# Modern methods for multiloop Feynman integrals

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*Analytic Tools for Feynman Integrals* (STMP, Springer 2013)

- Feynman integrals: basic notation, definitions and properties. Dimensional regularization. Alpha and Feynman parameters. Sector decomposition.

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- Some other methods (?)

Perturbation theory. Feynman rules. A graph $\Gamma = \{\mathcal{V}, \mathcal{L}, \pi_\pm\}$ with vertices and lines (edges), where $\mathcal{V}$ is the set of vertices, $\mathcal{L}$ is the set of lines, and $\pi_\pm : \mathcal{L} \to \mathcal{V}$ .

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$$F_\Gamma(a_1, a_2, \ldots) = \int \ldots \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots}$$

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Dimensional regularization: $d = 4 - 2\epsilon\,;$ $\mathrm{d}^4 k \to \mathrm{d}^d k$
$k = (k_0, \vec{k}) = (k_0, k_1, k_2, k_3)$
$k_1, k_2, \ldots$ are loop momenta;
$p_1, p_2, \ldots$ are momenta of the lines; they are linear combinations of $k_1, k_2, \ldots$ and external momenta $q_1, q_2, \ldots$

# The propagator as a building block

$$\frac{1}{k^2 - m^2 + i0} = \lim_{\delta \to 0} \frac{1}{k^2 - m^2 + i\delta} ,$$

$$k^2 = k_0^2 - \vec{k}^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$$

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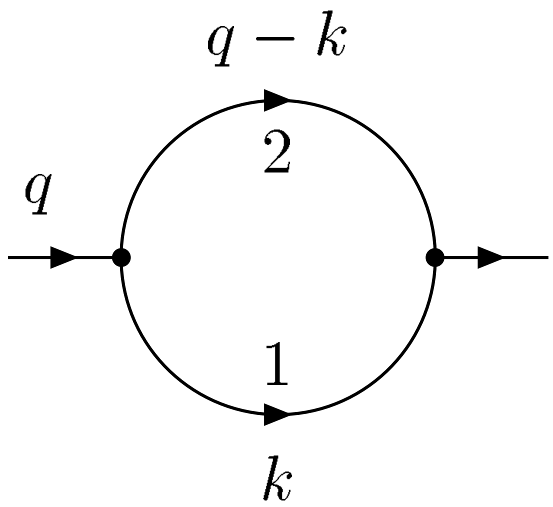
HQET, NRQCD,$\ldots \to$ other types of propagators, e.g.

$$\frac{1}{v \cdot k \pm i0} \;, \quad v = (1, \vec{0})$$

For example,



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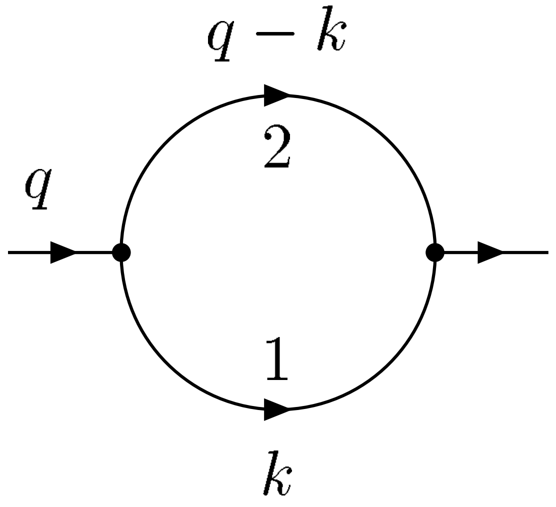
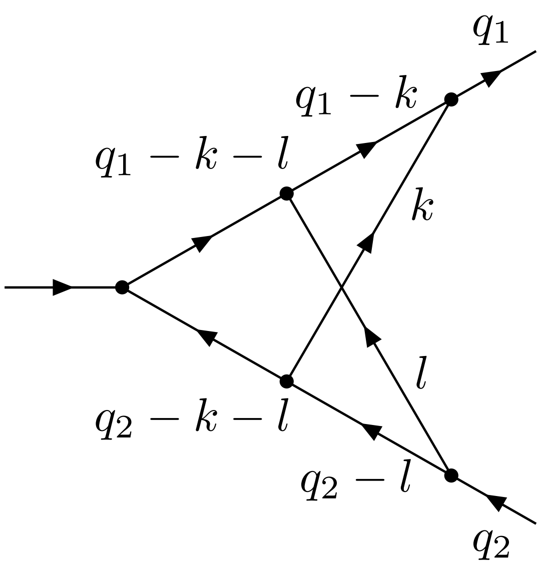



$$F_\Gamma(a_1, a_2, d) = \int \frac{\mathrm{d}^d k}{(m_1^2 - k^2 - i0)^{a_1} (m_2^2 - (q-k)^2 - i0)^{a_2}}$$

$$p_1^2 = p_2^2 = 0,\ Q^2 = -(p_1 - p_2)^2 = 2p_1 \cdot p_2$$



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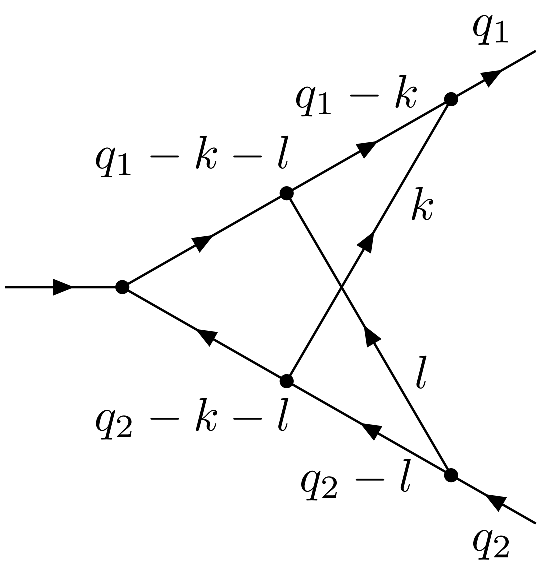


$$F_\Gamma(Q^2; a_1, \ldots, a_6, d) \;=\; \int\int \frac{\mathrm{d}^d k \, \mathrm{d}^d l}{[(k+l)^2 - 2q_1 \cdot (k+l)]^{a_1}}$$

$$\times \frac{1}{[(k+l)^2 - 2q_2 \cdot (k+l)]^{a_2} (k^2 - 2q_1 \cdot k)^{a_3} (l^2 - 2q_2 \cdot l)^{a_4} (k^2)^{a_5} (l^2)^{a_6}}$$

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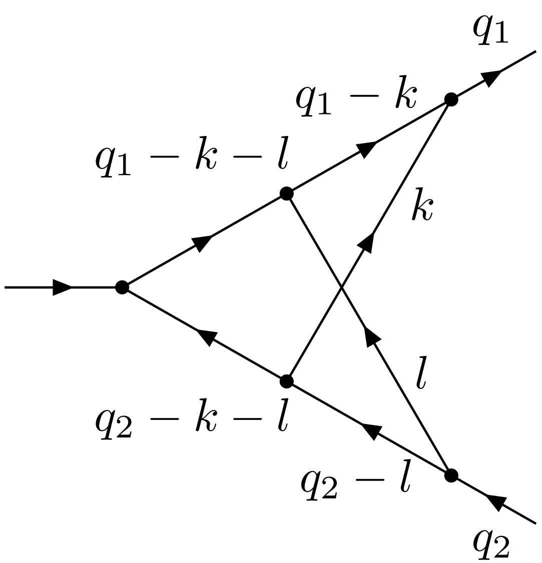


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$$\times \frac{(2k\cdot l)^{-a_7}}{[(k+l)^2 - 2q_2\cdot(k+l)]^{a_2}(k^2 - 2q_1\cdot k)^{a_3}(l^2 - 2q_2\cdot l)^{a_4}(k^2)^{a_5}(l^2)^{a_6}}$$

## UV, IR and collinear divergences

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$$F_\Gamma = \int \frac{\mathrm{d}^d k}{k^2 (k+p_1)^2 (k+p_2)^2}$$

at $p_1^2 = p_2^2 = 0$

Divergences $\rightarrow$ regularization

Divergences $\to$ regularization

Analytical regularization [E. Speer'68]

$$\frac{1}{(-k^2+m^2-i0)^a} \to \frac{1}{(-k^2+m^2-i0)^{a+\lambda}}$$

## Dimensional regularization

[G. ’t Hooft & M. Veltman’72]

[C.G. Bollini & J.J. Giambiagi’72; P. Breitenlohner & D. Maison’77]

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Informally, use alpha parameters

$$\frac{1}{(-k^2 + m^2 - i0)^a} = \frac{\mathrm{e}^{\mathrm{i}\pi a}}{\Gamma(a)} \int_0^\infty \alpha^{a-1} \mathrm{e}^{\mathrm{i}(k^2 - m^2)\alpha} \mathrm{d}\alpha$$

$$\frac{1}{(-v \cdot k - i0)^a} = \frac{\mathrm{e}^{\mathrm{i}\pi a}}{\Gamma(a)} \int_0^\infty \alpha^{a-1} \mathrm{e}^{\mathrm{i}(v \cdot k)\alpha} \mathrm{d}\alpha$$

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$$\int_{-\infty}^{+\infty} \mathrm{d}^1 k \, \mathrm{e}^{\pm \mathrm{i}\alpha k^2} \quad = \quad \sqrt{\frac{\pi}{\alpha}} \mathrm{e}^{\pm \mathrm{i}\pi/4} \; ,$$

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \mathsf{d}^1 k \, \mathrm{e}^{\pm \mathrm{i}\alpha k^2} &= \sqrt{\frac{\pi}{\alpha}} \mathrm{e}^{\pm \mathrm{i}\pi/4} \; , \\
\int \mathsf{d}^4 k \, \mathrm{e}^{\mathrm{i}\alpha k^2} &= \int\int\int\int \mathsf{d}k_0 \mathsf{d}k_1 \mathsf{d}k_2 \mathsf{d}k_3 \, \mathrm{e}^{\mathrm{i}\alpha k_0^2 - \mathrm{i}\alpha k_1^2 - \mathrm{i}\alpha k_2^2 - \mathrm{i}\alpha k_3^2}
\end{aligned}
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$$\int \mathrm{d}^d k \, \mathrm{e}^{\mathrm{i}\alpha k^2} = \left(\frac{\pi}{\alpha}\right)^{d/2} \mathrm{e}^{\mathrm{i}\pi/4 - (d-1)\mathrm{i}\pi/4} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(1-d/2)} \left(\frac{\pi}{\alpha}\right)^d \, ,$$

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&= \left(\frac{\pi}{\alpha}\right)^2 \mathrm{e}^{\mathrm{i}\pi/4 - 3\mathrm{i}\pi/4} = -\mathrm{i} \left(\frac{\pi}{\alpha}\right)^2 \, , \\
\int \mathrm{d}^d k \, \mathrm{e}^{\mathrm{i}\alpha k^2} &= \left(\frac{\pi}{\alpha}\right)^{d/2} \mathrm{e}^{\mathrm{i}\pi/4 - (d-1)\mathrm{i}\pi/4} = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(1-d/2)} \left(\frac{\pi}{\alpha}\right)^d \, , \\
\int \mathrm{d}^d k \, \mathrm{e}^{\mathrm{i}(\alpha k^2 - 2q\cdot k)} &= \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(1-d/2)} \left(\frac{\pi}{\alpha}\right)^d \mathrm{e}^{-\mathrm{i}q^2/\alpha}
\end{aligned}
$$

## Dimensional regularization:

when deriving alpha representations, apply this rule with $d = 4 - 2\epsilon$

$$\int \mathrm{d}^4 k \, \mathrm{e}^{\mathrm{i}(\alpha k^2 - 2q \cdot k)} = -\mathrm{i}\pi^2 \alpha^{-2} \mathrm{e}^{-\mathrm{i}q^2/\alpha}$$

$\longrightarrow$

$$\int \mathrm{d}^d k \, \mathrm{e}^{\mathrm{i}(\alpha k^2 - 2q \cdot k)} = \mathrm{e}^{\mathrm{i}\pi(1-d/2)/2} \pi^{d/2} \alpha^{-d/2} \mathrm{e}^{-\mathrm{i}q^2/\alpha}$$

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{a_1}(-(q-k)^2)^{a_2}} = \frac{\mathrm{e}^{\mathrm{i}\pi(a_1+a_2)}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \alpha_1^{a_1-1}\alpha_2^{a_2-1}$$

$$\times \int \mathrm{d}^d k\, \mathrm{e}^{\mathrm{i}[\alpha_1 k^2+\alpha_2(k^2+2q\cdot k+q^2)]}$$

$$= \frac{\mathrm{e}^{\mathrm{i}\pi(a_1+a_2+1-d/2)/2}\pi^{d/2}}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \frac{\alpha_1^{a_1-1}\alpha_2^{a_2-1}}{(\alpha_1+\alpha_2)^{d/2}} \mathrm{e}^{\mathrm{i}\alpha_1\alpha_2 q^2/(\alpha_1+\alpha_2)}$$

$\alpha_1 = \eta\xi,\ \alpha_2 = \eta(1-\xi)$, with the Jacobian $\eta$, integrate over $\eta$ and $\xi$

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{a_1}[-(q-k)^2]^{a_2}} = \mathrm{i}\pi^{d/2} \frac{G(a_1,a_2)}{(-q^2)^{a_1+a_2+\epsilon-2}} \; ,$$

$$G(a_1,a_2) = \frac{\Gamma(a_1+a_2+\epsilon-2)\Gamma(2-\epsilon-a_1)\Gamma(2-\epsilon-a_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(4-a_1-a_2-2\epsilon)}$$

Graph $\Gamma \to$ dimensionally regularized Feynman integral

$$F_\Gamma(a_1 \ldots, a_L; d) = \frac{e^{i\pi(a+h(1-d/2))/2} \pi^{hd/2}}{\prod_l \Gamma(a_l)} \times \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} e^{i\mathcal{V}/\mathcal{U} - i\sum m_l^2 \alpha_l} ,$$

where $a = \sum a_i$

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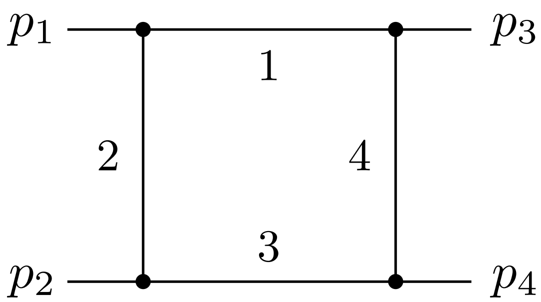
For a Feynman integral with $1/(m^2 - k^2 - i0)^{a_l}$ propagators,

$$\mathcal{U} = \sum_{\text{trees } T} \prod_{l \notin T} \alpha_l ,$$

$$\mathcal{V} = \sum_{2-\text{trees } T} \prod_{l \notin T} \alpha_l \left(q^T\right)^2 .$$

The massless box,
$p_i^2 = 0,\ i = 1, 2, 3, 4,\ s = (p_1 + p_2)^2,\ t = (p_1 + p_3)^2$



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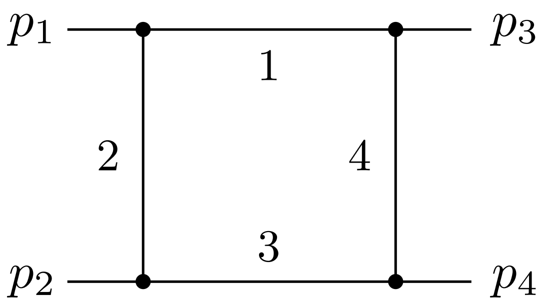


trees

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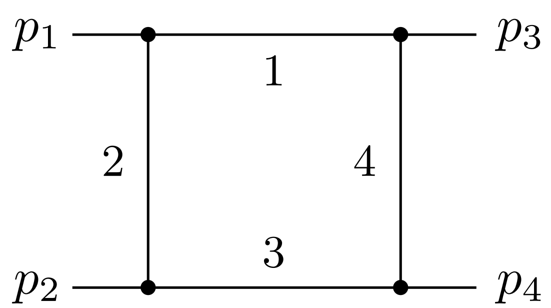


## trees

## 2-trees

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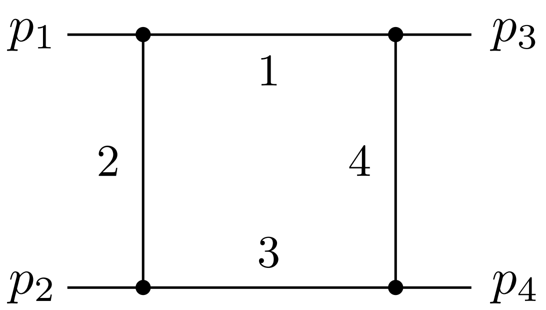
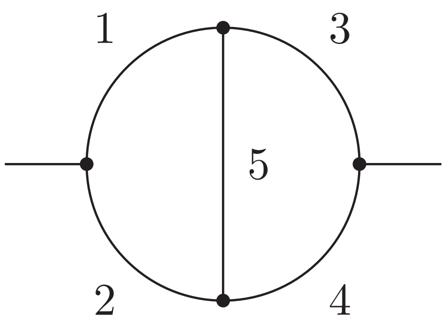



trees

2-trees

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\ ,\quad \mathcal{V} = s\alpha_1\alpha_3 + t\alpha_2\alpha_4\ .$$

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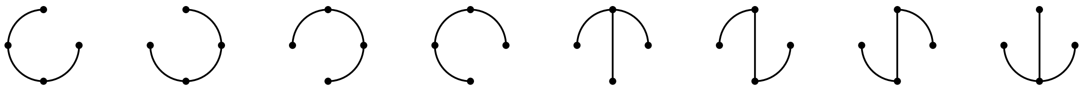
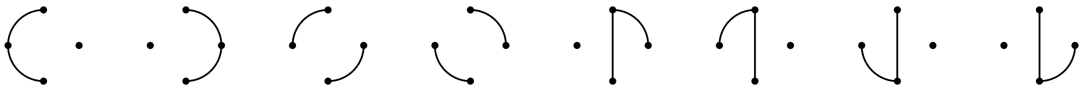


2-trees



$$\mathcal{U} = (\alpha_1+\alpha_2+\alpha_3+\alpha_4)\alpha_5 + (\alpha_1+\alpha_2)(\alpha_3+\alpha_4)\ ,$$

$$\mathcal{V} = [(\alpha_1+\alpha_2)\alpha_3\alpha_4 + \alpha_1\alpha_2(\alpha_3+\alpha_4) + (\alpha_1+\alpha_3)(\alpha_2+\alpha_4)\alpha_5]q^2$$

The code `UF.m` to evaluate $\mathcal{U}$ and $\mathcal{V}$
http://www-ttp.particle.uni-karlsruhe.de/∼asmirnov

Alpha representation $\rightarrow$

- Mathematical proofs (for Feynman integrals at Euclidean external momenta, $(\sum q_i)^2 < 0$) Analysis of convergence.

  [K. Hepp'66; P. Breitenlohner & D. Maison'77; E. Speer'68,'77]

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  Hepp sectors [K. Hepp'66]

$$\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_L$$

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# Recursively defined sector decompositions

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Primary sectors

$$\alpha_i \le \alpha_l \;, \quad l \neq i = 1, 2, \ldots, L \;,$$

with new variables

$$t_i = \begin{cases} \alpha_i/\alpha_l & \text{if } i \neq l \\ \alpha_l & \text{if } i = l \end{cases}$$

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The contribution of a primary sector

$$F_l \;=\; \int_0^1 \ldots \int_0^1 \left( \prod_{i \neq l} \mathrm{d}t_i \right) \frac{\mathcal{U}^{L-(h+1)d/2}}{\mathcal{V}^{L-hd/2}} \bigg|_{t_l=1}$$

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Strategy S [A.V. Smirnov & M.N. Tentyukov'08]
FIESTA
(Feynman Integral Evaluation by a Sector decomposiTion Approach)

http://www-ttp.particle.uni-karlsruhe.de/∼asmirnov

The usage of Speer’s sectors within FIESTA.

It turns out that, for Feynman integrals at Euclidean external momenta, Speer’ sectors are reproduced within Strategy S.

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A geometrical strategy: sector decomposition via computational geometry.

[ T. Kaneko & T. Ueda'10]

$\alpha = \eta\alpha'_l,\ l = 1, 2, \ldots, L-1,\ \eta = \sum_{l=1}^{L} \alpha_l$, integrate over $\eta$, introduce $\alpha'_L = 1 - \sum_{l=1}^{L-1} \alpha'_l$ by inserting an integration over $\alpha'_L$ with $\delta\left(\sum_{l=1}^{L} \alpha_l - 1\right)$, replace $\alpha'_l$ by $\alpha_l$:

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$$F_\Gamma(q_1,\ldots,q_n;d;a_1\ldots,a_L) \;=\; \frac{\left(\mathrm{i}\pi^{d/2}\right)^h \Gamma(a-hd/2)}{\prod_l \Gamma(a_l)}$$

$$\times \int_0^\infty \ldots \int_0^\infty \delta\left(\sum_{l=1}^L \alpha_l - 1\right) \frac{\prod_l \alpha_l^{a_l-1}\mathcal{U}^{a-(h+1)d/2}}{\left(-\mathcal{V}+\mathcal{U}\sum m_l^2\alpha_l\right)^{a-hd/2}} \mathrm{d}\alpha_1\ldots\mathrm{d}\alpha_L$$

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Cheng–Wu theorem:

$$\delta\left(\sum_{l=1}^L \alpha_l - 1\right) \to \delta\left(\sum_{l\in\nu}\alpha_l - 1\right) \to \delta\left(\alpha_l - 1\right)$$

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$$\times \int_0^\infty \ldots \int_0^\infty \delta\left(\sum_{l=1}^{L} \alpha_l - 1\right) \frac{\prod_l \alpha_l^{a_l - 1} \mathcal{U}^{a-(h+1)d/2}}{\left(-\mathcal{V} + \mathcal{U} \sum m_l^2 \alpha_l\right)^{a-hd/2}} \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_L$$

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Proof. Use $\eta = \sum_{l \in \nu} \alpha_l$ instead of $\eta = \sum_{l=1}^{L} \alpha_l$

$$\int\int \frac{\mathrm{d}^d k\, \mathrm{d}^d l}{(-k^2+m^2)^{\lambda_1}[-(k+l)^2]^{\lambda_2}(-l^2+m^2)^{\lambda_3}}$$

$$= \left(\mathrm{i}\pi^{d/2}\right)^2 \frac{\Gamma(\lambda_1+\lambda_2+\epsilon-2)\Gamma(\lambda_2+\lambda_3+\epsilon-2)\Gamma(2-\epsilon-\lambda_2)}{\Gamma(\lambda_1)\Gamma(\lambda_3)}$$

$$\times \frac{\Gamma(\lambda_1+\lambda_2+\lambda_3+2\epsilon-4)}{\Gamma(\lambda_1+2\lambda_2+\lambda_3+2\epsilon-4)\Gamma(2-\epsilon)(m^2)^{\lambda_1+\lambda_2+\lambda_3+2\epsilon-4}}$$

choose $\delta\left(\alpha_1+\alpha_3-1\right)$

Feynman parameters:

$$\frac{1}{(m_1^2 - p_1^2)^{a_1}(m_2^2 - p_2^2)^{a_2}}$$

$$= \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \int_0^1 \frac{\mathrm{d}\xi\, \xi^{a_1-1}(1-\xi)^{a_2-1}}{[(m_1^2 - p_1^2)\xi + (m_2^2 - p_2^2)(1-\xi)]^{a_1+a_2}}$$

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$$\frac{1}{\prod A_l^{a_l}} = \frac{\Gamma(\sum a_l)}{\prod \Gamma(a_l)} \int_0^1 \mathrm{d}\xi_1 \ldots \int_0^1 \mathrm{d}\xi_L \prod_l \xi_l^{a_l-1} \frac{\delta\left(\sum \xi_l - 1\right)}{\left(\sum A_l \xi_l\right)^{\sum a_l}}$$