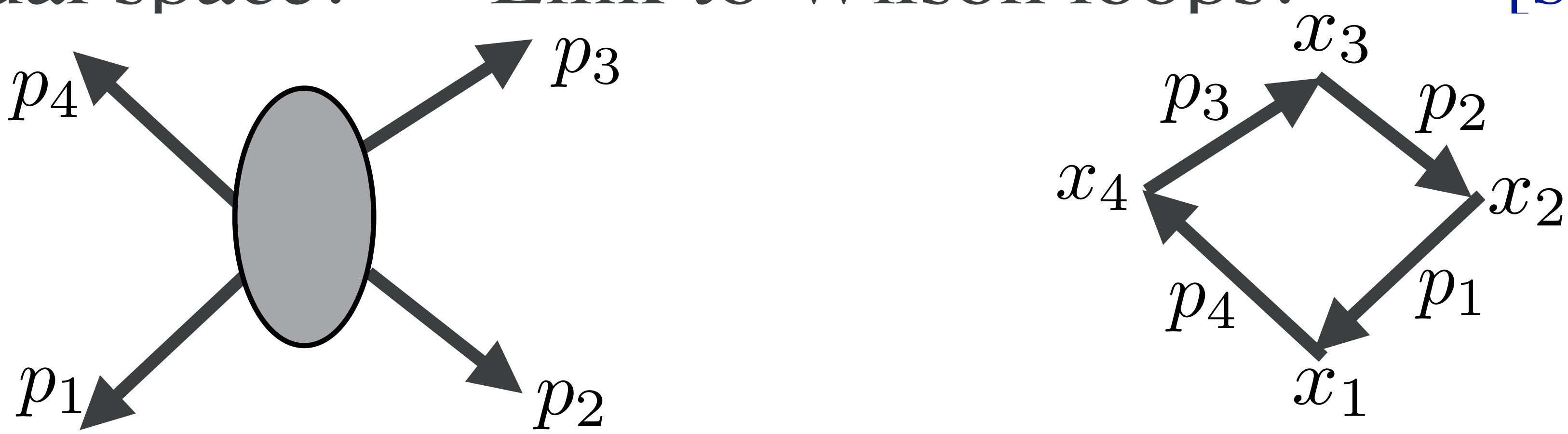
Dual coordinates

- Some properties:
  - ➡ The integral can only depend on distances
    $$x_{ij}^2 = (x_i - x_j)^2 (= (p_i + p_{i+1} + \ldots + p_{j-1})^2)$$
  - ➡ Dual coordinates make momentum conservation manifest.
  - ➡ The on-shellness constraint must however be imposed by hand.
- A side remark: The kinematics is encoded in a polygon in dual space! $\longrightarrow$ Link to Wilson loops! [See Henn's lecture]



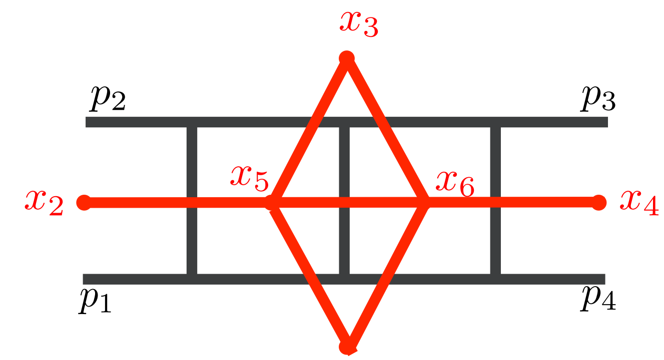
Dual coordinates

- Some properties:
  - ➡ The integral can only depend on distances
    $$x_{ij}^2 = (x_i - x_j)^2 (= (p_i + p_{i+1} + \ldots + p_{j-1})^2)$$
  - ➡ Dual coordinates make momentum conservation manifest.
  - ➡ The on-shellness constraint must however be imposed by hand.
- A side remark: The kinematics is encoded in a polygon in dual space! $\longrightarrow$ Link to Wilson loops! [See Henn's lecture]



Dual conformal invariance [Drummond, Henn, Smirnov, Sokatchev]

- Some integrals can exhibit an unexpected symmetry in dual coordinates!



$$\int \frac{\mathrm{d}^4 x_5\, \mathrm{d}^4 x_6}{\pi^4} \frac{(x_{13}^2)^2\, x_{24}^2}{x_{51}^2\, x_{52}^2\, x_{53}^2\, x_{56}^2 x_{63}^2\, x_{64}^2\, x_{61}^2}$$

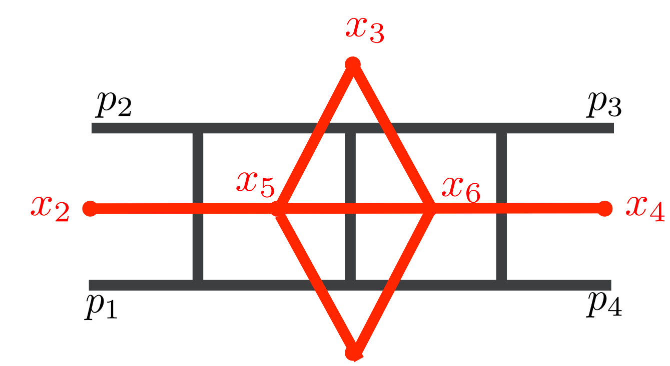
with all external legs massive (integral is finite!)

Dual conformal invariance

[Drummond, Henn, Smirnov, Sokatchev]

- Some integrals can exhibit an unexpected symmetry in dual coordinates!

$$\int \frac{d^4x_5\, d^4x_6}{\pi^4} \frac{(x_{13}^2)^2\, x_{24}^2}{x_{51}^2\, x_{52}^2\, x_{53}^2\, x_{56}^2 x_{63}^2\, x_{64}^2\, x_{61}^2}$$

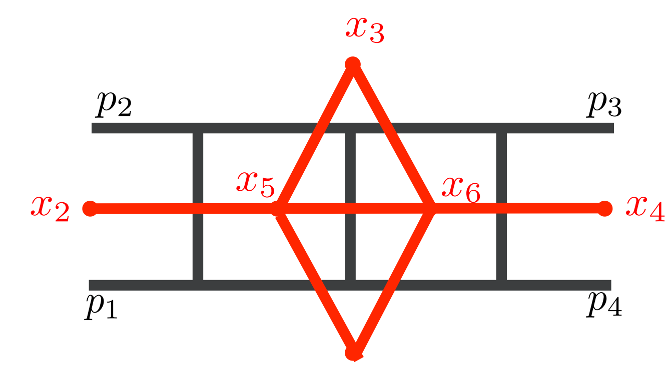


with all external legs massive (integral is finite!)

- Translational and rotational invariance is manifest.

Dual conformal invariance [Drummond, Henn, Smirnov, Sokatchev]

- Some integrals can exhibit an unexpected symmetry in dual coordinates!



$$\int \frac{d^4x_5\, d^4x_6}{\pi^4} \frac{(x_{13}^2)^2\, x_{24}^2}{x_{51}^2\, x_{52}^2\, x_{53}^2\, x_{56}^2 x_{63}^2\, x_{64}^2\, x_{61}^2}$$

with all external legs massive (integral is finite!)

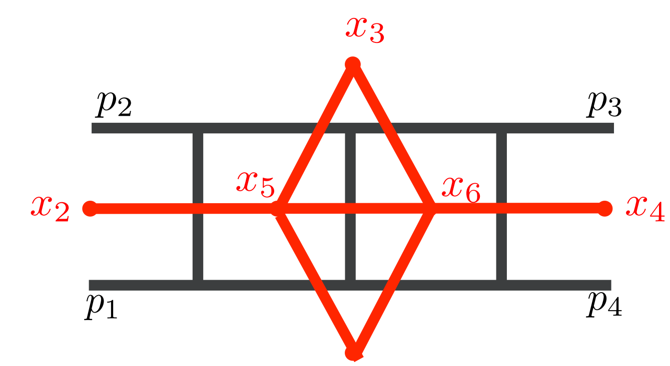
- Translational and rotational invariance is manifest.
- Dilatation invariance $x_i \to \lambda\, x_i$.

Dual conformal invariance

[Drummond, Henn, Smirnov, Sokatchev]

- Some integrals can exhibit an unexpected symmetry in dual coordinates!

$$\int \frac{d^4x_5\, d^4x_6}{\pi^4} \frac{(x_{13}^2)^2\, x_{24}^2}{x_{51}^2\, x_{52}^2\, x_{53}^2\, x_{56}^2 x_{63}^2\, x_{64}^2\, x_{61}^2}$$



with all external legs massive (integral is finite!)

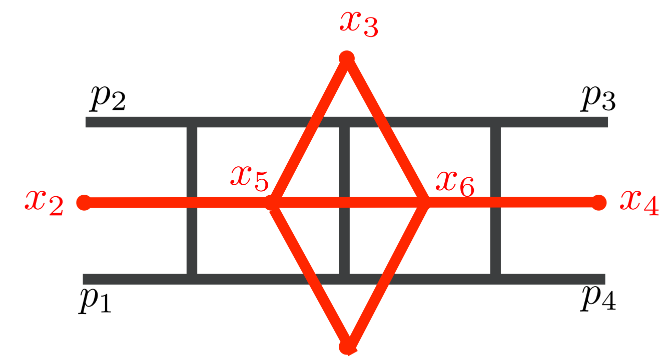
- Translational and rotational invariance is manifest.
- Dilatation invariance $x_i \to \lambda\, x_i$.
- Inversion invariance $x_i \to x_i/x_i^2$,

$$x_{ij}^2 \to x_{ij}^2/(x_i^2 x_j^2) \qquad d^4x_i \to d^4x_i/(x_i^2)^4$$

Dual conformal invariance [Drummond, Henn, Smirnov, Sokatchev]

- Some integrals can exhibit an unexpected symmetry in dual coordinates!

$$\int \frac{d^4x_5\, d^4x_6}{\pi^4} \frac{(x_{13}^2)^2\, x_{24}^2}{x_{51}^2\, x_{52}^2\, x_{53}^2\, x_{56}^2 x_{63}^2\, x_{64}^2\, x_{61}^2}$$



with all external legs massive (integral is finite!)

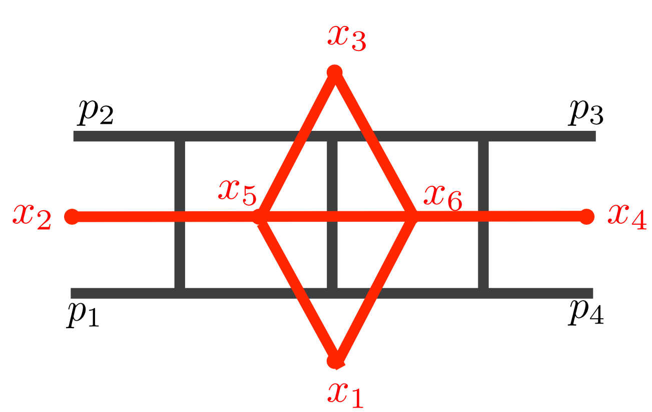
- Translational and rotational invariance is manifest.
- Dilatation invariance $x_i \to \lambda\, x_i$.
- Inversion invariance $x_i \to x_i/x_i^2$,

$$x_{ij}^2 \to x_{ij}^2/(x_i^2 x_j^2) \qquad d^4x_i \to d^4x_i/(x_i^2)^4$$

- In total, we get a conformal symmetry group!

Dual conformal invariance [Drummond, Henn, Smirnov, Sokatchev]

- The integral is a (dual) conformal invariant.
- A conformal invariant can only depend on conformal cross ratios:
$$\frac{x_{ij}^2 x_{kl}^2}{x_{il}^2 x_{kj}^2}$$
- For a 4-mass box, there are only two independent cross ratios:
$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$



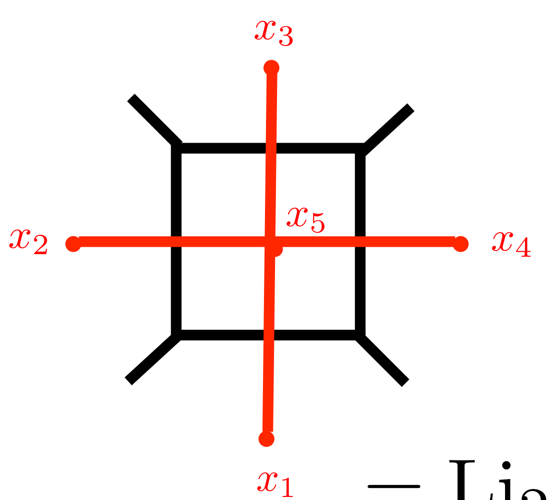
$$= \Phi(u, v)$$

- N.B.: This was naively a function of 6 scales!

Dual conformal invariance [Drummond, Henn, Smirnov, Sokatchev]

- Simplest example: The one-loop 4-mass box:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$



$$\alpha_\pm \equiv \frac{2u}{1+u-v\pm\sqrt{(1-u-v)^2-4uv}}$$

$$= \mathrm{Li}_2(1-\alpha^+) - \mathrm{Li}_2(1-\alpha^-) + 1/2 \ln v \, \ln \frac{\alpha^+}{\alpha^-} - 1/2 \ln u \, \ln v$$

- N.B.: Divergences in general destroy dual conformal invariance! [Why?]
- Exercise: Proof that every (finite) n-gon in D=n is dual conformal invariant.

Summary

- Contraints:
  - ➡ Momentum conservation.
  - ➡ On-shellness.
  - ➡ Gram determinant.

  } $3n-10$ independent variables in 4 dimensions

- For massless theories:

  Spinor helicity formalism solves the on-shellness constraint.
- For planar graphs:

  Dual coordinates solve the momentum conservation constraint.

Kinematics

Momentum twistors

Momentum twistors [Hodges]

- Define 4-component objects transforming under SU(2,2)

$$Z_i = \begin{pmatrix} \lambda_i \\ \bar{\mu}_i \end{pmatrix} \qquad \bar{\mu}_i^{\dot{a}} = i\, x_i^{a\dot{a}}\, \lambda_{ia}$$

- Such objects are called twistors.
- Twistors are the spinorial representation of the conformal group.
- The point x is said to incident to the twistor Z.
- Momentum twistors have nice properties:
  - ➡ They solve the momentum conservation constraint.
  - ➡ They solve the on-shellness constraint.
  - ➡ They even solve the Gram determinant constraint!
  - ➡ Kinematic configurations are described by geometric configurations in twistor space.

Twistor space in a nutshell

- We consider the space $\mathbb{C}^4$ transforming under SU(2,2).
- We can define 'dual twistors' $\overline{Z}_i$ as the objects transforming in the complex conjugate representation.
- Then there are two invariant forms on this space:

$$Z_i \cdot \overline{Z}_j = \langle ij \rangle + [ij] \qquad \langle i\ j\ k\ l \rangle = \epsilon_{IJKL}\, Z_i^I\, Z_j^J\, Z_k^K\, Z_l^L$$

Twistor space in a nutshell

- We consider the space $\mathbb{C}^4$ transforming under SU(2,2).
- We can define 'dual twistors' $\overline{Z}_i$ as the objects transforming in the complex conjugate representation.
- Then there are two invariant forms on this space:

$$Z_i \cdot \overline{Z}_j = \langle ij \rangle + [ij] \qquad \langle i\ j\ k\ l \rangle = \epsilon_{IJKL}\, Z_i^I\, Z_j^J\, Z_k^K\, Z_l^L$$

- This allows us to give an interpretation to dual twistors: Consider the locus of all twistors Z satisfying $Z \cdot \overline{Z}_i = 0$ for some fixed $\overline{Z}_i$.

Twistor space in a nutshell

- We consider the space $\mathbb{C}^4$ transforming under SU(2,2).
- We can define 'dual twistors' $\overline{Z}_i$ as the objects transforming in the complex conjugate representation.
- Then there are two invariant forms on this space:

$$Z_i \cdot \overline{Z}_j = \langle ij \rangle + [ij] \qquad \langle i\; j\; k\; l \rangle = \epsilon_{IJKL}\, Z_i^I\, Z_j^J\, Z_k^K\, Z_l^L$$

- This allows us to give an interpretation to dual twistors: Consider the locus of all twistors Z satisfying $Z \cdot \overline{Z}_i = 0$ for some fixed $\overline{Z}_i$.

➡ Dual twistors are hyperplanes in twistor space!

Incidence relation

- We want to link twistor space to Minkowski space
  - ➡ Incidence relation:

$$Z_i = \begin{pmatrix} \lambda_i \\ \bar{\mu}_i \end{pmatrix} \qquad\qquad \bar{\mu}_i^{\dot{a}} = i\, x_i^{a\dot{a}}\, \lambda_{ia}$$

  - ➡ The incidence relation is invariant under a rescaling of Z, and we should therefore rather work with a projective space $\mathbb{CP}^3$.

Incidence relation

- We want to link twistor space to Minkowski space
  - ➡ Incidence relation:

$$Z_i = \begin{pmatrix} \lambda_i \\ \bar{\mu}_i \end{pmatrix} \qquad\qquad \bar{\mu}_i^{\dot{a}} = i\, x_i^{a\dot{a}}\, \lambda_{ia}$$

  - ➡ The incidence relation is invariant under a rescaling of Z, and we should therefore rather work with a projective space $\mathbb{CP}^3$.
- Given a twistor Z incident to x, which points in Minkowski space is it incident to?

Incidence relation

- We want to link twistor space to Minkowski space
  - ➡ Incidence relation:

$$Z_i = \begin{pmatrix} \lambda_i \\ \bar{\mu}_i \end{pmatrix} \qquad\qquad \bar{\mu}_i^{\dot{a}} = i\, x_i^{a\dot{a}}\, \lambda_{ia}$$

  - ➡ The incidence relation is invariant under a rescaling of Z, and we should therefore rather work with a projective space $\mathbb{CP}^3$.
- Given a twistor Z incident to x, which points in Minkowski space is it incident to?
  - ➡ Answer: The light-ray through x in the direction $p^{a\dot{a}} = \lambda^a \bar{\lambda}^{\dot{a}}$

Incidence relation

- We want to link twistor space to Minkowski space
  - ➡ Incidence relation:

$$Z_i = \begin{pmatrix} \lambda_i \\ \bar{\mu}_i \end{pmatrix} \qquad \bar{\mu}_i^{\dot{a}} = i\, x_i^{a\dot{a}}\, \lambda_{ia}$$

  - ➡ The incidence relation is invariant under a rescaling of Z, and we should therefore rather work with a projective space $\mathbb{CP}^3$.

- Given a twistor Z incident to x, which points in Minkowski space is it incident to?

➡ Answer: The light-ray through x in the direction $p^{a\dot{a}} = \lambda^a \bar{\lambda}^{\dot{a}}$

➡ Proof: $y^{a\dot{a}} = x_{a\dot{a}} + t\, \lambda^a \bar{\lambda}^{\dot{a}}$

Incidence relation

- We want to link twistor space to Minkowski space
  - ➡ Incidence relation:

$$Z_i = \begin{pmatrix} \lambda_i \\ \bar{\mu}_i \end{pmatrix} \qquad \bar{\mu}_i^{\dot{a}} = i\, x_i^{a\dot{a}}\, \lambda_{ia}$$

  - ➡ The incidence relation is invariant under a rescaling of Z, and we should therefore rather work with a projective space $\mathbb{CP}^3$.
- Given a twistor Z incident to x, which points in Minkowski space is it incident to?
  - ➡ **Answer:** The light-ray through x in the direction $p^{a\dot{a}} = \lambda^a \bar{\lambda}^{\dot{a}}$
  - ➡ **Proof:** $y^{a\dot{a}} = x_{a\dot{a}} + t\, \lambda^a \bar{\lambda}^{\dot{a}}$

$$i\, y^{a\dot{a}} \lambda_a = i\, x^{a\dot{a}} \lambda_a + i\, t\, \langle \lambda \lambda \rangle \bar{\lambda}^{\dot{a}} = i\, x^{a\dot{a}} \lambda_a = \bar{\mu}^{\dot{a}}$$

Incidence relation

- We want to link twistor space to Minkowski space
  - ➡ Incidence relation:

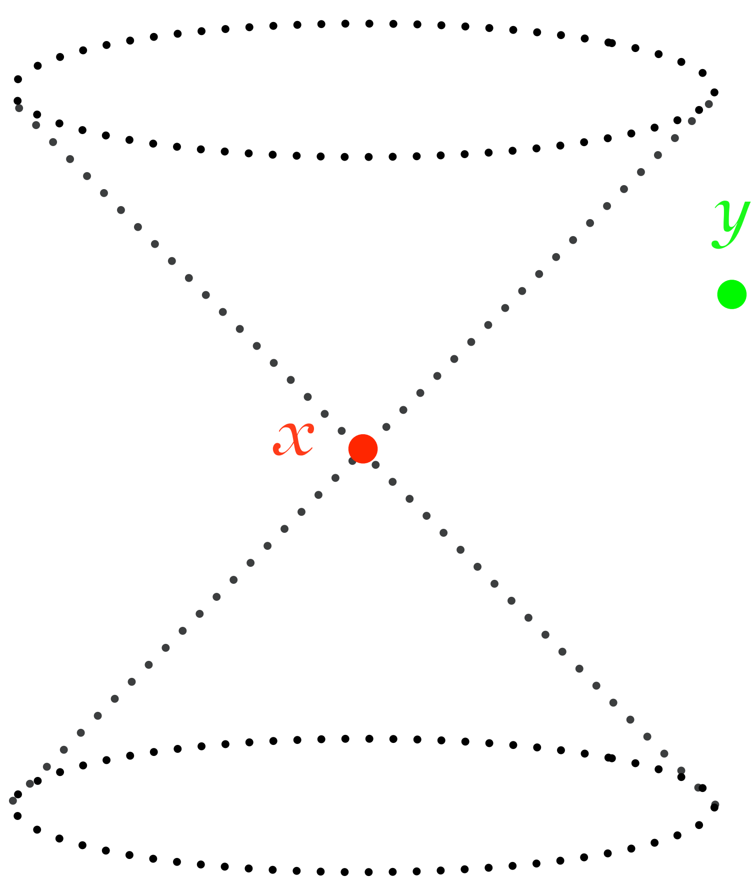
$$Z_i = \begin{pmatrix} \lambda_i \\ \bar{\mu}_i \end{pmatrix} \qquad\qquad \bar{\mu}_i^{\dot{a}} = i\, x_i^{a\dot{a}}\, \lambda_{ia}$$

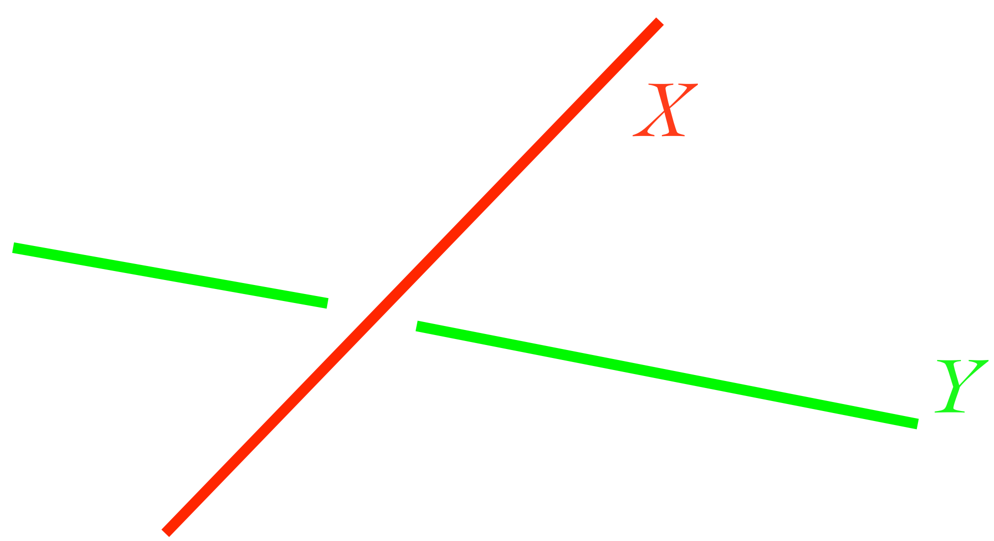
  - ➡ The incidence relation is invariant under a rescaling of Z, and we should therefore rather work with a projective space $\mathbb{CP}^3$.
- Given a twistor Z incident to x, which points in Minkowski space is it incident to?
  - ➡ Answer: The light-ray through x in the direction $p^{a\dot{a}} = \lambda^a \bar{\lambda}^{\dot{a}}$
- Conversely, a line in twistor space corresponds to a point in Minkowski space!

Twistor ‘dictionary’

points in Minkowski space ⟷ lines in twistor space

light rays in Minkowski space ⟷ points in twistor space

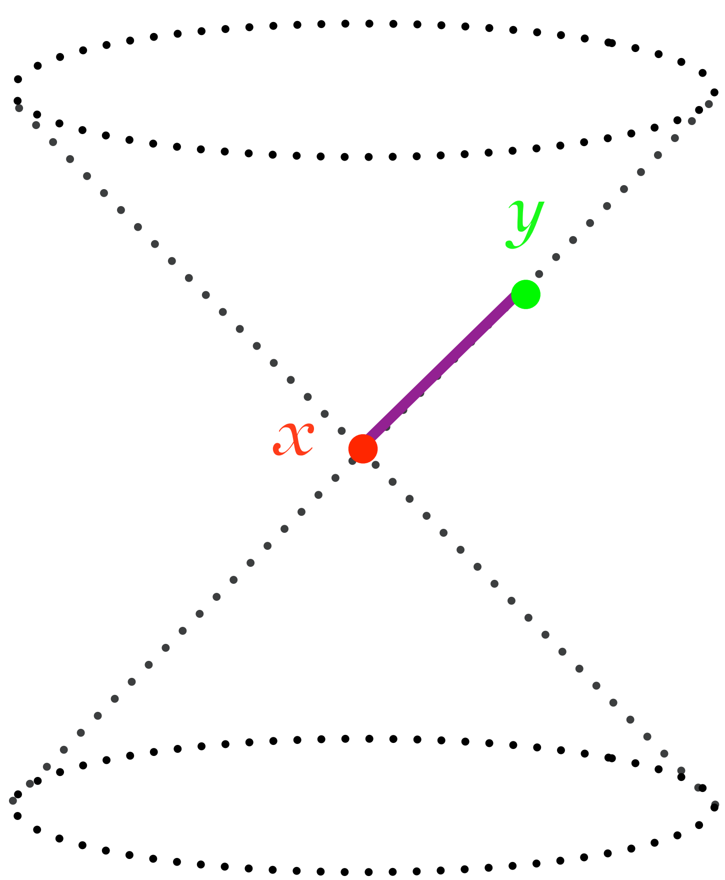


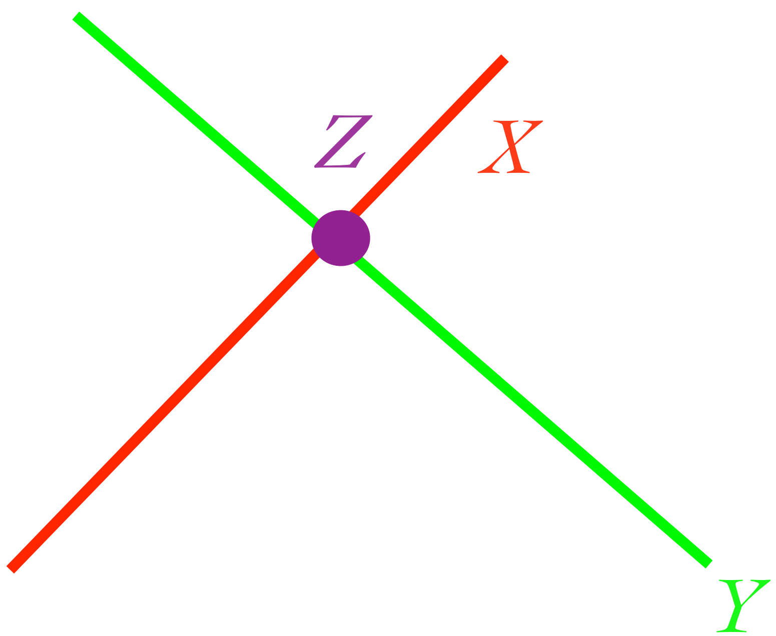


Twistor 'dictionary'

points in Minkowski space ⟷ lines in twistor space

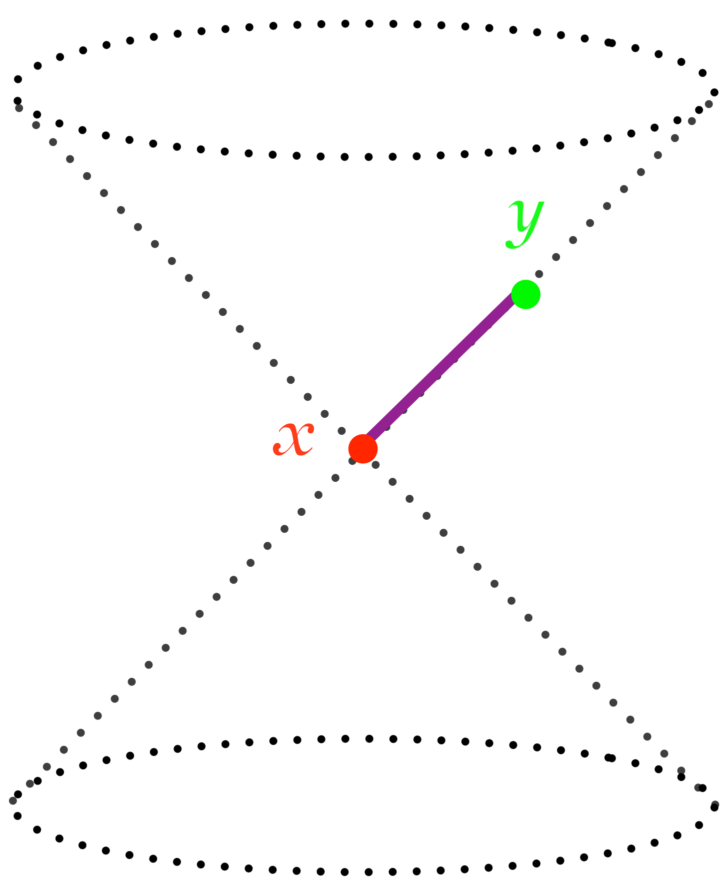
light rays in Minkowski space ⟷ points in twistor space

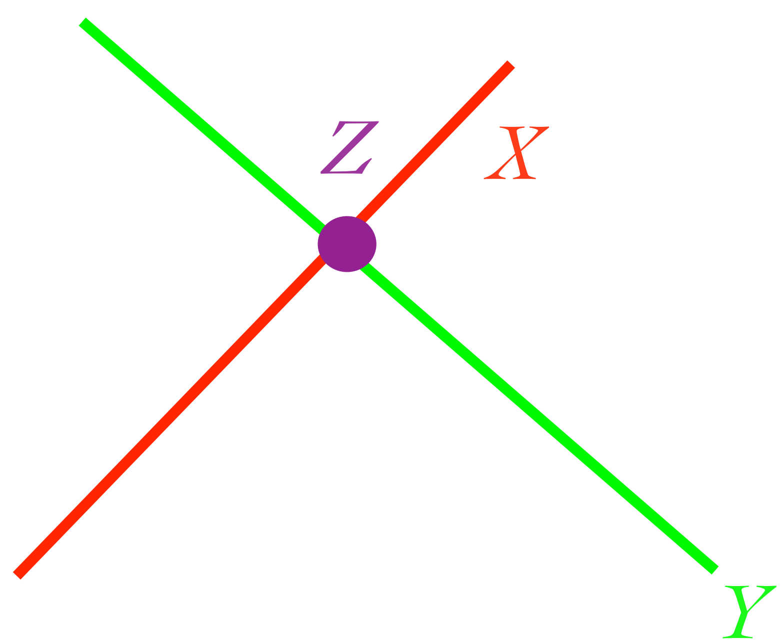




Twistor ‘dictionary’

points in Minkowski space	⟷	lines in twistor space
light rays in Minkowski space	⟷	points in twistor space
light-like distances	⟷	intersection of lines



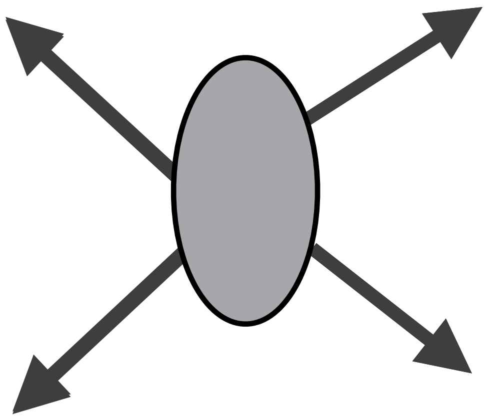


Twistor geometry

- Notation:
  - ➡ The line passing L through Z_1 and Z_2 :
  $$L = Z_1 \wedge Z_2 = -Z_2 \wedge Z_1$$
  - ➡ The plane P passing through Z_1, Z_2 and Z_3 :
  $$P = Z_1 \wedge Z_2 \wedge Z_3$$
- Geometric statements are now encoded in the 'twistor bracket':
  - ➡ $L_1 = Z_1 \wedge Z_2$ and $L_2 = Z_3 \wedge Z_4$ intersect iff
  $$\langle L_1 L_2 \rangle \equiv \langle 1234 \rangle = 0$$
  - ➡ Z lies on the plane $P = Z_1 \wedge Z_2 \wedge Z_3$ iff
  $$\langle Z P \rangle \equiv \langle Z123 \rangle = 0$$
- Exercise: Proof this!

Momentum twistors

- Consider amplitude with massless external legs:



- An n-point massless amplitudes can be given by n momentum twistors.
- All the constraints in 4 dimensions are now trivial!
- The the point x_i in dual space is associated the line

$$X_i = Z_i \wedge Z_{i-1}$$

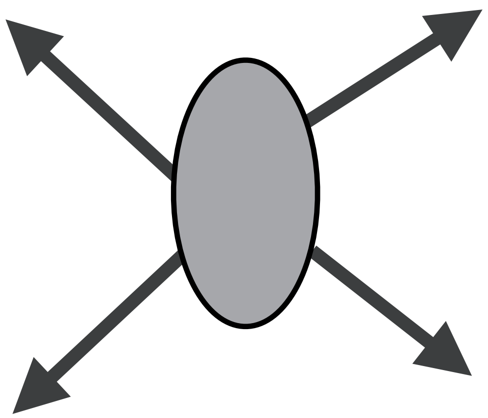
- Dual momentum twistors ('planes') are given by

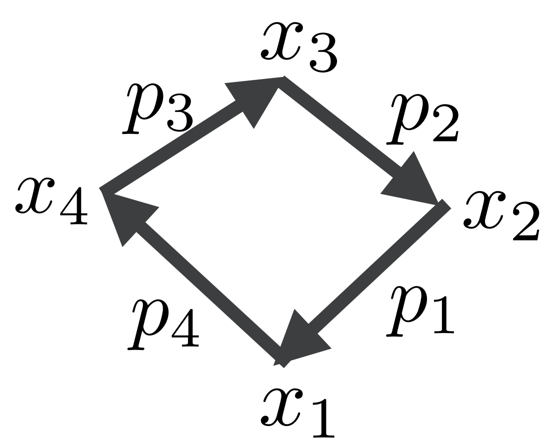
$$\overline{Z}_i = Z_{i-1} \wedge Z_i \wedge Z_{i-1}$$

[Why?]

Momentum twistors

- Consider amplitude with massless external legs:





- An *n*-point massless amplitudes can be given by *n* momentum twistors.
- All the constraints in 4 dimensions are now trivial!
- The the point x_i in dual space is associated the line

$$X_i = Z_i \wedge Z_{i-1}$$

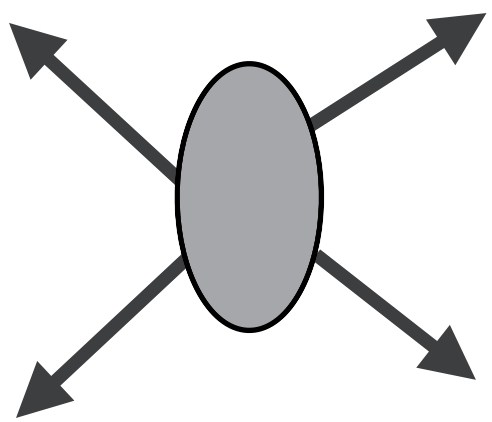
- Dual momentum twistors ('planes') are given by

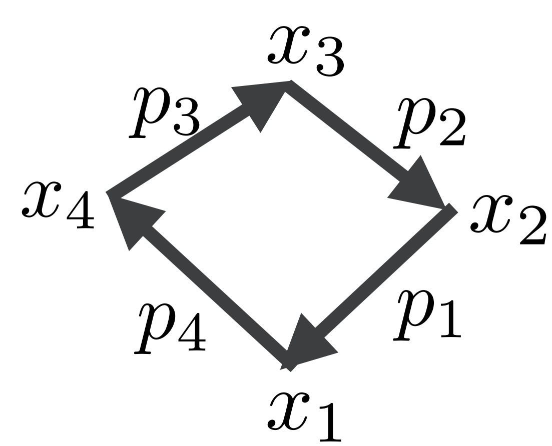
$$\overline{Z}_i = Z_{i-1} \wedge Z_i \wedge Z_{i-1}$$

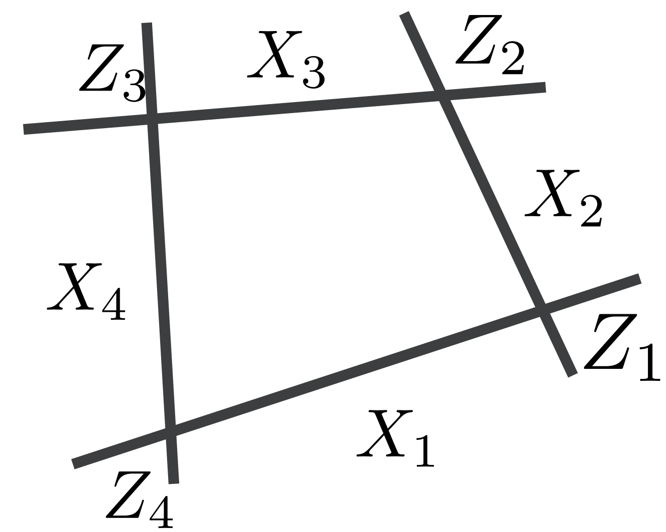
[Why?]

Momentum twistors

- Consider amplitude with massless external legs:







- An n-point massless amplitudes can be given by n momentum twistors.
- All the constraints in 4 dimensions are now trivial!
- The the point x_i in dual space is associated the line

$$X_i = Z_i \wedge Z_{i-1}$$

- Dual momentum twistors ('planes') are given by

$$\overline{Z}_i = Z_{i-1} \wedge Z_i \wedge Z_{i-1}$$

[Why?]

Momentum twistors

- Distances in dual space are now expressed through twistor and spinor brackets:

$$x_{ij}^2 = \frac{\langle i-1\; i\; j-1\; j\rangle}{\langle i-1\; i\rangle\,\langle j-1\; j\rangle} = \frac{\langle X_i\; X_j\rangle}{\langle i-1\; i\rangle\,\langle j-1\; j\rangle}$$

- N.B.: Spinor brackets must cancel out from dual conformal quantities:

$$\frac{x_{ij}^2 x_{kl}^2}{x_{il}^2 x_{kj}^2} = \frac{\langle X_i\; X_j\rangle\langle X_k\; X_l\rangle}{\langle X_i\; X_l\rangle\langle X_k\; X_j\rangle} = \frac{\langle Z_{i-1}Z_iZ_{j-1}Z_j\rangle\langle Z_{k-1}Z_kZ_{l-1}Z_l\rangle}{\langle Z_{i-1}Z_iZ_{l-1}Z_l\rangle\langle Z_{k-1}Z_kZ_{j-1}Z_j\rangle}$$

Momentum twistors

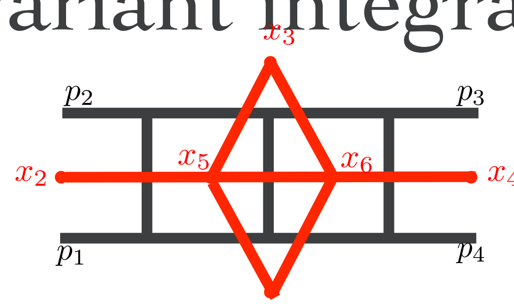
- Distances in dual space are now expressed through twistor and spinor brackets:

$$x_{ij}^2 = \frac{\langle i-1\; i\; j-1\; j\rangle}{\langle i-1\; i\rangle\,\langle j-1\; j\rangle} = \frac{\langle X_i\; X_j\rangle}{\langle i-1\; i\rangle\,\langle j-1\; j\rangle}$$

- N.B.: Spinor brackets must cancel out from dual conformal quantities:

$$\frac{x_{ij}^2 x_{kl}^2}{x_{il}^2 x_{kj}^2} = \frac{\langle X_i\; X_j\rangle\langle X_k\; X_l\rangle}{\langle X_i\; X_l\rangle\langle X_k\; X_j\rangle} = \frac{\langle Z_{i-1}Z_iZ_{j-1}Z_j\rangle\langle Z_{k-1}Z_kZ_{l-1}Z_l\rangle}{\langle Z_{i-1}Z_iZ_{l-1}Z_l\rangle\langle Z_{k-1}Z_kZ_{j-1}Z_j\rangle}$$

- Dual conformal invariant integrals can be written directly in twistor space:



[See Caron-Huot's lecture]

$$\int \frac{\mathrm{d}^4x_5\,\mathrm{d}^4x_6}{\pi^4}\,\frac{(x_{13}^2)^2\,x_{24}^2}{x_{51}^2\,x_{52}^2\,x_{53}^2\,x_{56}^2x_{63}^2\,x_{64}^2\,x_{61}^2}$$

Momentum twistors

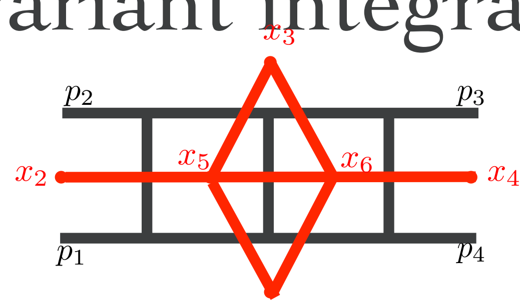
- Distances in dual space are now expressed through twistor and spinor brackets:

$$x_{ij}^2 = \frac{\langle i-1\ i\ j-1\ j\rangle}{\langle i-1\ i\rangle\langle j-1\ j\rangle} = \frac{\langle X_i\ X_j\rangle}{\langle i-1\ i\rangle\langle j-1\ j\rangle}$$

- N.B.: Spinor brackets must cancel out from dual conformal quantities:

$$\frac{x_{ij}^2 x_{kl}^2}{x_{il}^2 x_{kj}^2} = \frac{\langle X_i\ X_j\rangle\langle X_k\ X_l\rangle}{\langle X_i\ X_l\rangle\langle X_k\ X_j\rangle} = \frac{\langle Z_{i-1}Z_iZ_{j-1}Z_j\rangle\langle Z_{k-1}Z_kZ_{l-1}Z_l\rangle}{\langle Z_{i-1}Z_iZ_{l-1}Z_l\rangle\langle Z_{k-1}Z_kZ_{j-1}Z_j\rangle}$$

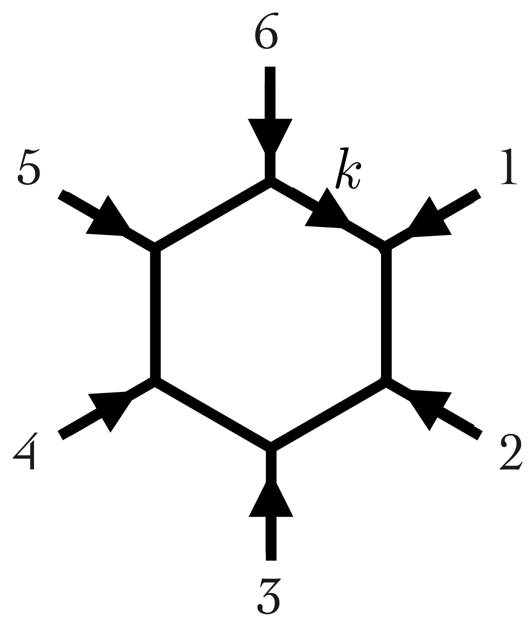
- Dual conformal invariant integrals can be written directly in twistor space:



[See Caron-Huot's lecture]

$$\int \frac{dZ_{AB}dZ_{CD}}{\pi^4}\frac{\langle 1234\rangle^3}{\langle AB41\rangle\langle AB12\rangle\langle AB23\rangle\langle ABCD\rangle\langle CD23\rangle\langle CD34\rangle\langle CD41\rangle}$$

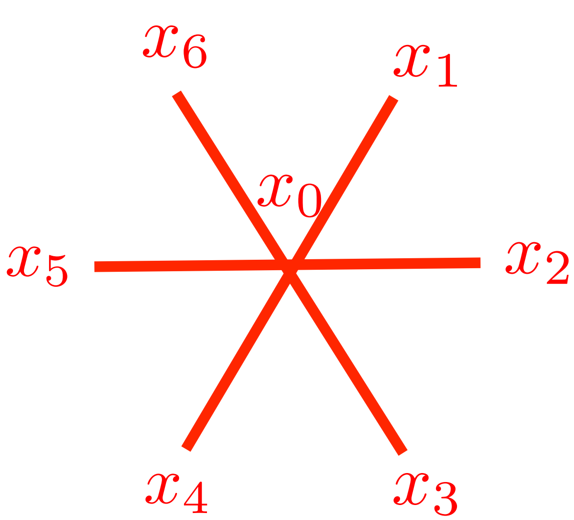
Example: Hexagons in 6 dimensions



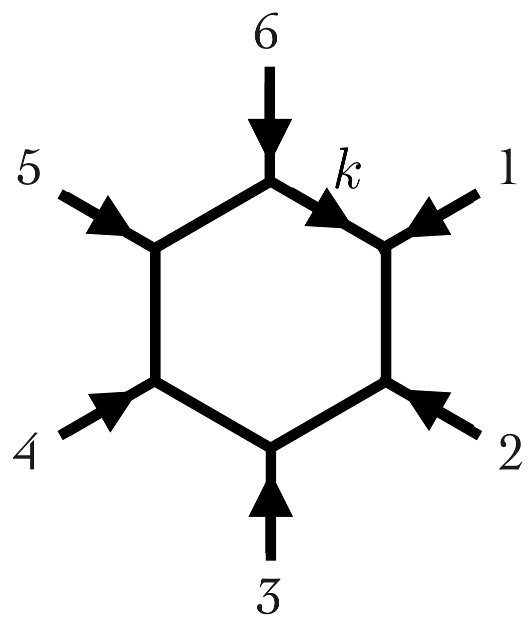
$$I_6^{D=6} = \int \frac{\mathrm{d}^6 k}{i\pi^3} \prod_{i=0}^{5} \frac{1}{D_i} \,,$$

$D_0 = k^2$ and $D_i = (k+p_i)^2$, for $i = 1, \ldots, 5$.

- ➡ This integral is finite!
- ➡ Function of 9 scales.



Example: Hexagons in 6 dimensions



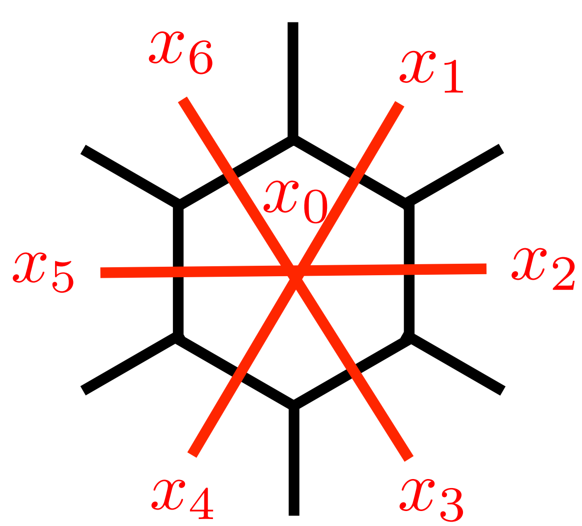
$$I_6^{D=6} = \int \frac{d^6 k}{i\pi^3} \prod_{i=0}^{5} \frac{1}{D_i}\,,$$

$$D_0 = k^2 \quad \text{and} \quad D_i = (k+p_i)^2, \quad \text{for} \quad i = 1, \ldots, 5\,.$$

➡ This integral is finite!

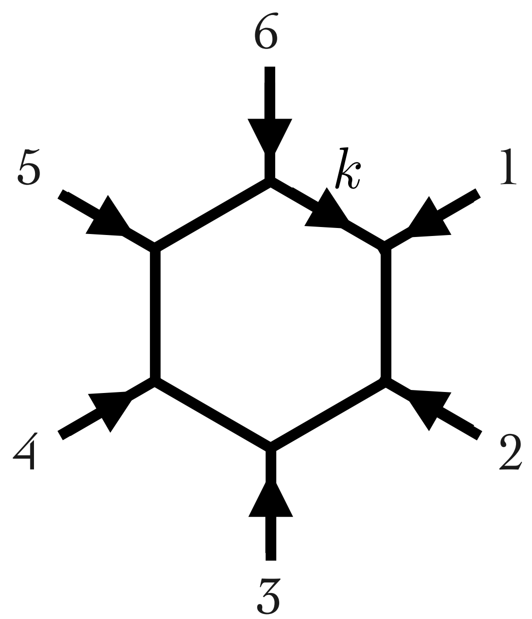
➡ Function of 9 scales.

- Integral in dual coordinates



$$I_6^{D=6} = \int \frac{d^6 x_0}{i\pi^3} \frac{x_{13}^2 x_{24}^2 x_{36}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2 x_{05}^2 x_{06}^2}$$

Example: Hexagons in 6 dimensions



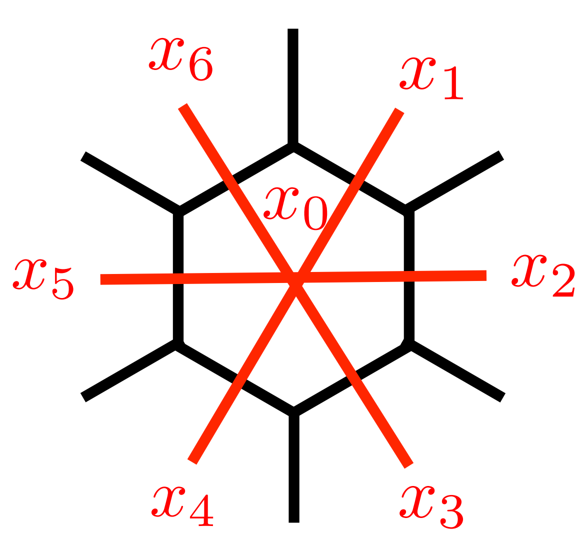
$$I_6^{D=6} = \int \frac{\mathrm{d}^6 k}{i\pi^3} \prod_{i=0}^{5} \frac{1}{D_i}\,,$$

$$D_0 = k^2 \quad \text{and} \quad D_i = (k+p_i)^2, \quad \text{for} \quad i = 1, \ldots, 5\,.$$

➡ This integral is finite!

➡ Function of 9 scales.

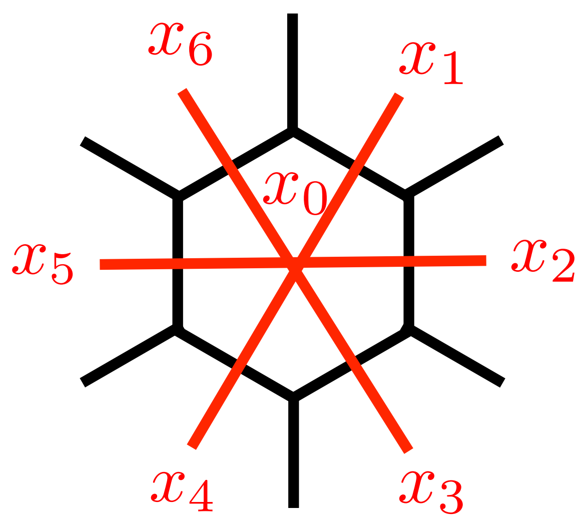
- Integral in dual coordinates



$$I_6^{D=6} = \int \frac{\mathrm{d}^6 x_0}{i\pi^3} \frac{x_{13}^2 x_{24}^2 x_{36}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2 x_{05}^2 x_{06}^2}$$

➡ The integral is dual conformally invariant in 6 dimensions!

Example: Hexagons in 6 dimensions

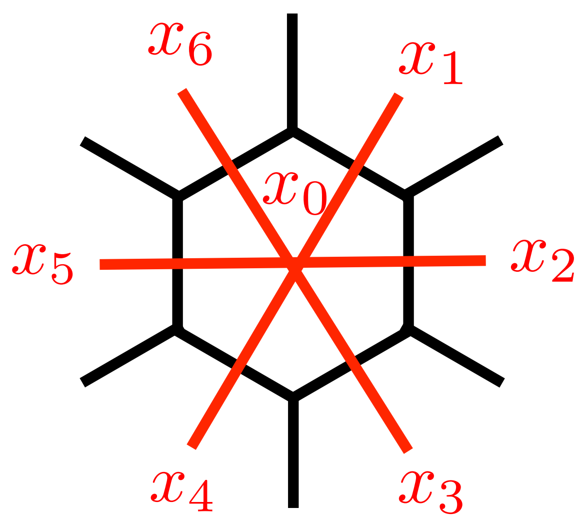


$$I_6^{D=6} = \int \frac{\mathrm{d}^6 x_0}{i\pi^3} \frac{x_{13}^2 x_{24}^2 x_{36}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2 x_{05}^2 x_{06}^2}$$

- There are 3 independent cross ratios we can form:

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2} \qquad u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2} \qquad u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}$$

Example: Hexagons in 6 dimensions



$$I_6^{D=6} = \int \frac{d^6 x_0}{i\pi^3} \frac{x_{13}^2 x_{24}^2 x_{36}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2 x_{05}^2 x_{06}^2}$$

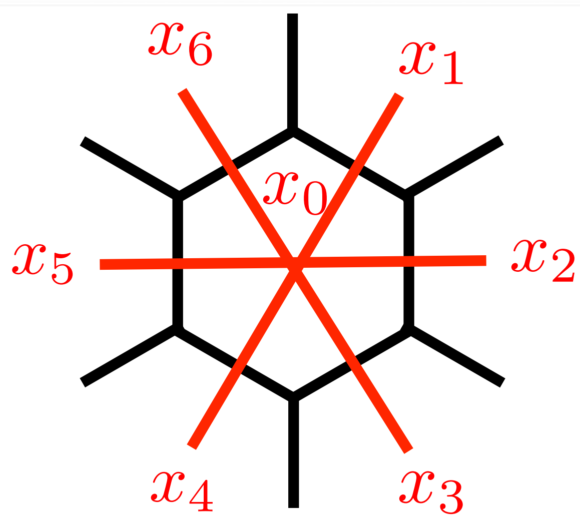
- There are 3 independent cross ratios we can form:

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2} \qquad u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2} \qquad u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}$$

- Instead of having to deal with the 9 scales, we 'only' have 3 cross ratios:

$$I_6^{D=6} = \Phi(u_1, u_2, u_3)$$

Example: Hexagons in 6 dimensions

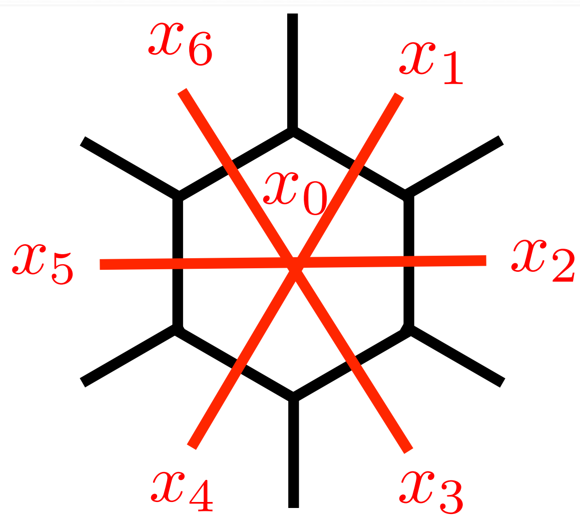


$$I_6^{D=6} = \Phi(u_1, u_2, u_3)$$

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2} \qquad u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2} \qquad u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}$$

- We are in 6 dimensions, so no twistors a priori.

Example: Hexagons in 6 dimensions

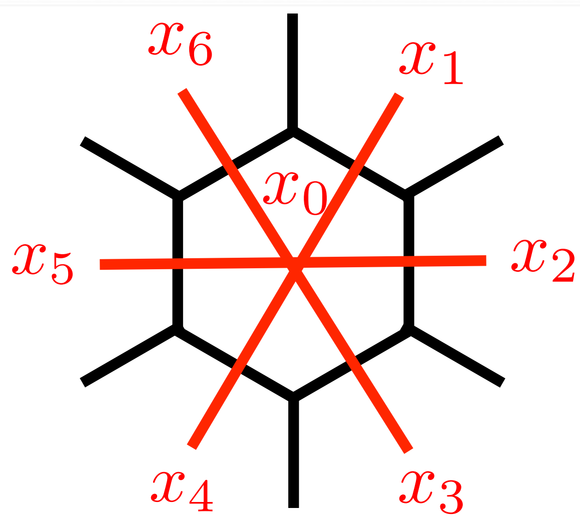


$$I_6^{D=6} = \Phi(u_1, u_2, u_3)$$

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2} \qquad u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2} \qquad u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}$$

- We are in 6 dimensions, so no twistors a priori.
- However, the only thing that matters are the cross ratios:
  - ➡ We can choose our momenta to lie in a 4D subspace.

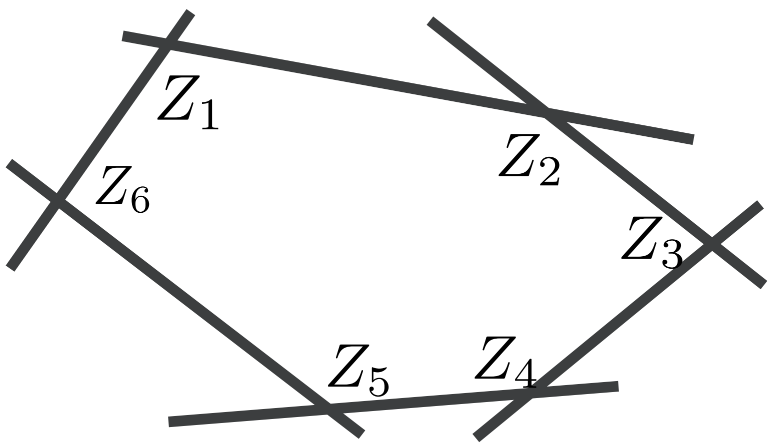
Example: Hexagons in 6 dimensions



$$I_6^{D=6} = \Phi(u_1, u_2, u_3)$$

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{36}^2 x_{41}^2} \qquad u_2 = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2} \qquad u_3 = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}$$

- We are in 6 dimensions, so no twistors a priori.
- However, the only thing that matters are the cross ratios:
  - ➡ We can choose our momenta to lie in a 4D subspace.



➡ space time cross ratios:

$$u_1 = \frac{\langle X_1 X_3\rangle\langle X_4 X_6\rangle}{\langle X_3 X_6\rangle\langle X_4 X_1\rangle} = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 2356\rangle\langle 3461\rangle}$$

➡ new cross ratios:

$$x_1^+ = -\frac{\langle 6345\rangle\langle 1245\rangle}{\langle 6145\rangle\langle 2345\rangle}$$

Example: Hexagons in 6 dimensions

$$\frac{1}{\sqrt{\Delta}}\left[-2\sum_{i=1}^{3} L_3(x_i^+,x_i^-) + \frac{1}{3}\left(\sum_{i=1}^{3}\ell_1(x_i^+)-\ell_1(x_i^-)\right)^3 + \frac{\pi^2}{3}\chi\sum_{i=1}^{3}(\ell_1(x_i^+)-\ell_1(x_i^-))\right],$$

$$x_i^\pm = u_i x^\pm, \quad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3},$$

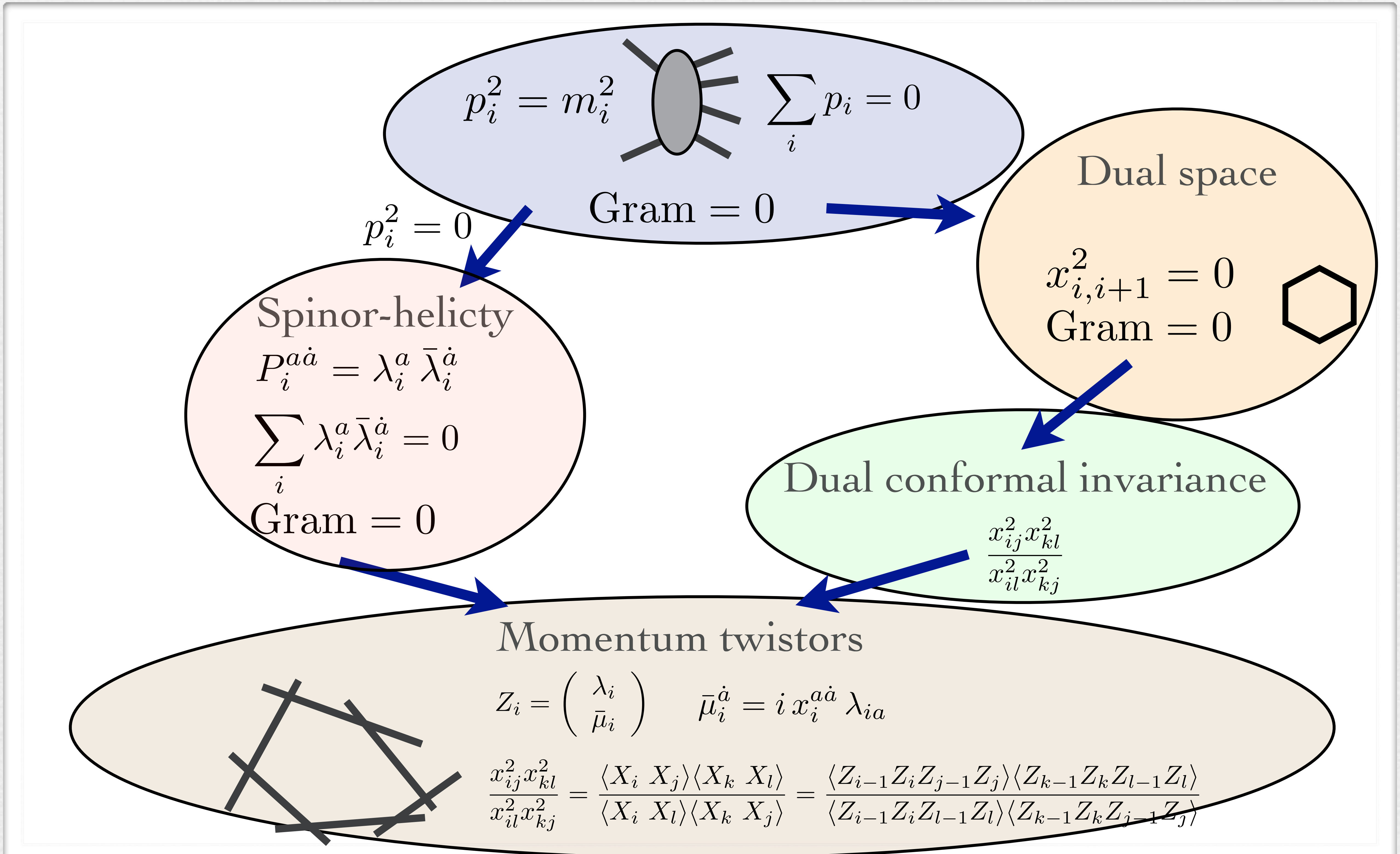
$$\Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3.$$

$$L_3(x^+,x^-) = \sum_{k=0}^{2}\frac{(-1)^k}{(2k)!!}\ln^k(x^+x^-)(\ell_{3-k}(x^+)-\ell_{3-k}(x^-)),$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\mathrm{Li}_n(1/x)\right),$$

[Dixon, Drummond, Henn;
Del Duca, CD, Smirnov]

Summary lecture 1



Summary lecture 1