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Mellin transformation, Mellin integrals as a tool for Feynman integrals: [M.C. Bergère & Y.-M.P. Lam'74]

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Systematic evaluation of dimensionally regularized Feynman integrals (in particular, systematic resolution of the singularities in ϵ) [V.A. Smirnov'99, J.B. Tausk'99]

The basic formula:

$$\frac{1}{(X+Y)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{Y^{z}}{X^{\lambda+z}} \Gamma(\lambda+z)\Gamma(-z) \; .$$

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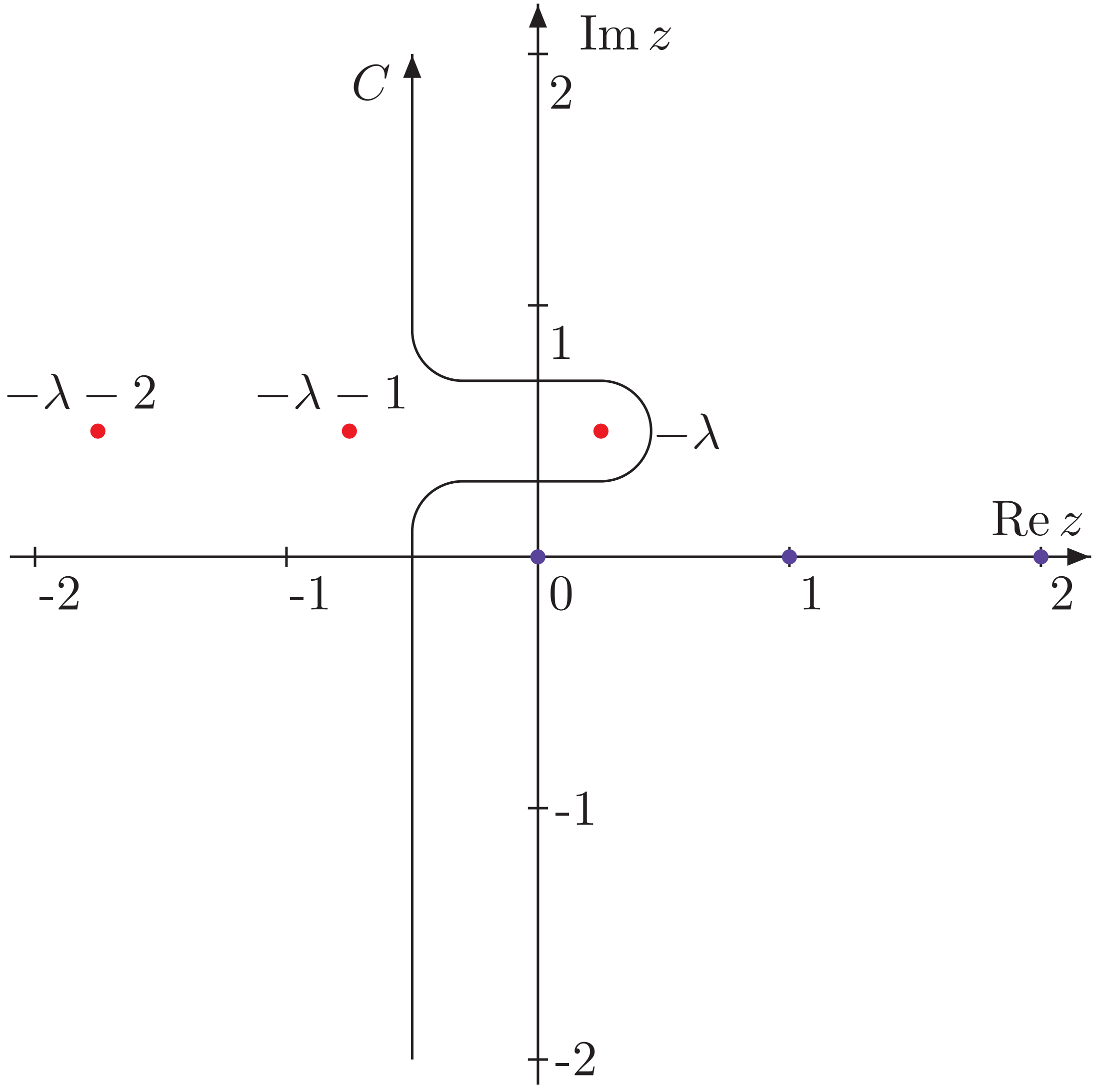
The poles with a $\Gamma(\ldots+z)$ dependence are to the left of the contour and the poles with a $\Gamma(\ldots-z)$ dependence are to the right


Im z
C
2
1
$-\lambda - 2$
$-\lambda - 1$
$-\lambda$
Re z
-2
-1
0
1
2
-1
-2

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- Expand in a Laurent series in ϵ
- Evaluate expanded MB integrals

The simplest possibility:

$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \frac{(m^2)^z}{(-k^2)^{\lambda+z}}\Gamma(\lambda+z)\Gamma(-z)$$

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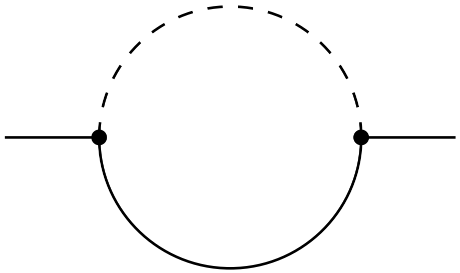
Example 1



$$F_\Gamma(q^2, m^2; a_1, a_2, d) = \int \frac{\mathrm{d}^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

$$F_\Gamma = \frac{1}{\Gamma(a_1)} \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z (m^2)^z \Gamma(a_1 + z)\Gamma(-z)$$

$$\times \int \frac{\mathrm{d}^d k}{(-k^2)^{a_1+z}(-(q-k)^2)^{a_2}}$$

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$$\int \frac{\mathrm{d}^d k}{(-k^2)^{a_1+z}[-(q-k)^2]^{a_2}} = \mathrm{i}\pi^{d/2} \frac{G(a_1+z, a_2)}{(-q^2)^{a_1+a_2+\epsilon-2+z}} \; ,$$

$$G(a_1,a_2) = \frac{\Gamma(a_1+a_2+\epsilon-2)\Gamma(2-\epsilon-a_1)\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4-a_1-a_2-2\epsilon)}$$

$$
\begin{aligned}
F_\Gamma(q^2, m^2; a_1, a_2, d) \quad &= \quad \frac{\mathrm{i}\pi^{d/2}\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\epsilon-2}} \\
&\times \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \left(\frac{m^2}{-q^2} \right)^z \Gamma(a_1+a_2+\epsilon-2+z) \\
&\times \frac{\Gamma(2-\epsilon-a_1-z)\Gamma(-z)}{\Gamma(4-2\epsilon-a_1-a_2-z)}
\end{aligned}
$$

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$$\times \frac{\Gamma(2-\epsilon-a_1-z)\Gamma(-z)}{\Gamma(4-2\epsilon-a_1-a_2-z)}$$

Unambiguous prescriptions for contours:
the poles with a $\Gamma(\ldots + z)$ dependence are to the left and the poles with a $\Gamma(\ldots - z)$ dependence are to the right of a contour

Strategy A

[V.A. Smirnov'99]

$$F_\Gamma(q^2, m^2; 1, 1, d) = \frac{\mathrm{i}\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon}$$

$$\times \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

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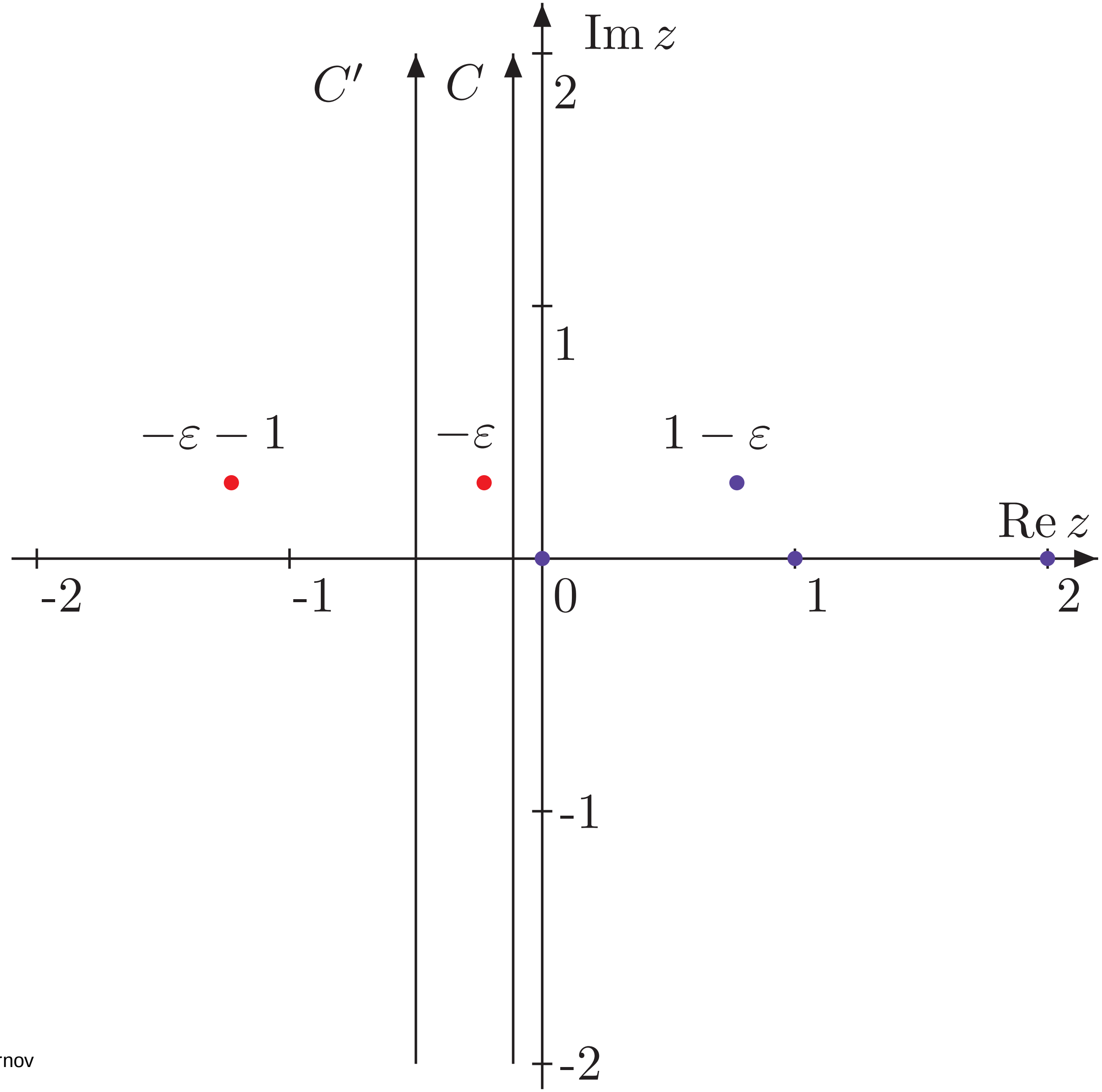
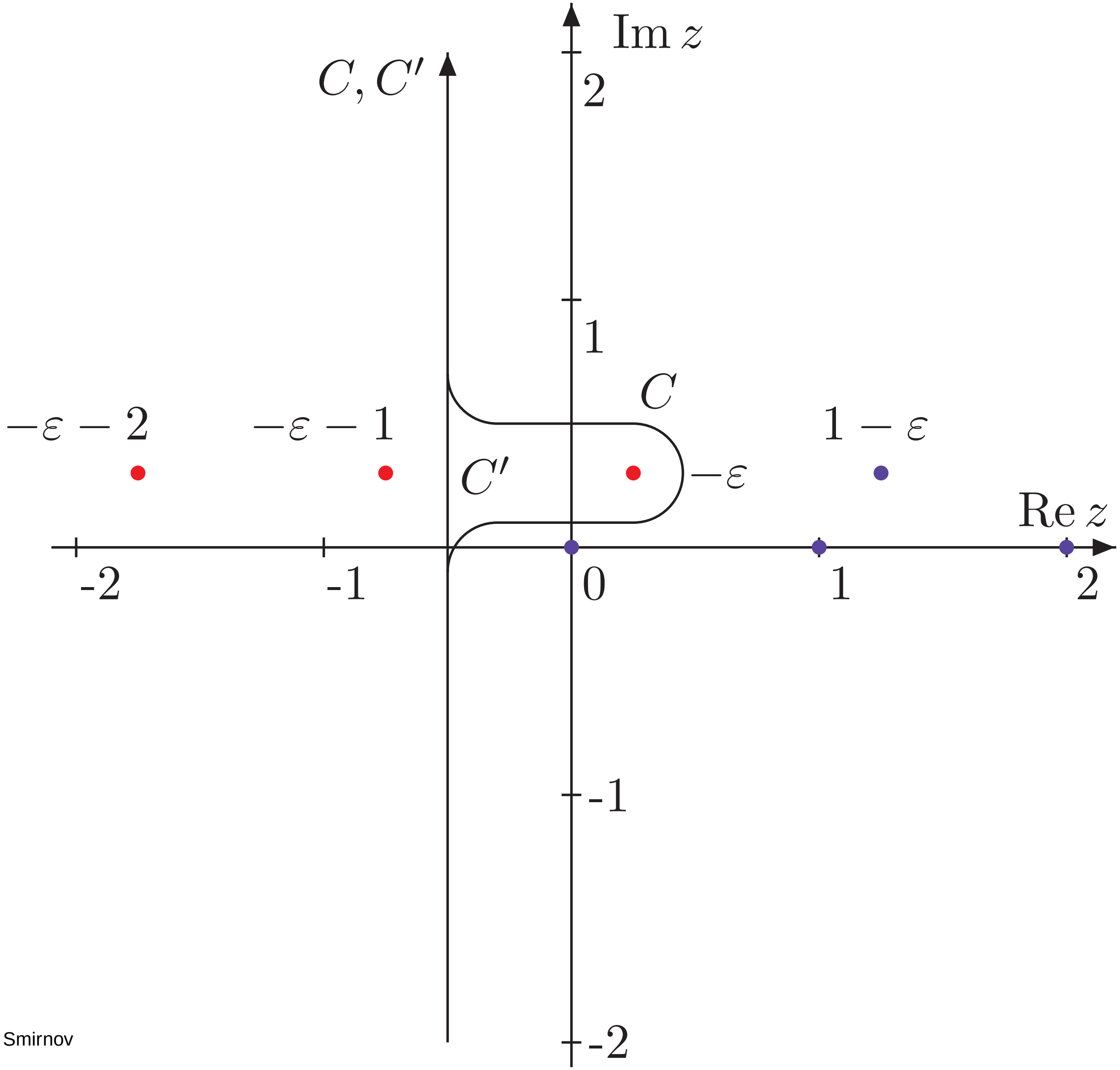
$\Gamma(\epsilon+z)\Gamma(-z) \rightarrow$ a singularity in ϵ


Im z
C'
C
2
1
$-\varepsilon-1$
$-\varepsilon$
$1-\varepsilon$
Re z
-2
-1
0
1
2
-1
-2



Im z
C, C'
2
1
C
$-\varepsilon - 2$
$-\varepsilon - 1$
$1 - \varepsilon$
C'
$-\varepsilon$
Re z
-2
-1
0
1
2
-1
-2

Take a residue at $z = -\epsilon$:

$$\mathrm{i}\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon (1-\epsilon)}$$

and shift the contour:

$$\frac{\mathrm{i}\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \frac{1}{2\pi \mathrm{i}} \int_{C'} \mathrm{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

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$$\Gamma(\epsilon+z)\Gamma(-z) \to \Gamma(\epsilon)$$

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$$\Gamma(\epsilon{\color{red}+z})\Gamma({\color{blue}-z}) \to \Gamma(\epsilon)$$

NB:

$$\Gamma(\epsilon+z)\Gamma(1-\epsilon-z) = -\Gamma(1+\epsilon+z)\Gamma(-\epsilon-z)$$

The integral can be expanded in ϵ, e.g., the value at $\epsilon = 0$ is

$$\frac{1}{2\pi \mathrm{i}} \int_{C'} f(z) \mathsf{d}z = -\sum_{n=0} \mathsf{res}_{z=n} f(z)$$

$$= +\sum_{n=1} \mathsf{res}_{z=n} f(z)$$

$$= 1 - \left(1 - \frac{m^2}{q^2}\right) \ln\left(1 - \frac{q^2}{m^2}\right)$$

where

$$f(z) = \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(z)\Gamma(-z)\Gamma(1-z)}{\Gamma(2-z)} = \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(z)\Gamma(-z)}{(1-z)}$$

Strategy A in a modified form

[A.V. Smirnov and V.A. Smirnov'09]

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Choose a straight contour C_0 for which the gamma functions in the numerator of the integrand are spoiled at $\epsilon = 0$ in a minimal way, i.e. the initial rules for choosing a contour are changed in a minimal way

$$\Gamma(\epsilon + z)\Gamma(-z)\Gamma(1 - \epsilon - z) \to \Gamma(z)\Gamma(-z)\Gamma(1 - z)$$

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Let us choose C_0 with $\text{Re}z = -1/4$.

Then $\Gamma(\epsilon + z)$ which transforms into $\Gamma(z)$ at $\epsilon = 0$ is spoiled.
$\Gamma(\epsilon + z) \to \Gamma^{(1)}(\epsilon + z)$

$\Gamma^{(1)}(\epsilon + z)$ means that the rule $\text{Re}(\epsilon + z) > 0$ when crossing the real axis is changed to $-1 < \text{Re}(\epsilon + z) < 0$

We do not need to spoil it more, e.g., by $\Gamma(\epsilon + z) \to \Gamma^{(2)}(\epsilon + z)$ with the rule $-2 <$ Re$(\epsilon + z) < -1$, etc.

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$$f(z,\epsilon) = \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon+z)\Gamma(-z)\Gamma(1-\epsilon-z)}{\Gamma(2-2\epsilon-z)}$$

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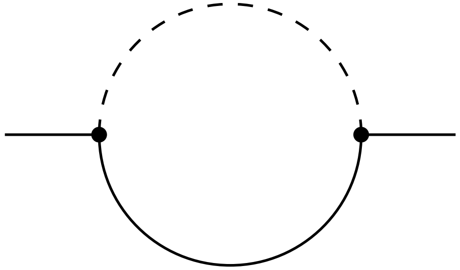
Then

$$\frac{1}{2\pi \mathrm{i}}\int_C f(z,\epsilon)\mathrm{d}z = \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\epsilon)\mathrm{d}z + \left(\frac{1}{2\pi \mathrm{i}}\int_C f(z,\epsilon)\mathrm{d}z - \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\epsilon)\mathrm{d}z\right) = \frac{1}{2\pi \mathrm{i}}\int_{C_0} f(z,\epsilon)\mathrm{d}z + \text{res}_{z=\epsilon} f(z,\epsilon)$$

Strategy B

Example 1

[J.B. Tausk'99, C. Anastasiou & A. Daleo'05, Czakon'05]



$$F_\Gamma(q^2, m^2; 1, 1, d) \;=\; \frac{\mathrm{i}\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \frac{1}{2\pi \mathrm{i}} \int_C \mathrm{d}z\, f(z, \epsilon)$$

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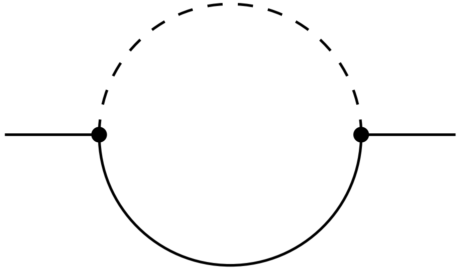

$$F_\Gamma(q^2, m^2; 1, 1, d) = \frac{i\pi^{d/2}\Gamma(1-\epsilon)}{(-q^2)^\epsilon} \frac{1}{2\pi i} \int_C dz f(z, \epsilon)$$

Take ϵ real. Choose ϵ and a straight contour such that the the arguments of the gamma functions are positive when crossing the real axis.

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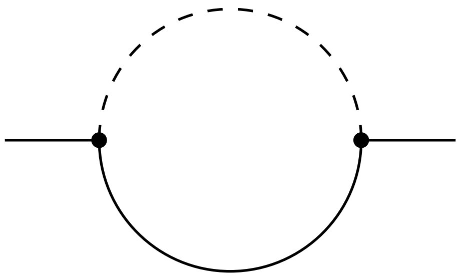

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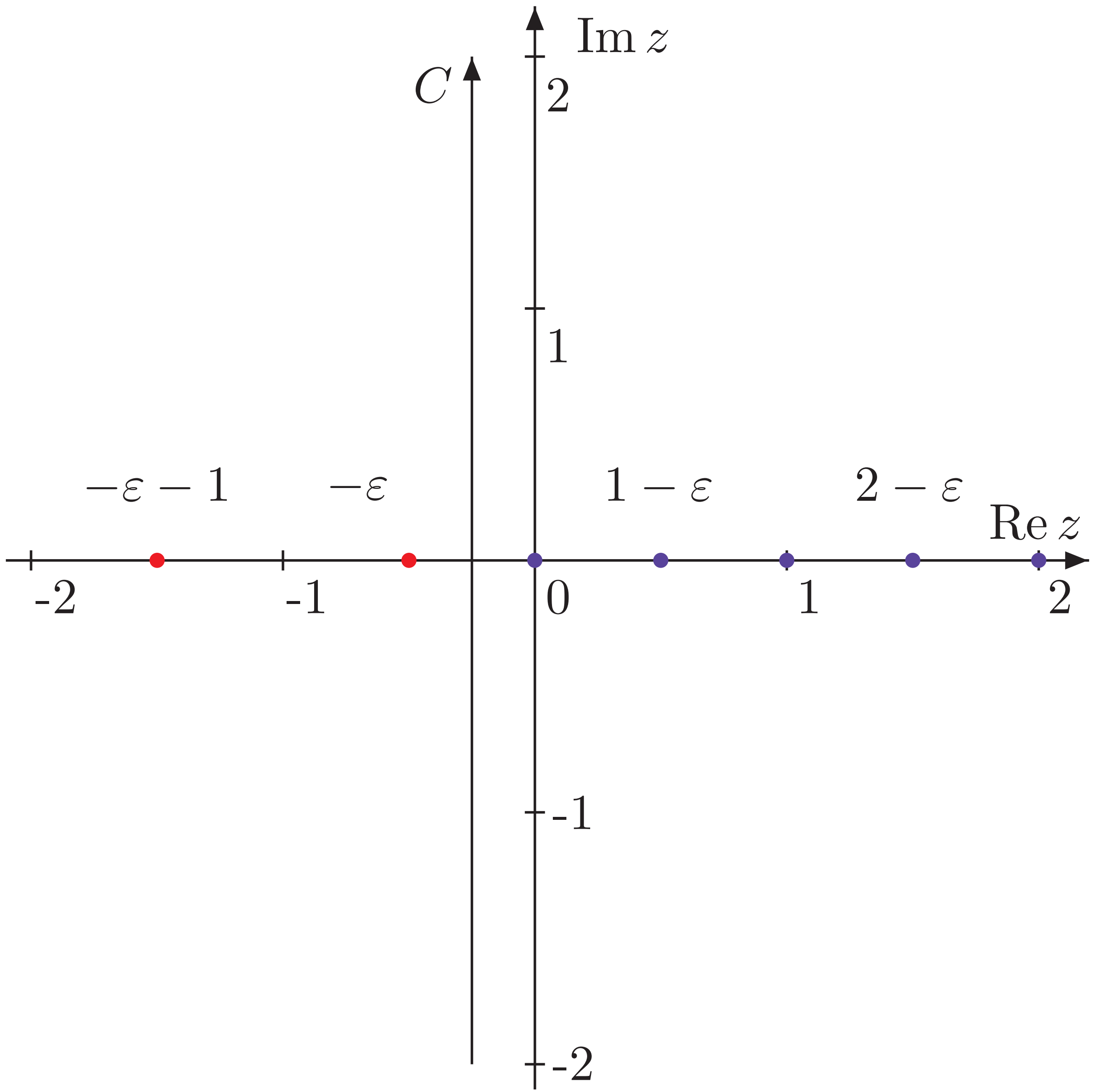
Take ϵ real. Choose ϵ and a straight contour such that the the arguments of the gamma functions are positive when crossing the real axis.

For example, take $\epsilon = 1/2$, Re$z = -1/4$. The contour is kept fixed. Tend ϵ to zero.


Im z
C
2
1
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-2
-1
0
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2
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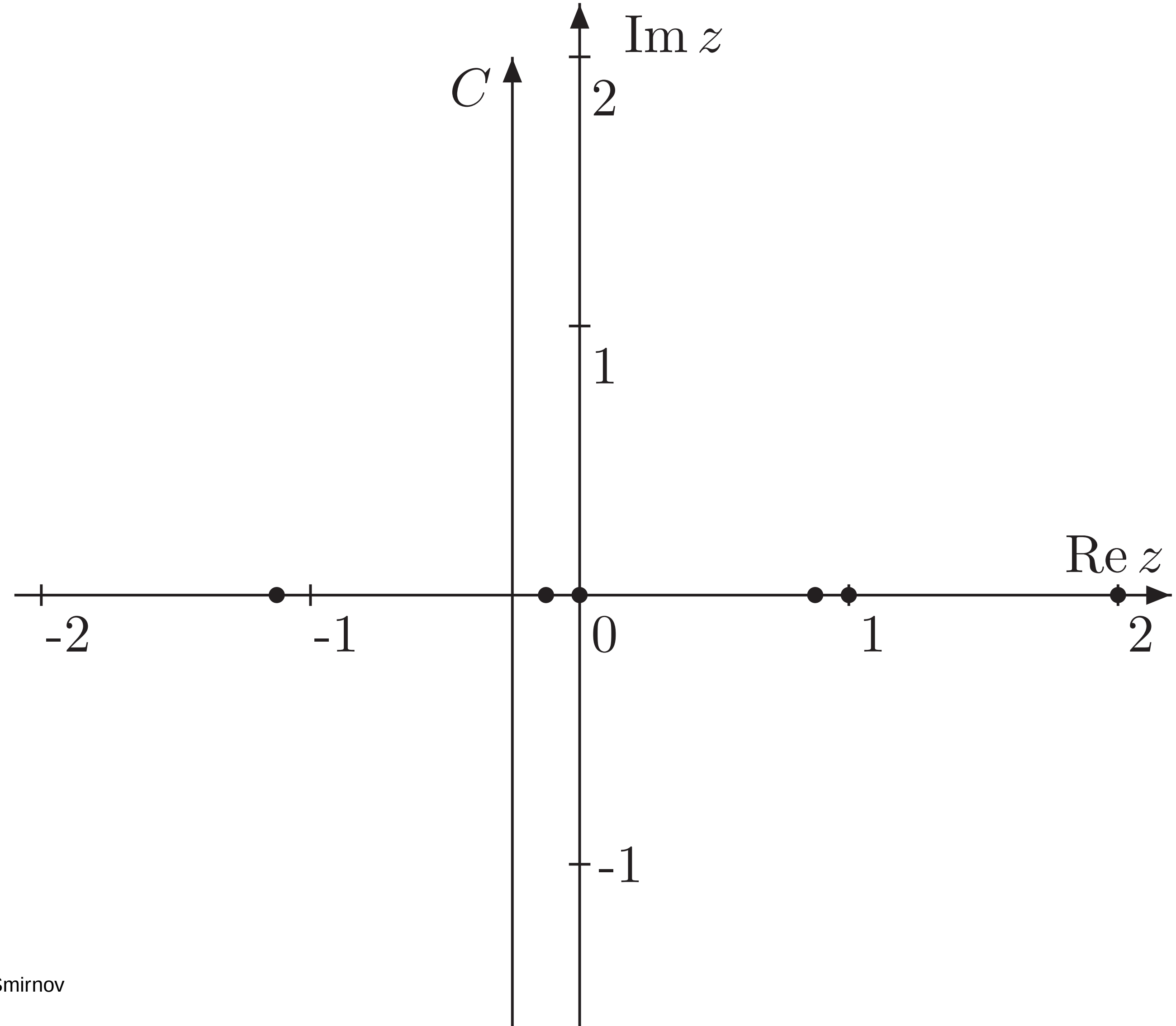
-2

Whenever a pole of some gamma function is crossed add a residue and tend ϵ to zero further



$$\frac{1}{2\pi \mathrm{i}} \int_C f(z,\epsilon) \mathrm{d}z = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\, z=-1/4} f(z,\epsilon) \mathrm{d}z + \mathrm{res}_{z=\epsilon} f(z,\epsilon)$$

General recipes for resolving the singularity structure in ϵ.

$$\frac{1}{(2\pi i)^n}\int_{-i\infty}^{+i\infty}\ldots\int_{-i\infty}^{+i\infty}\frac{\prod_i \Gamma\left(a_i+b_i\epsilon+\sum_j c_{ij}z_j\right)}{\prod_i \Gamma\left(a'_i+b'_i\epsilon+\sum_j c'_{ij}z_j\right)}\prod_k x_k^{d_k}\prod_{l=1}^{n}\mathrm{d}z_l$$

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Two strategies: Strategy A and Strategy B

- Strategy B [J.B. Tausk'99, C. Anastasiou & A. Daleo'05, M. Czakon'05]

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Let $\epsilon \to 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.

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Let $\epsilon \to 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.

For every resulting residue, which involves one integration less, apply a similar procedure, etc.

Two algorithmic descriptions [C. Anastasiou & A. Daleo'05, M. Czakon'05]

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The Czakon's version MB.m implemented in `Mathematica` is public.
http://projects.hepforge.org/mbtools/

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Strategy B: straight contours in the beginning
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Let $\prod \Gamma(A_i)$ with $A_i = a_i + b_i\epsilon + \sum_j c_{ij} z_j$

be the numerator of a multiple MB integral

Let $\sigma(x) = [(1-x)_+]$ where $[\ldots]$ is the integer part of a number and $x_+ = x$ for $x > 0$ and 0 otherwise.

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Choose contours, i.e. Rez_i, for which

$$\sum_i \sigma\left(\text{Re}A_i|_{\epsilon=0}\right) \equiv \sum_i \sigma\left(a_i + \sum_j c_{ij}\text{Re}z_j\right)$$

is minimal

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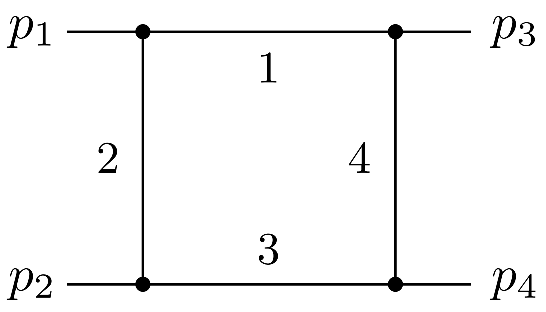
MBresolve.m
http://projects.hepforge.org/mbtools/

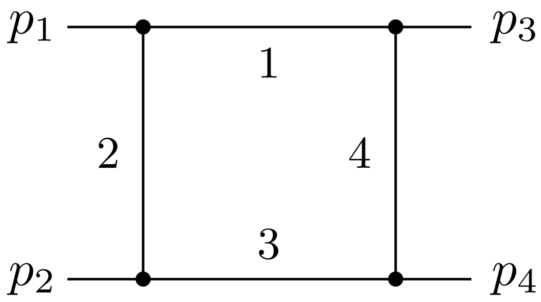
- Derive an MB representation
- Check it
- Resolve the singularity structure in ϵ. The goal is to obtain a sum of MB integrals where one may expand integrands in Laurent series in ϵ
- Expand in a Laurent series in ϵ
- Evaluate expanded MB integrals

Example 2. The massless on-shell box diagram, i.e. with $p_i^2 = 0, \; i = 1, 2, 3, 4$



How to derive MB representations

Example 2. The massless on-shell box diagram, i.e. with $p_i^2 = 0,\ i = 1, 2, 3, 4$



$$F_\Gamma(s, t; a_1, a_2, a_3, a_4, d)$$

$$= \int \frac{\mathrm{d}^d k}{(-k^2)^{a_1}[-(k+p_1)^2]^{a_2}[-(k+p_1+p_2)^2]^{a_3}[-(k-p_3)^2]^{a_4}} ,$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \;, \quad \mathcal{V} = t\alpha_1\alpha_3 + s\alpha_2\alpha_4 \;.$$

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$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \mathrm{i}\pi^{d/2}\frac{\Gamma(a+\epsilon-2)}{\prod\Gamma(a_l)}$$

$$\times \int_0^\infty \ldots \int_0^\infty \frac{\delta\left(\sum_{l=1}^4 \alpha_l - 1\right)}{(-t\alpha_1\alpha_3 - s\alpha_2\alpha_4)^{a+\epsilon-2}} \prod_l \alpha_l^{a_l-1} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_4 \;,$$

$a = a_1 + \ldots + a_4$.

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \; , \quad \mathcal{V} = t\alpha_1\alpha_3 + s\alpha_2\alpha_4 \; .$$

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$a = a_1 + \ldots + a_4$.

Introduce new variables by $\alpha_1 = \eta_1\xi_1, \; \alpha_2 = \eta_1(1-\xi_1), \; \alpha_3 = \eta_2\xi_2, \; \alpha_4 = \eta_2(1-\xi_2)$, with the Jacobian $\eta_1\eta_2$

$$
\begin{aligned}
&F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) \\
&= \mathrm{i}\pi^{d/2}\frac{\Gamma(a+\epsilon-2)\Gamma(2-\epsilon-a_1-a_2)\Gamma(2-\epsilon-a_3-a_4)}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)} \\
&\times \int_0^1\int_0^1 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2-t(1-\xi_1)(1-\xi_2)-\mathrm{i}0]^{a+\epsilon-2}}\,\mathrm{d}\xi_1\mathrm{d}\xi_2
\end{aligned}
$$

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d)$$

$$= \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\epsilon-2)\Gamma(2-\epsilon-a_1-a_2)\Gamma(2-\epsilon-a_3-a_4)}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)}$$

$$\times \int_0^1 \int_0^1 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1-\xi_1)(1-\xi_2) - \mathrm{i}0]^{a+\epsilon-2}} \,\mathrm{d}\xi_1 \mathrm{d}\xi_2$$

Apply the basic formula to separate $-s\xi_1\xi_2$ and $-t(1-\xi_1)(1-\xi_2)$ in the denominator

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d)$$
$$= \mathrm{i}\pi^{d/2}\frac{\Gamma(a+\epsilon-2)\Gamma(2-\epsilon-a_1-a_2)\Gamma(2-\epsilon-a_3-a_4)}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)}$$
$$\times \int_0^1\int_0^1 \frac{\xi_1^{a_1-1}(1-\xi_1)^{a_2-1}\xi_2^{a_3-1}(1-\xi_2)^{a_4-1}}{[-s\xi_1\xi_2-t(1-\xi_1)(1-\xi_2)-\mathrm{i}0]^{a+\epsilon-2}}\,\mathrm{d}\xi_1\mathrm{d}\xi_2$$

Apply the basic formula to separate $-s\xi_1\xi_2$ and $-t(1-\xi_1)(1-\xi_2)$ in the denominator

Change the order of integration over z and ξ-parameters, evaluate parametric integrals in terms of gamma functions

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \frac{\mathrm{i}\pi^{d/2}}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)(-s)^{a+\epsilon-2}}$$

$$\times\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z\left(\frac{t}{s}\right)^z\Gamma(a+\epsilon-2+z)\Gamma(a_2+z)\Gamma(a_4+z)\Gamma(-z)$$

$$\times\Gamma(2-a_1-a_2-a_4-\epsilon-z)\Gamma(2-a_2-a_3-a_4-\epsilon-z)$$

$$F_\Gamma(s,t;a_1,a_2,a_3,a_4,d) = \frac{\mathrm{i}\pi^{d/2}}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)(-s)^{a+\epsilon-2}}$$

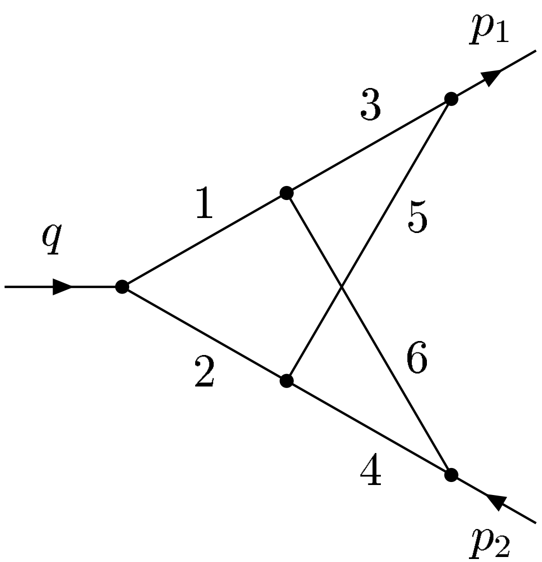
$$\times\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z\left(\frac{t}{s}\right)^z\Gamma(a+\epsilon-2+z)\Gamma(a_2+z)\Gamma(a_4+z)\Gamma(-z)$$

$$\times\Gamma(2-a_1-a_2-a_4-\epsilon-z)\Gamma(2-a_2-a_3-a_4-\epsilon-z)$$

$$F_\Gamma(s,t;1,1,1,1,d) = \frac{\mathrm{i}\pi^{d/2}}{\Gamma(-2\epsilon)(-s)^{2+\epsilon}}$$

$$\times\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z\left(\frac{t}{s}\right)^z\Gamma(2+\epsilon+z)\Gamma(1+z)^2\Gamma(-1-\epsilon-z)^2\Gamma(-z)$$

Example 3. Non-planar two-loop massless vertex diagram with $p_1^2 = p_2^2 = 0$, $Q^2 = -(p_1 - p_2)^2 = 2p_1 \cdot p_2$



$$F_\Gamma(Q^2; a_1, \ldots, a_6, d) = \int\int \frac{\mathrm{d}^d k \, \mathrm{d}^d l}{[(k+l)^2 - 2p_1 \cdot (k+l)]^{a_1}}$$

$$\times \frac{1}{[(k+l)^2 - 2p_2 \cdot (k+l)]^{a_2} (k^2 - 2p_1 \cdot k)^{a_3} (l^2 - 2p_2 \cdot l)^{a_4} (k^2)^{a_5} (l^2)^{a_6}}$$

$$\frac{1}{(k^2 - 2p_1 \cdot k)^{a_3} (k^2)^{a_5}} = \frac{(-1)^{a_3+a_5} \Gamma(a_3 + a_5)}{\Gamma(a_3)\Gamma(a_5)} \times \int_0^1 \frac{\mathrm{d}\xi_1 \, \xi_1^{a_3-1} (1-\xi_1)^{a_5-1}}{[-(k - \xi_1 p_1)^2 - \mathrm{i}0]^{a_3+a_5}}$$

and, similarly, for the second pair, with the replacements

$$\xi_1 \to \xi_2, \; p_1 \to p_2, \; k \to l, \; a_3 \to a_4, \; a_5 \to a_6$$

Change the integration variable $l \to r = k + l$ and integrate over k by means of our massless one-loop formula

$$\int \frac{\mathrm{d}k}{[-(k-\xi_1 p_1)^2]^{a_3+a_5}[-(r-\xi_2 p_2-k)^2]^{a_4+a_6}}$$

$$= \mathrm{i}\pi^{d/2} \frac{G(a_3+a_5, a_4+a_6)}{[-(r-\xi_1 p_1-\xi_2 p_2)^2]^{a_3+a_4+a_5+a_6+\epsilon-2}}$$

Apply Feynman parametric formula to the propagators 1 and 2 and the propagator arising from the previous integration, with a resulting integral over r evaluated in terms of gamma functions:

$$\int \frac{\mathrm{d}^d r}{[-(r^2-Q^2 A(\xi_1,\xi_2,\xi_3,\xi_4))]^{a+\epsilon-2}}$$

$$= \mathrm{i}\pi^{d/2} \frac{\Gamma(a+2\epsilon-4)}{\Gamma(a+\epsilon-2)} \frac{1}{(Q^2)^{a+2\epsilon-4} A(\xi_1,\xi_2,\xi_3,\xi_4)^{a+2\epsilon-4}}$$

where $a = a_1 + \ldots + a_6$ and

$$A(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_3\xi_4 + (1-\xi_3-\xi_4)[\xi_2\xi_3(1-\xi_1) + \xi_1\xi_4(1-\xi_2)]$$

Gonsalves'83:

$$F_\Gamma(Q^2; a_1, \ldots, a_6, d) \;=\; \frac{(-1)^a \left(\mathrm{i}\pi^{d/2}\right)^2 \Gamma(2-\epsilon-a_{35})\Gamma(2-\epsilon-a_{46})}{(Q^2)^{a+2\epsilon-4} \prod \Gamma(a_l)\; \Gamma(4-2\epsilon-a_{3456})}$$

$$\times\Gamma(a+2\epsilon-4) \int_0^1 \mathrm{d}\xi_1 \ldots \int_0^1 \mathrm{d}\xi_4\, \xi_1^{a_3-1}(1-\xi_1)^{a_5-1}\xi_2^{a_4-1}(1-\xi_2)^{a_6-1}$$

$$\times\xi_3^{a_1-1}\xi_4^{a_2-1}(1-\xi_3-\xi_4)_+^{a_{3456}+\epsilon-3} A(\xi_1, \xi_2, \xi_3, \xi_4)^{4-2\epsilon-a}$$

$$\frac{\Gamma(a+2\epsilon-4)}{[\eta\xi(1-\xi)+(1-\eta)(\xi\xi_2(1-\xi_1)+(1-\xi)\xi_1(1-\xi_2))]^{a+2\epsilon-4}}$$

$$= \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_1\, \Gamma(-z_1)\eta^{z_1}\xi^{z_1}(1-\xi)^{z_1}}{(1-\eta)^{a+2\epsilon-4+z_1}}$$

$$\times \frac{\Gamma(a+2\epsilon-4+z_1)}{[\xi\xi_2(1-\xi_1)+(1-\xi)\xi_1(1-\xi_2)]^{a+2\epsilon-4+z_1}}$$

The last line $\rightarrow$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z_2\, \Gamma(a+2\epsilon-4+z_1+z_2)\Gamma(-z_2)\xi^{z_2}\xi_2^{z_2}(1-\xi_1)^{z_2}}{(1-\xi)^{a+2\epsilon-4+z_1+z_2}\xi_1^{a+2\epsilon-4+z_1+z_2}(1-\xi_2)^{a+2\epsilon-4+z_1+z_2}}$$

$$F_\Gamma(Q^2;a_1,\ldots,a_6,d) = \frac{(-1)^a\left(\mathrm{i}\pi^{d/2}\right)^2\Gamma(2-\epsilon-a_{35})}{(Q^2)^{a+2\epsilon-4}\Gamma(6-3\epsilon-a)\prod\Gamma(a_l)}$$

$$\times\frac{\Gamma(2-\epsilon-a_{46})}{\Gamma(4-2\epsilon-a_{3456})}\frac{1}{(2\pi\mathrm{i})^2}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}z_1\mathrm{d}z_2\Gamma(a+2\epsilon-4+z_1+z_2)$$

$$\times\Gamma(-z_1)\Gamma(-z_2)\Gamma(a_4+z_2)\Gamma(a_5+z_2)\Gamma(a_1+z_1+z_2)$$

$$\times\frac{\Gamma(2-\epsilon-a_{12}-z_1)\Gamma(4-2\epsilon+a_2-a-z_2)}{\Gamma(4-2\epsilon-a_{1235}-z_1)\Gamma(4-2\epsilon-a_{1246}-z_1)}$$

$$\times\Gamma(4-2\epsilon+a_3-a-z_1-z_2)\Gamma(4-2\epsilon+a_6-a-z_1-z_2)\;,$$

where $a_{3456}=a_3+a_4+a_5+a_6$, etc.

The massless box diagram with two legs on shell, $p_3^2 = p_4^2 = 0$, and two legs off shell, $p_1^2, p_2^2 \neq 0$

$$B_{1100} = \mathrm{i}\pi^{d/2} \frac{\Gamma(a+\epsilon-2)}{\prod \Gamma(a_l)}$$

$$\times \int_0^\infty \ldots \int_0^\infty \left(\prod_{l=1}^4 \alpha_l^{a_l-1} \mathrm{d}\alpha_l \right) \delta\left(\sum_{l=1}^4 \alpha_l - 1 \right)$$

$$\times (-s\alpha_1\alpha_3 - t\alpha_2\alpha_4 - p_1^2\alpha_1\alpha_2 - p_2^2\alpha_2\alpha_3 - \mathrm{i}0)^{2-a-\epsilon}$$

Apply

$$\frac{1}{(X_1+\ldots+X_n)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{(2\pi \mathrm{i})^{n-1}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \ldots \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z_2 \ldots \mathrm{d}z_n \prod_{i=2}^{n} X_i^{z_i}$$

$$\times X_1^{-\lambda-z_2-\ldots-z_n} \Gamma(\lambda+z_2+\ldots+z_n) \prod_{i=2}^{n} \Gamma(-z_i)$$

Separate terms with p_1^2 and p_2^2, turn to new variables by

$$\alpha_1=\eta_1\xi_1,\ \alpha_2=\eta_1(1-\xi_1),\ \alpha_3=\eta_2\xi_2,\ \alpha_4=\eta_2(1-\xi_2)$$

and evaluate integrals over parameters to obtain a three fold MB representation

$$B_{1100} = \frac{\mathrm{i}\pi^{d/2}}{\Gamma(4-2\epsilon-a)\prod\Gamma(a_l)(-s)^{a+\epsilon-2}}$$

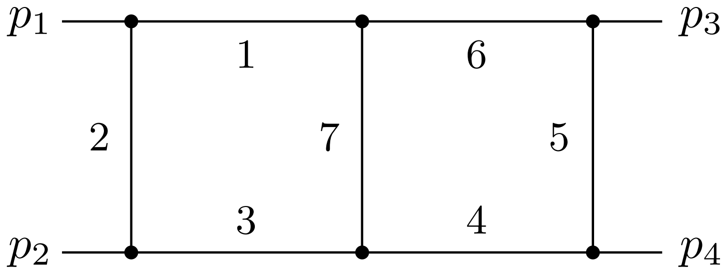
$$\times \frac{1}{(2\pi\mathrm{i})^3} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z_2 \mathrm{d}z_3 \mathrm{d}z_4 \frac{(-p_1^2)^{z_2}(-p_2^2)^{z_3}(-t)^{z_4}}{(-s)^{z_2+z_3+z_4}}$$

$$\times\Gamma(a+\epsilon-2+z_2+z_3+z_4)\Gamma(a_2+z_2+z_3+z_4)\Gamma(a_4+z_4)$$

$$\times\Gamma(2-\epsilon-a_{234}-z_3-z_4)\Gamma(2-\epsilon-a_{124}-z_2-z_4)$$

$$\times\Gamma(-z_2)\Gamma(-z_3)\Gamma(-z_4) \ .$$

Double box with irreducible numerator $(k + p_1 + p_2 + p_4)^2$



$$B_2(s,t;a_1,\ldots,a_8,\epsilon) \;=\; \int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}}$$

$$\times \frac{[(k+p_1+p_2+p_4)^2]^{-a_8}}{[(l+p_1+p_2)^2]^{a_4}[(l+p_1+p_2+p_4)^2]^{a_5}(l^2)^{a_6}[(k-l)^2]^{a_7}}$$

$$B_2(s,t;a_1,\ldots,a_8,\epsilon) \;=\; \int \frac{\mathrm{d}^d k\,[(k+p_1+p_2+p_4)^2]^{-a_8}}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}}$$

$$\times B_{1100}(s,(k+p_1+p_2+p_4)^2,k^2,(k+p_1+p_2)^2;a_6,a_7,a_4,a_5,d)$$

$$B_2(s,t;a_1,\ldots,a_8,\epsilon) \;\; = \;\; \int \frac{\mathrm{d}^d k \, [(k+p_1+p_2+p_4)^2]^{-a_8}}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}}$$

$$\times B_{1100}(s,(k+p_1+p_2+p_4)^2,k^2,(k+p_1+p_2)^2;a_6,a_7,a_4,a_5,d)$$

After using the threefold MB representation for B_{1100} and changing the order of integration we obtain an on-shell box integral with indices shifted by z-variables. Apply then the onefold representation for the this box.

$$B_2(s,t;a_1,\ldots,a_8,\epsilon) = \int \frac{\mathrm{d}^d k \, [(k+p_1+p_2+p_4)^2]^{-a_8}}{(k^2)^{a_1}[(k+p_1)^2]^{a_2}[(k+p_1+p_2)^2]^{a_3}}$$
$$\times B_{1100}(s,(k+p_1+p_2+p_4)^2,k^2,(k+p_1+p_2)^2;a_6,a_7,a_4,a_5,d)$$

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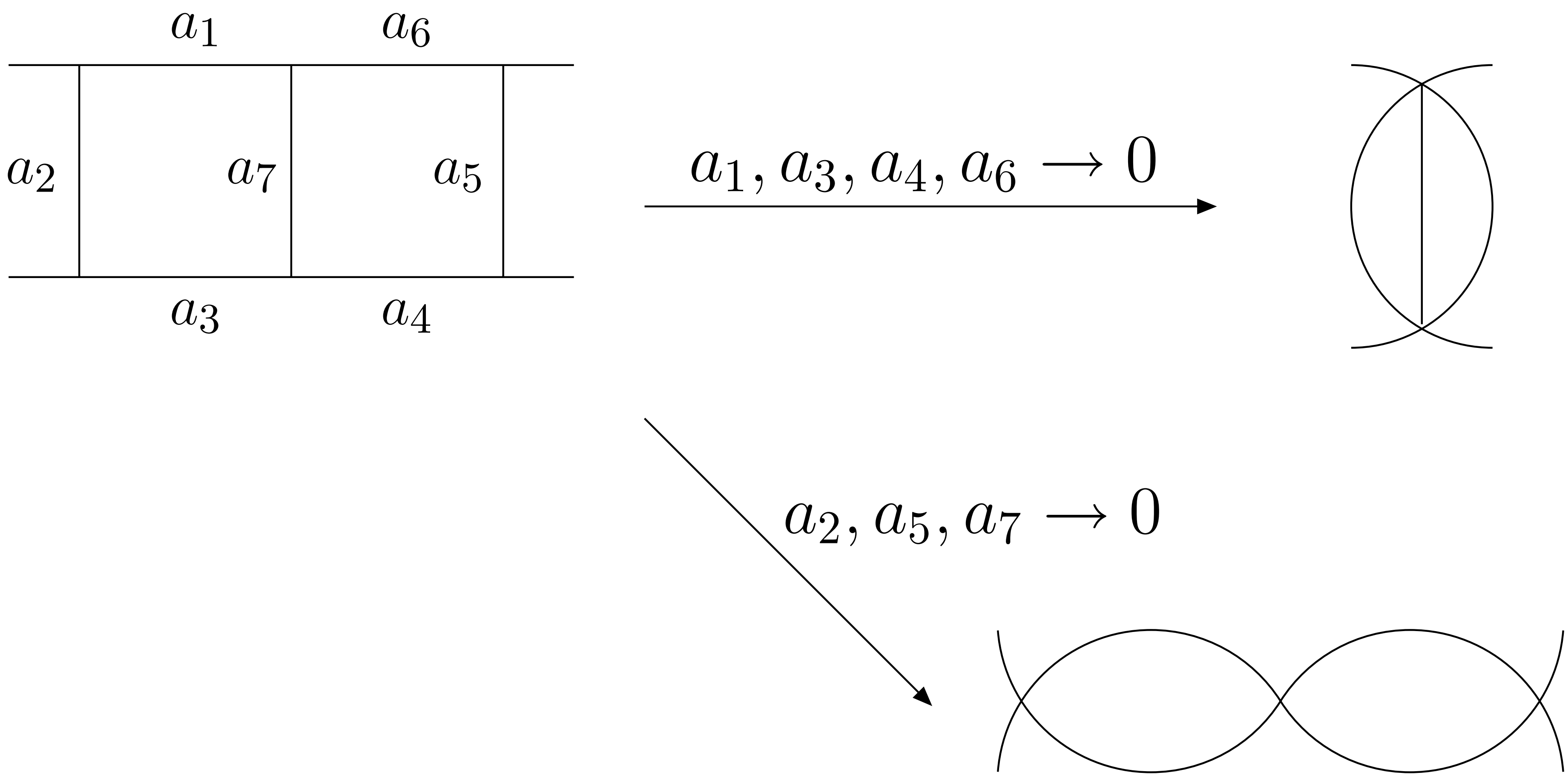
The loop by loop derivation of MB representations.

`AMBRE`

[J. Gluza, K. Kajda & T. Riemann'07]

How to check a given MB representation



The first Barnes lemma

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z)$$
$$= \frac{\Gamma(\lambda_1 + \lambda_3)\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_3)\Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}$$

How to evaluate MB integrals after expanding in ϵ

The first Barnes lemma

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z)$$

$$= \frac{\Gamma(\lambda_1 + \lambda_3)\Gamma(\lambda_1 + \lambda_4)\Gamma(\lambda_2 + \lambda_3)\Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)}$$

Multiple corollaries, e.g.,

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \Gamma(\lambda_1 + z)\Gamma^*(\lambda_2 + z)\Gamma(-\lambda_2 - z)\Gamma(\lambda_3 - z)$$

$$= \Gamma(\lambda_1 - \lambda_2)\Gamma(\lambda_2 + \lambda_3)\left[\psi(\lambda_1 - \lambda_2) - \psi(\lambda_1 + \lambda_3)\right]$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z}{z}\, \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z)$$

$$= \frac{\Gamma(2-\lambda_1-\lambda_3)\Gamma(1-\lambda_2-\lambda_3)\Gamma(\lambda_1+\lambda_3-1)\Gamma(\lambda_2+\lambda_3)}{\Gamma(1-\lambda_1)\Gamma(1-\lambda_2)}$$

$$\times \left[\Gamma(1-\lambda_1)\Gamma(1-\lambda_2) - \Gamma(2-\lambda_1-\lambda_2-\lambda_3)\Gamma(\lambda_3)\right]$$

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}z}{z} \Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 - z)\Gamma(\lambda_4 - z)$$

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$$\times \left[\Gamma(1-\lambda_1)\Gamma(1-\lambda_2) - \Gamma(2-\lambda_1-\lambda_2-\lambda_3)\Gamma(\lambda_3)\right]$$

The second Barnes lemma

$$\frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}z \, \frac{\Gamma(\lambda_1 + z)\Gamma(\lambda_2 + z)\Gamma(\lambda_3 + z)\Gamma(\lambda_4 - z)\Gamma(\lambda_5 - z)}{\Gamma(\lambda_6 + z)}$$

$$= \frac{\Gamma(\lambda_1+\lambda_4)\Gamma(\lambda_2+\lambda_4)\Gamma(\lambda_3+\lambda_4)\Gamma(\lambda_1+\lambda_5)}{\Gamma(\lambda_1+\lambda_2+\lambda_4+\lambda_5)\Gamma(\lambda_1+\lambda_3+\lambda_4+\lambda_5)}$$

$$\times \frac{\Gamma(\lambda_2+\lambda_5)\Gamma(\lambda_3+\lambda_5)}{\Gamma(\lambda_2+\lambda_3+\lambda_4+\lambda_5)} \, , \qquad \lambda_6 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$$

Transform a given multiple MB integral originating after expanding in ϵ into multiple series.

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Summing up series with nested sums

$$S_i(n) = \sum_{j=1}^{n} \frac{1}{j^i} \;, \quad S_{ik}(n) = \sum_{j=1}^{n} \frac{S_k(j)}{j^i} \;,$$

$$S_{ikl}(n) = \sum_{j=1}^{n} \frac{S_{kl}(j)}{j^i} \;, \quad S_{iklm}(n) = \sum_{j=1}^{n} \frac{S_{klm}(j)}{j^i}$$

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For example, with one index:

$$\begin{aligned} \psi(n) &= S_1(n-1) - \gamma_{\rm E} \;, \\ \psi^{(k)}(n) &= (-1)^k k! \left(S_{k+1}(n-1) - \zeta(k+1) \right) \;, \quad k = 1, 2, \ldots \;, \end{aligned}$$

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SUMMER [J.A.M. Vermaseren'00]

XSummer [S. Moch and P. Uwer'00]

Harmonic polylogarithms (HPL)

$H_{a_1,a_2,\ldots,a_n}(x) \equiv H(a_1, a_2, \ldots, a_n; x)$, with $a_i = 1, 0, -1$

[E. Remiddi & J.A.M. Vermaseren'00]

are generalizations of the usual polylogarithms $\mathrm{Li}_a\,(z)$ and Nielsen polylogarithms $S_{a,b}(z)$

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HPL implemented in `Mathematica` [D. Maitre'06]

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Consider, for example, the two-loop nonplanar vertex diagram at $p_1^2 = p_2^2 = 0$ and derive an MB representation loop by loop.

Poles in ϵ can arise not only locally but also from an integration over large z.

$$\frac{1}{2\pi i}\int_C \frac{\Gamma(1+2\epsilon+z)\Gamma(-z)}{1+\epsilon+z}(-1)^z \mathrm{d}z$$

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when $y \to +\infty$

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The integrand behaves like

$$2\pi \frac{1}{y^{1-2\epsilon}}$$

MB tools at http://projects.hepforge.org/mbtools/:

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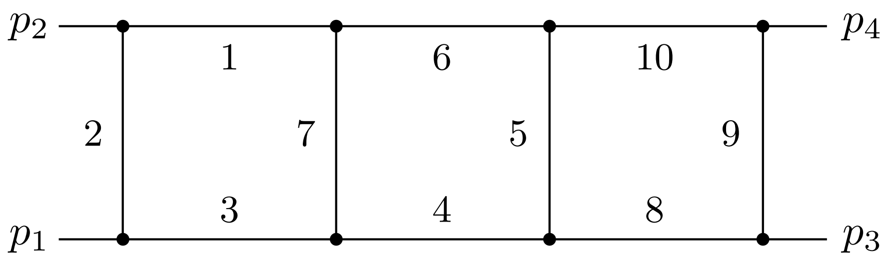
barnesroutines.m [D. Kosower'08]
(applying Barnes lemmas automatically)

backup slides



The general planar triple box Feynman integral

$$T(a_1,\ldots,a_{10};s,t;\epsilon) \;=\; \int\int\int \frac{\mathrm{d}^d k\,\mathrm{d}^d l\,\mathrm{d}^d r}{[k^2]^{a_1}[(k+p_2)^2]^{a_2}}$$

$$\times \frac{1}{[(k+p_1+p_2)^2]^{a_3}[(l+p_1+p_2)^2]^{a_4}[(r-l)^2]^{a_5}[l^2]^{a_6}[(k-l)^2]^{a_7}}$$

$$\times \frac{1}{[(r+p_1+p_2)^2]^{a_8}[(r+p_1+p_2+p_3)^2]^{a_9}[r^2]^{a_{10}}}$$

General 7fold MB representation:

$$T(a_1,\ldots,a_{10};s,t,m^2;\epsilon) \quad = \quad \frac{\left(i\pi^{d/2}\right)^3 (-1)^a}{\prod_{j=2,5,7,8,9,10}\Gamma(a_j)\Gamma(4-a_{589(10)}-2\epsilon)(-s)^{a-6+3\epsilon}}$$

$$\times \frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty} \mathrm{d}w \prod_{j=2}^{7} \mathrm{d}z_j \left(\frac{t}{s}\right)^w \frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}$$

$$\times \frac{\Gamma(2-a_1-a_2-\epsilon+z_2)\Gamma(2-a_2-a_3-\epsilon+z_3)\Gamma(a_7+w-z_4)}{\Gamma(4-a_1-a_2-a_3-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)}$$

$$\times\Gamma(+a_1+a_2+a_3-2+\epsilon+z_4)\Gamma(w+z_2+z_3+z_4-z_7)\Gamma(-z_5)\Gamma(-z_6)$$

$$\times\Gamma(2-a_5-a_9-a_{10}-\epsilon-z_5-z_7)\Gamma(2-a_5-a_8-a_9-\epsilon-z_6-z_7)$$

$$\times\Gamma(a_4+a_6+a_7-2+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(a_9+z_7)$$

$$\times\Gamma(4-a_4-a_6-a_7-2\epsilon+z_5+z_6+z_7)$$

$$\times\Gamma(2-a_6-a_7-\epsilon-w-z_2+z_5+z_7)\Gamma(2-a_4-a_7-\epsilon-w-z_3+z_6+z_7)$$

$$\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(a_5+a_8+a_9+a_{10}-2+\epsilon+z_5+z_6+z_7)\,,$$

General 7fold MB representation:

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$$\times\frac{1}{(2\pi i)^7}\int_{-i\infty}^{+i\infty}\mathrm{d}w\prod_{j=2}^{7}\mathrm{d}z_j\left(\frac{t}{s}\right)^w\frac{\Gamma(a_2+w)\Gamma(-w)\Gamma(z_2+z_4)\Gamma(z_3+z_4)}{\Gamma(a_1+z_3+z_4)\Gamma(a_3+z_2+z_4)}$$

$$\times\frac{\Gamma(2-a_1-a_2-\epsilon+z_2)\Gamma(2-a_2-a_3-\epsilon+z_3)\Gamma(a_7+w-z_4)}{\Gamma(4-a_1-a_2-a_3-2\epsilon+w-z_4)\Gamma(a_6-z_5)\Gamma(a_4-z_6)}$$

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$$\times\Gamma(a_5+z_5+z_6+z_7)\Gamma(a_5+a_8+a_9+a_{10}-2+\epsilon+z_5+z_6+z_7)\,,$$

$$
\begin{aligned}
& T(1,1,\ldots,1;s,t;\epsilon) \\
= & \frac{\left(i\pi^{d/2}\right)^3}{\Gamma(-2\epsilon)(-s)^{4+3\epsilon}} \frac{1}{(2\pi i)^7} \int_{-i\infty}^{+i\infty} \mathrm{d}w \prod_{j=2}^{7} \mathrm{d}z_j \left(\frac{t}{s}\right)^w \frac{\Gamma(1+w)\Gamma(-w)}{\Gamma(1-2\epsilon+w-z_4)} \\
\times & \frac{\Gamma(-\epsilon+z_2)\Gamma(-\epsilon+z_3)\Gamma(1+w-z_4)\Gamma(-z_2-z_3-z_4)\Gamma(1+\epsilon+z_4)}{\Gamma(1+z_2+z_4)\Gamma(1+z_3+z_4)} \\
\times & \frac{\Gamma(z_2+z_4)\Gamma(z_3+z_4)\Gamma(-z_5)\Gamma(-z_6)\Gamma(w+z_2+z_3+z_4-z_7)}{\Gamma(1-z_5)\Gamma(1-z_6)\Gamma(1-2\epsilon+z_5+z_6+z_7)} \\
\times & \Gamma(-1-\epsilon-z_5-z_7)\Gamma(-1-\epsilon-z_6-z_7)\Gamma(1+z_7) \\
\times & \Gamma(1+\epsilon+w-z_4-z_5-z_6-z_7)\Gamma(-\epsilon-w-z_2+z_5+z_7) \\
\times & \Gamma(-\epsilon-w-z_3+z_6+z_7)\Gamma(1+z_5+z_6+z_7)\Gamma(2+\epsilon+z_5+z_6+z_7)
\end{aligned}
$$

Result

[V.A. Smirnov'03]

$$T(1,1,\ldots,1;s,t;\epsilon) = -\frac{\left(i\pi^{d/2}e^{-\gamma_{\rm E}\epsilon}\right)^3}{s^3(-t)^{1+3\epsilon}} \sum_{i=0}^{6} \frac{c_j(x,L)}{\epsilon^j} \,,$$

where $x = -t/s$, $L = \ln(s/t)$, and

$$c_6 = \frac{16}{9}\,, \quad c_5 = -\frac{5}{3}L\,, \quad c_4 = -\frac{3}{2}\pi^2\,,$$

$$c_3 = 3(H_{0,0,1}(x) + LH_{0,1}(x)) + \frac{3}{2}(L^2+\pi^2)H_1(x) - \frac{11}{12}\pi^2 L - \frac{131}{9}\zeta_3\,,$$

$$c_2 = -3\left(17H_{0,0,0,1}(x) + H_{0,0,1,1}(x) + H_{0,1,0,1}(x) + H_{1,0,0,1}(x)\right)$$

$$-L\left(37H_{0,0,1}(x) + 3H_{0,1,1}(x) + 3H_{1,0,1}(x)\right) - \frac{3}{2}(L^2+\pi^2)H_{1,1}(x)$$

$$-\left(\frac{23}{2}L^2 + 8\pi^2\right)H_{0,1}(x) - \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right)H_1(x) + \frac{49}{3}\zeta_3 L - \frac{1411}{1080}\pi^4\,,$$

$$
\begin{aligned}
c_1 &= 3\left(81H_{0,0,0,0,1}(x) + 41H_{0,0,0,1,1}(x) + 37H_{0,0,1,0,1}(x) + H_{0,0,1,1,1}(x)\right.\\
&+33H_{0,1,0,0,1}(x) + H_{0,1,0,1,1}(x) + H_{0,1,1,0,1}(x) + 29H_{1,0,0,0,1}(x)\\
&\left.+H_{1,0,0,1,1}(x) + H_{1,0,1,0,1}(x) + H_{1,1,0,0,1}(x)\right) + L\left(177H_{0,0,0,1}(x) + 85H_{0,0,1,1}(x)\right.\\
&\left.+73H_{0,1,0,1}(x) + 3H_{0,1,1,1}(x) + 61H_{1,0,0,1}(x) + 3H_{1,0,1,1}(x) + 3H_{1,1,0,1}(x)\right)\\
&+\left(\frac{119}{2}L^2 + \frac{139}{12}\pi^2\right) H_{0,0,1}(x) + \left(\frac{47}{2}L^2 + 20\pi^2\right) H_{0,1,1}(x)\\
&+\left(\frac{35}{2}L^2 + 14\pi^2\right) H_{1,0,1}(x) + \frac{3}{2}\left(L^2 + \pi^2\right) H_{1,1,1}(x)\\
&+\left(\frac{23}{2}L^3 + \frac{83}{12}\pi^2 L - 96\zeta_3\right) H_{0,1}(x) + \left(\frac{3}{2}L^3 + \pi^2 L - 3\zeta_3\right) H_{1,1}(x)\\
&+\left(\frac{9}{8}L^4 + \frac{25}{8}\pi^2 L^2 - 58\zeta_3 L + \frac{13}{8}\pi^4\right) H_1(x) - \frac{503}{1440}\pi^4 L + \frac{73}{4}\pi^2\zeta_3 - \frac{301}{15}\zeta_5\,,
\end{aligned}
$$

$$
\begin{aligned}
c_0 = & -\left(951 H_{0,0,0,0,0,1}(x) + 819 H_{0,0,0,0,1,1}(x) + 699 H_{0,0,0,1,0,1}(x) + 195 H_{0,0,0,1,1,1}(x)\right.\\
& +547 H_{0,0,1,0,0,1}(x) + 231 H_{0,0,1,0,1,1}(x) + 159 H_{0,0,1,1,0,1}(x) + 3 H_{0,0,1,1,1,1}(x)\\
& +363 H_{0,1,0,0,0,1}(x) + 267 H_{0,1,0,0,1,1}(x) + 195 H_{0,1,0,1,0,1}(x) + 3 H_{0,1,0,1,1,1}(x)\\
& +123 H_{0,1,1,0,0,1}(x) + 3 H_{0,1,1,0,1,1}(x) + 3 H_{0,1,1,1,0,1}(x) + 147 H_{1,0,0,0,0,1}(x)\\
& +303 H_{1,0,0,0,1,1}(x) + 231 H_{1,0,0,1,0,1}(x) + 3 H_{1,0,0,1,1,1}(x) + 159 H_{1,0,1,0,0,1}(x)\\
& +3 H_{1,0,1,0,1,1}(x) + 3 H_{1,0,1,1,0,1}(x) + 87 H_{1,1,0,0,0,1}(x) + 3 H_{1,1,0,0,1,1}(x)\\
& \left. +3 H_{1,1,0,1,0,1}(x) + 3 H_{1,1,1,0,0,1}(x)\right)\\
& -L\left(729 H_{0,0,0,0,1}(x) + 537 H_{0,0,0,1,1}(x) + 445 H_{0,0,1,0,1}(x) + 133 H_{0,0,1,1,1}(x)\right.\\
& +321 H_{0,1,0,0,1}(x) + 169 H_{0,1,0,1,1}(x) + 97 H_{0,1,1,0,1}(x) + 3 H_{0,1,1,1,1}(x)\\
& +165 H_{1,0,0,0,1}(x) + 205 H_{1,0,0,1,1}(x) + 133 H_{1,0,1,0,1}(x) + 3 H_{1,0,1,1,1}(x)\\
& \left. +61 H_{1,1,0,0,1}(x) + 3 H_{1,1,0,1,1}(x) + 3 H_{1,1,1,0,1}(x)\right)\\
& -\left(\frac{531}{2} L^2 + \frac{89}{4}\pi^2\right) H_{0,0,0,1}(x) - \left(\frac{311}{2} L^2 + \frac{619}{12}\pi^2\right) H_{0,0,1,1}(x)\\
& -\left(\frac{247}{2} L^2 + \frac{307}{12}\pi^2\right) H_{0,1,0,1}(x) - \left(\frac{71}{2} L^2 + 32\pi^2\right) H_{0,1,1,1}(x)
\end{aligned}
$$

$$
-\left(\frac{151}{2}L^2-\frac{197}{12}\pi^2\right)H_{1,0,0,1}(x)-\left(\frac{107}{2}L^2+50\pi^2\right)H_{1,0,1,1}(x)
$$

$$
-\left(\frac{35}{2}L^2+14\pi^2\right)H_{1,1,0,1}(x)-\frac{3}{2}\left(L^2+\pi^2\right)H_{1,1,1,1}(x)
$$

$$
-\left(\frac{119}{2}L^3+\frac{317}{12}\pi^2L-455\zeta_3\right)H_{0,0,1}(x)-\left(\frac{47}{2}L^3+\frac{179}{12}\pi^2L\right.
$$

$$
\left.-120\zeta_3\right)H_{0,1,1}(x)-\left(\frac{35}{2}L^3+\frac{35}{12}\pi^2L-156\zeta_3\right)H_{1,0,1}(x)-\left(\frac{3}{2}L^3+\pi^2L\right.
$$

$$
\left.-3\zeta_3\right)H_{1,1,1}(x)-\left(\frac{69}{8}L^4+\frac{101}{8}\pi^2L^2-291\zeta_3L+\frac{559}{90}\pi^4\right)H_{0,1}(x)
$$

$$
-\left(\frac{9}{8}L^4+\frac{25}{8}\pi^2L^2-58\zeta_3L+\frac{13}{8}\pi^4\right)H_{1,1}(x)
$$

$$
-\left(\frac{27}{40}L^5+\frac{25}{8}\pi^2L^3-\frac{183}{2}\zeta_3L^2+\frac{131}{60}\pi^4L-\frac{37}{12}\pi^2\zeta_3+57\zeta_5\right)H_1(x)
$$

$$
+\left(\frac{223}{12}\pi^2\zeta_3+149\zeta_5\right)L+\frac{167}{9}\zeta_3^2-\frac{624607}{544320}\pi^6\,.
$$

‘Inverse Feynman parameters’

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Integrating over a MB variable (not by a Barnes lemma)

$$\frac{1}{2\pi i}\int_C \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(-z)x^z\text{d}z$$

$$= \frac{\Gamma(a)\Gamma(b)\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+b+c)}\,{}_2F_1(a;b;a+b+c;1-x)$$

$$= \Gamma(a)\Gamma(b+c)\int_0^1 t^{b-1}(1-t)^{a+c-1}(1-t+tx)^{-a}\text{d}t$$

- Strategy A [V.A. Smirnov'99]

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Identifying key gamma functions (responsible for the generation of poles in ϵ).

Let $\Gamma(A_i)$ with $A_i = a_i + b_i\epsilon + \sum_j c_{ij} z_j$
be one of the key gamma functions. Consider ϵ real.

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'Changing the nature' of these key gamma functions (i.e. changing rules for the contours)

Re$A_i > 0 \;\; \rightarrow -1 <$ Re $A_i < 0$
$\Gamma(A_i) \rightarrow \Gamma^{(1)}(A_i)$

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Taking residues and shifting contours.

For each resulting residue, which involves one integration less, apply a similar procedure, etc.