Scattering Amplitudes at Strong coupling

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Scattering Amplitudes At Strong Coupling

Outline

- Perturbative results plus prescription at strong coupling (I)
- $\mathcal{N}=4$ example and minimal surfaces in AdS_3 (II)
- $Y-$system for Scattering Amplitudes (III)
- Correlation with local operators (IV)

Motivations

We will be interested in gluon scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills.

Motivation: It can give non trivial information about more realistic theories but is more tractable. In the last years, many tools become available.

- Perturbative computations are easier (as you will see in this school!).
- The strong coupling regime can be studied, by means of the gauge/string duality, through a weakly coupled string sigma model.

$\mathcal{N} = 4$ Super-Yang Mills

- Most symmetric four dimensional quantum field theory.
- $SU(N)$ gauge group $\rightarrow$ fixed Lagrangian.
- Parametrized by N_c and g_{YM}.
- We will focus in the planar limit: $N_c \gg 1$, $\lambda = g_{YM}^2 N_c$ fixed:

$$A(g_{YM}, N_c) \rightarrow A(\lambda)$$

- We will scatter gluons in this theory!

Gluon state $|\mathcal{G}\rangle = |h, k^\mu, a\rangle$

$$A_n^{L,Full} \sim \sum_\rho Tr(T^{a_{\rho(1)}} \ldots T^{a_{\rho(n)}}) A_n^{(L)}(\rho(1), \ldots, \rho(2))$$

- Leading N_c color ordered $n-$points amplitude at L loops: $A_n^{(L)}$
- $A_n^{(L)}$ depends only on the kinematics and the helicities of the gluons.

What about the helicities?

- SuSy Ward identities $\rightarrow A(\pm,+,+,+,...,+)=0$.
- First non trivial, $A(-,-,+,+,...+)$, MHV amplitudes.

Lorentz structure already captured by the tree level amplitude!

- Reduced $M_n^{(L)}(\epsilon)=A_n^{(L)}(\epsilon)/A_n^{(0)}$, only function of the kinematical invariants.

- The amplitudes are IR divergent so we need to introduce a regulator.
- Dimensional regularization $D = 4 - 2\epsilon \rightarrow A_n^{(L)}(\epsilon) = 1/\epsilon^{2L} + \ldots$
- IR divergences "exponentiate":

$$\mathcal{M}_N = 1 + \lambda M_N^{(1)} + \lambda^2 M_N^{(2)} + \ldots$$

$$\mathcal{M}_N = e^{f(\lambda) M_N^{(1)}} e^{R_N(k_i,\lambda)}$$

- All IR-divergences are contained in the first factor $M^{(1)} = \frac{1}{\epsilon^2} + \ldots.$
- $R_N(k_i, \lambda)$: finite piece, depends only on cross-ratios and the coupling constant.

$f(\lambda)$: Cusp anomalous dimension!

- Very well understood in the AdS/CFT.
- It appears in a lot of computations!
- Using integrability an equation was written, which computes it at all values of the coupling!!

$$f(\lambda) = \lambda + ..., \quad \lambda \ll 1; \qquad f(\lambda) = \frac{\sqrt{\lambda}}{\pi} + ..., \quad \lambda \gg 1$$

Question: How do we compute amplitudes at strong coupling?!

AdS/CFT duality

Four dimensional maximally SUSY Yang-Mills (g_{YM}, N)	$\Leftrightarrow$	Type IIB string theory on $AdS_5 \times S^5$ (g_s, R)

$$\sqrt{\lambda} \equiv \sqrt{g_{YM}^2 N} = \frac{R^2}{\alpha'} \qquad \frac{1}{N} \approx g_s$$

- The AdS/CFT is a very powerful computational tool!

Consider $F(\lambda)$

- The gauge theory is only good/reliable for $\lambda \ll 1$

$$F(\lambda) = F^{(0)} + \lambda F^{(1)} + \lambda^2 F^{(2)} + ...$$

- Systematic way to compute these terms, but the complexity grows really fast!

- What to do for large values of λ? use AdS/CFT! ($R \approx \lambda^{1/4}$)

$$F(\lambda) = \sqrt{\lambda}\, \boxed{\tilde{F}^{(0)}} + \tilde{F}^{(1)} + \frac{1}{\sqrt{\lambda}} \tilde{F}^{(2)} + \ldots$$

Some geometrical computation!

- In $\mathcal{N}=4$ SYM we have the luxury of the AdS/CFT duality.
- We can compute quantities of $\mathcal{N}=4$ SYM at strong coupling by doing geometrical computations on AdS.

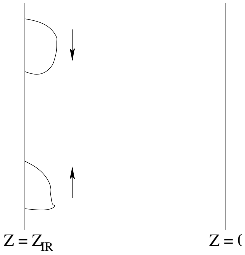
Disclaimer:

- Sometimes hard to build the AdS/CFT dictionary!
- Scattering amplitudes: First we need to introduce a regulator!

String theory set up

$$ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{z^2}$$

- Regulator: Place a D-brane extended along x_{3+1} and located at some large z_{IR}.



- The asymptotic states are open strings ending on the D-brane.
- Consider the scattering of these open strings.
- The proper momentum of these strings, $k_{pr} = k\frac{z_{IR}}{R}$ is very large, so we are interested in the regime of fixed angle and very high momentum.

This regime was considered in flat space (Gross and Mende)

Key feature

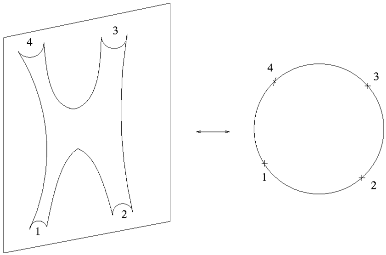
The amplitude is dominated by a saddle point of the classical action.

$\Downarrow$

We need to consider a classical string on AdS

- Important difference: k doesn't need to be too large.

World-sheet with the topology of a disk with vertex operator insertions (corresponding to external states)



- Near each vertex operator, the momentum of the external state fixes the form of the solution.
- In the boundary of the world-sheet $z = z_{IR}$

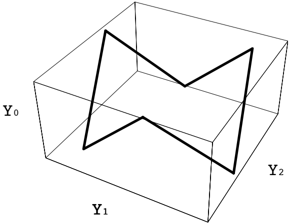
- "T-duality": $ds^2 = w^2(z) dx_\mu dx^\mu \to \partial_\alpha y^\mu = i w^2(z) \epsilon_{\alpha\beta} \partial_\beta x^\mu$
- Boundary conditions: x^μ carries momentum $k^\mu \to y^\mu$ has winding $\Delta y^\mu = 2\pi k^\mu$.
- After a change of coordinates $r = R^2/z$ we end up again with AdS_5

$$ds^2 = R^2 \frac{dy_{3+1}^2 + dr^2}{r^2}$$

What happened to the world-sheet?

Its boundary is a sequence of lines constructed as follows:

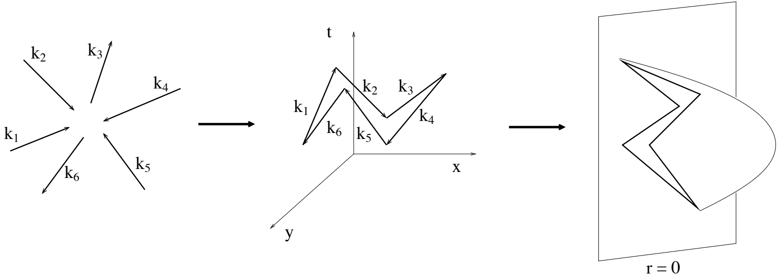
- For each particle with momentum k^μ draw a segment joining two points separated by $\Delta y^\mu = 2\pi k^\mu$



- Concatenate the segments according to the ordering of the insertions on the disk (particular color ordering)
- Momentum conservation: Closed diagram.
- Massless gluons $\rightarrow$ light-like edges.

- As $z_{IR} \rightarrow \infty$ the boundary of the world-sheet moves to $r = 0$.

- $A_n(\lambda)$ at strong coupling $\rightarrow$ Minimal area problem in AdS!



$$A_n \approx e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}, \qquad \lambda \gg 1$$

Prescription

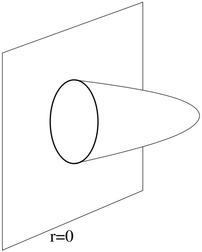
$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- $\mathcal{A}_n$: Leading exponential behavior of the $n-$point scattering amplitude.
- $A_{min}(k_1^\mu, k_2^\mu, ..., k_n^\mu)$: Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

Comments:

- Prefactors are subleading in $1/\sqrt{\lambda}$, and we don't compute them.
- In particular our computation is blind to helicity, etc.

- Remember a similar problem: Expectation value of Wilson loops at strong coupling (Maldacena, Rey)



- $ds^2 = R^2 \frac{dx_{3+1}^2 + dz^2}{r^2}$
- We need to consider the minimal area ending (at $r = 0$) on the Wilson loop.

$$\langle W \rangle \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- Our computation is exactly equivalent to the computation of the vev of a WL given by a sequence of null segments!!

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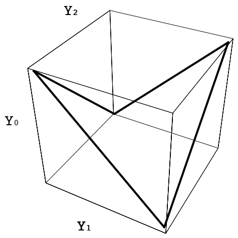
Lecture 2: Four gluons amplitude (N=4, sorry!) and minimal surfaces in AdS_3

Four point amplitude at strong coupling

Consider $k_1 + k_3 \to k_2 + k_4$

$$s = -(k_1 + k_2)^2$$
$$t = -(k_1 + k_4)^2$$



Need to find the minimal surface ending on such sequence of light-like segments.

- Use Poincare coordinates (r, y_0, y_1, y_2) and parametrize the surface by its projection to (y_1, y_2) plane.
- S_{NG}: Action for two fields $r(y_1, y_2), y_0(y_1, y_2)$. *E.g.* if $s = t$ the fields live on a square parametrized by y_1, y_2.

$$S_{NG} = \frac{R^2}{2\pi} \int dy_1 dy_2 \frac{\sqrt{1 + (\partial_i r)^2 - (\partial_i y_0)^2 - (\partial_1 r \partial_2 y_0 - \partial_2 r \partial_1 y_0)^2}}{r^2}$$

- By scale invariance, edges of the square at $y_1, y_2 = \pm 1$

Boundary conditions

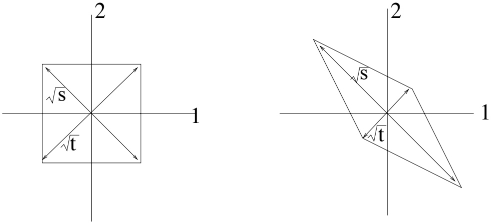
$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \quad y_0(\pm 1, y_2) = \pm y_2, \quad y_0(y_1, \pm 1) = \pm y_1$$

- Put the action in Mathematica. Deduce the Euler-Lagrange equations.
- Start making educated guesses
- Hope to be lucky!

$$y_0(y_1, y_2) = y_1 y_2, \quad r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)}$$

- Easily seen to satisfy all the boundary conditions and actually solves the eoms!
- However, $s = t$ is somehow a boring case...

- We would like to capture the kinematical dependence of the amplitude. We need to consider $s \neq t$.
- The square will be deformed to a rhombus



(a) (b)

Embedding coordinates

$$-Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1$$

$$Y^\mu = \frac{y^\mu}{r}, \quad \mu = 0, ..., 3$$

$$Y_{-1} + Y_4 = \frac{1}{r}, \quad Y_{-1} - Y_4 = \frac{r^2 + y_\mu y^\mu}{r}$$

Embedding coordinates surface

$$Y_0 Y_{-1} = Y_1 Y_2 \qquad Y_3 = Y_4 = 0$$

- We can perform $SO(2,4)$ transformations and get new solutions. This is a "dual" conformal symmetry.
- *e.g.* a boost in the $0-4$ direction gives a new solution with $s \neq t$.

Conformal gauge action

$$iS = -\frac{R^2}{2\pi}\int du_1 du_2 \frac{1}{2}\frac{\partial r \partial r + \partial y_\mu \partial y^\mu}{r^2}$$

Solution for the rhombus

$$r = \frac{a}{\cosh u_1 \cosh u_2 + b \sinh u_1 \sinh u_2},$$
$$y_0 = r\sqrt{1+b^2}\sinh u_1 \sinh u_2$$
$$y_1 = r \sinh u_1 \cosh u_2, \qquad y_2 = r \cosh u_1 \sinh u_2$$

- The parameters a and b encode the kinematical information.

$$-s(2\pi)^2 = \frac{8a^2}{(1-b)^2}, \qquad -t(2\pi)^2 = \frac{8a^2}{(1+b)^2}$$

Let's compute the area...

- Small problem: The area diverges!
- Dimensional reduction scheme: Theory in $D = 4 - 2\epsilon$ dimensions but with 16 supercharges.
- For integer D this is exactly the low energy theory living on $Dp-$branes ($p = D - 1 = 3 - 2\epsilon$)

Gravity dual

$$ds^2 = h^{-1/2} dx_D^2 + h^{1/2} \left(dr^2 + r^2 d\Omega_{9-D}^2 \right), \qquad h = \frac{R^4}{r^{8-D}}$$

T-dual coordinates

$$ds^2 = R^2\left(\frac{dy_D^2 + dr^2}{r^{2+\epsilon}}\right) \to S_\epsilon = \frac{R^2}{2\pi}\int \frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}$$

- Presence of ϵ will make the integrals convergent.
- The eoms will depend on ϵ but if we plug the original solution into the new action, the answer is accurate enough.
- plugging everything into the action...

$$iS = -\frac{\sqrt{\lambda}}{2\pi a^\epsilon}\left(\frac{\pi\Gamma\left[-\frac{\epsilon}{2}\right]^2}{\Gamma\left[\frac{1-\epsilon}{2}\right]}\,{}_2F_1\left(\frac{1}{2},-\frac{\epsilon}{2},\frac{1-\epsilon}{2};b^2\right)+1/2\right)+\mathcal{O}(\epsilon)$$

- Just expand in powers of ϵ...

Final answer

$$\mathcal{A} = e^{iS} = \exp\left[iS_{div} + \frac{\sqrt{\lambda}}{8\pi}\left(\log\frac{s}{t}\right)^2 + \tilde{C} \right]$$

$$S_{div} \approx -\frac{1}{\epsilon^2}\frac{1}{2\pi}\sqrt{\lambda}$$

- Should be compared to the field theory expectations:

$$\mathcal{A} \sim \mathcal{A}_{div} \exp\left\{ \frac{f(\lambda)}{8}(\ln s/t)^2 + R \right\}$$

$$\mathcal{A}_{div} \approx \exp\left\{ -\frac{1}{2\epsilon^2} f(\lambda) \right\}$$

- Exactly agrees with field theory expectations!
- After using the correct strong coupling limit of $f(\lambda)$ and
- With $R = 0$ (or a constant)

Reason: R can depend only on cross-ration, but for $N = 4$ we can construct none!
Question: What about $N > 4$??

- For $N = 5$, again, we cannot construct cross-ratios.
- Starting from $N = 6$ the answer will be non-trivial!

How do we find these minimal surfaces!?

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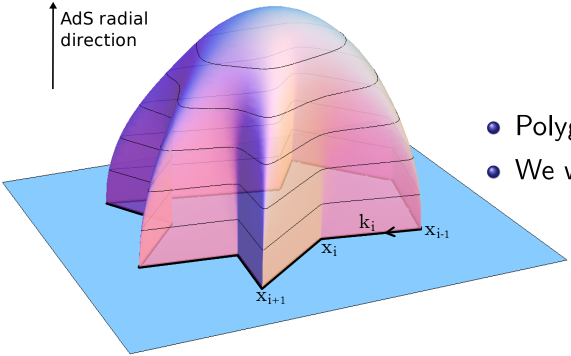
Lecture 3: Minimal surfaces in AdS_3

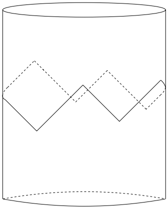
Math problem: Minimal surfaces/Soap bubbles in AdS



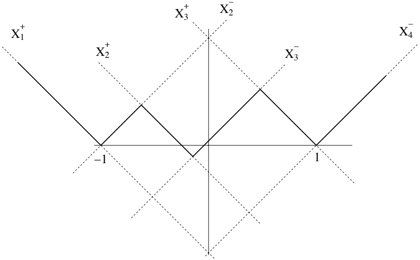
- Polygon parametrized by x_i.
- We want $A_{min}(x_i)$

- $Y_s = \frac{(x_i - x_j)^2 (x_k - x_l)^2}{(x_i - x_k)^2 (x_j - x_l)^2}$ are cross-ratios.
- N gluons: $3N - 15$ of them. We want $A_{min}(Y_s)$.
- For AdS_3, only $N - 6$ cross-ratios.

Surface on AdS_3: Ends in a $2D$ polygon, *e.g.* in the cylinder.



- Zig-zagged Wilson loop of $N = 2n$ sides
- Parametrized by n coordinates x_i^+ and n coordinates x_i^- .
- We can build $2n - 6$ cross ratios: $\frac{x_{ij}^+ x_{kl}^+}{x_{ik}^+ x_{jl}^+}$



- Minimal surfaces on AdS_3 $\leftrightarrow$ Classical strings on AdS_3.

Classical strings on AdS_3

$$ds^2 = \frac{dr^2 + dx^+ dx^-}{r^2}$$

Embedding coordinates: $-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = -1$

$$\begin{aligned} X_{-1} + X_2 &= 1/r, \quad X_1 \pm X_0 = \frac{x^\pm}{r} \\ X_{-1} - X_2 &= \frac{r^2 + x^+ x^-}{r^2} \end{aligned}$$

Boundary of AdS_3 $(r \to 0)$:

- $X's$ very large: $-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = 0$.
- X and λX are identified.

Point at the boundary

$$x^\pm = \frac{X_1 \pm X_0}{X_{-1} \pm X_2}$$

Classical strings on AdS_3

Strings on AdS_3 : $\vec{X}(z,\bar{z})$, $\quad \vec{X}.\vec{X} = -X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = -1$

$$\mathit{Eoms} : \partial\bar{\partial}\vec{X} - (\partial\vec{X}.\bar{\partial}\vec{X})\vec{X} = 0, \qquad \mathit{Virasoro} : \partial\vec{X}.\partial\vec{X} = \bar{\partial}\vec{X}.\bar{\partial}\vec{X} = 0$$

Pohlmeyer kind of reduction $\rightarrow$ generalized Sinh-Gordon

$$\alpha(z,\bar{z}) = \log(\partial\vec{X}.\bar{\partial}\vec{X}), \qquad p^2 = \partial^2\vec{X}.\partial^2\vec{X}$$

$$\downarrow$$

$$p = p(z), \qquad \partial\bar{\partial}\alpha - e^{\alpha} + |p(z)|^2 e^{-\alpha} = 0$$

- $\alpha(z,\bar{z})$ and $p(z)$ invariant under conformal transformations.
- Area of the world sheet: $\mathcal{A} = \int e^{\alpha} d^2 z$

Standard form of the sinh-Gordon equation: go to the $w-$plane

$$dw = \sqrt{p(z)}dz, \quad \hat{\alpha} = \alpha - \frac{1}{4}\log p\bar{p} \to \partial_w \bar{\partial}_{\bar{w}} \hat{\alpha} = sinh\hat{\alpha}$$

- Simpler equation in a more complicated space.
- Convenient to understand some features of the solution.

$$\mathcal{A} = \int e^{\alpha} d^2 z = \int e^{\hat{\alpha}} d^2 w$$

Generalized Sinh-Gordon $\rightarrow$ Strings on AdS_3?

- From α, p construct flat connections $B_{L,R}$ and solve two linear auxiliary problems.

$$(d + B^L)\psi^L_a = 0 \qquad B^L_z = \begin{pmatrix} \partial\alpha & e^{\alpha} \\ e^{-\alpha}p(z) & -\partial\alpha \end{pmatrix}$$
$$(d + B^R)\psi^R_{\dot a} = 0$$

Important: $\partial B_{\bar z} - \bar\partial B_z + [B_z, B_{\bar z}] = 0$

Space-time coordinates

$$X_{a,\dot a} = \begin{pmatrix} X_{-1} + X_2 & X_1 - X_0 \\ X_1 + X_0 & X_{-1} - X_2 \end{pmatrix} = \psi^L_a \psi^R_{\dot a}$$

One can check that X constructed that way has all the correct properties.

- Flat connection: $\partial B_{\bar{z}} - \bar{\partial} B_z + [B_z, B_{\bar{z}}] = 0$.
- We can introduce a spectral parameter, and the connection is still flat!!

$$B_z \to B_z(\zeta) = \begin{pmatrix} \partial\alpha & 0 \\ 0 & -\partial\alpha \end{pmatrix} + \frac{1}{\zeta} \begin{pmatrix} 0 & e^{\alpha} \\ e^{-\alpha} p(z) & 0 \end{pmatrix}$$

$$B_{\bar{z}} \to B_z(\zeta) = \begin{pmatrix} -\bar{\partial}\alpha & 0 \\ 0 & \bar{\partial}\alpha \end{pmatrix} + \zeta \begin{pmatrix} 0 & \bar{p}(\bar{z}) e^{-\alpha} \\ e^{\alpha} & 0 \end{pmatrix}$$

- Sign of integrability!
- Unified picture:
  - B^L is simply $B(\zeta = 1)$.
  - B^R is simply $B(\zeta = i)$

Bonus: Relation to Hitchin equations ($\zeta = e^{\theta}$)

$$B_z = A_z + \Phi \to B_z(\theta) = A_z + e^{-\theta}\Phi$$
$$B_{\bar{z}} = A_{\bar{z}} + \Phi^* \to B_{\bar{z}}(\theta) = A_{\bar{z}} + e^{\theta}\Phi^*$$

- Additional symmetry: $B(\theta + i\pi) = \sigma_3 B(\theta)\sigma_3$

Hitchin equations

- Consider self-dual YM in 4d reduced to 2d

$A_{1,2} \to A_{1,2}$: 2d gauge field, $A_{3,4} \to \Phi, \Phi^*$: Higgs field.

Hitchin equations

$$F^{(4)} = *F^{(4)} \qquad \to \qquad \begin{aligned} D_{\bar{z}}\Phi = D_z\Phi^* &= 0 \\ F_{z\bar{z}} + [\Phi, \Phi^*] &= 0 \end{aligned}$$

- Flatness for all θ implies the above equations!
- Particular case of $SU(2)$ Hitchin equations.

- What is $p(z)$ for our problem?
- What are the boundary conditions for α?

Four cusps solution ($n=2$): $p(z)=1, \quad \hat{\alpha}=0$

For the solutions relevant to scattering amplitudes we require:

- $p(z)$ to be a polynomial.
- $\hat{\alpha}$ to decay at infinity, where the boundary is located (we approach the vacuum solution).

We are interested in the regularized area:

$$\mathcal{A}=\int e^{\hat{\alpha}}d^2w \rightarrow \mathcal{A}_{reg}=\int (e^{\hat{\alpha}}-1)d^2w, \quad \rightarrow \quad \mathcal{A}=\mathcal{A}_{div}+\mathcal{A}_{reg}$$

Consider a generic polynomial of degree $n-2$

$$p(z) = z^{n-2} + c_{n-4} z^{n-4} + \dots + c_1 z + c_0$$

- We have used translations and re-scalings in order to fix the first two coefficients to one and zero.
- For a polynomial of degree $n-2$ we are left with $2n-6$ (real) variables.
- This is exactly the number of invariant cross-ratios in two dimensions for the scattering of $2n$ gluons!

$$\text{Null polygons of } 2n \text{ sides} \quad \Leftrightarrow \quad p^{n-2}(z) \text{ and } \hat{\alpha}(z,\bar{z}) \to 0$$

Solution close to the boundary?

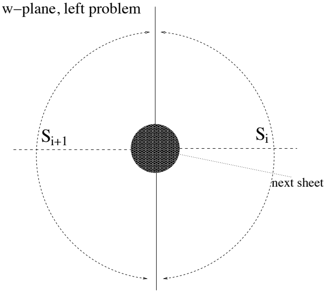
Is this really what we want?!

- We want to understand the solution close to the boundary $|z| \gg 1$
- $p(z) = z^{n-2} \rightarrow w \approx z^{n/2}$
- As we go around the $z-$plane once, in the w-plane we go around $n/2$ times.
- As $|w| \gg 1 \rightarrow \hat{\alpha} \approx 0$. The linear problems simplify drastically! and we can write a general solution.

Solution close to the boundary?

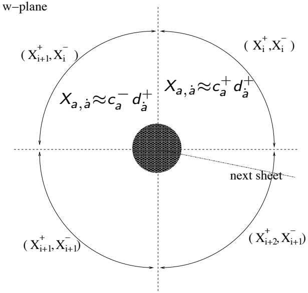
- Gral solution of the left problem:

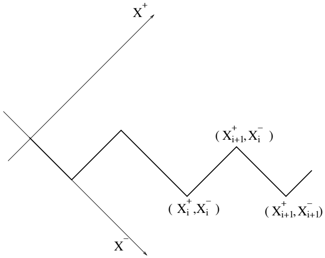
$$\psi_a^L = c_a^+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{w+\bar{w}} + c_a^- \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-(w+\bar{w})}, \qquad (\psi^L_{a,\alpha})$$



- Focus in the left problem.
- Each sheet divided into two regions/sectors, $\pm Re(w) > 0$
- In each sector, one of the two solutions dominates.

The right-problem is similar: $\psi_{\dot{a}}^R = d_{\dot{a}}^+ \begin{pmatrix} e^{\frac{w-\bar{w}}{i}} \\ 0 \end{pmatrix} + d_{\dot{a}}^- \begin{pmatrix} 0 \\ e^{-\frac{w-\bar{w}}{i}} \end{pmatrix}$





- The $w-$plane is divided into quadrants.
- At each quadrant, one solution of each problem is dominant.
- The whole region corresponds to a single point in space-time (at the boundary), a cusp.
- As we cross one of the axis, the dominant solution L or R changes and we jump to the next cusp.
- At each step only one changes $\rightarrow$ in $R^{1,1}$ only the x^+ or x^- coordinate changes
- As we go around the $w-$plane $n/2$ times, we get the $2n$ cusps!

Comments

- The locations of the cusps can be written as: $X^i_{a,\dot{a}} = \lambda^i_a \tilde{\lambda}^i_{\dot{a}}$
- The distance between consecutive cusps is null, as only one dominant solution changes:

$$x_{ij}^2 = \epsilon^{ab}\epsilon^{\dot{a}\dot{b}} X^i_{a\dot{a}} X^j_{b\dot{b}} \equiv X^i \cdot X^j$$

  Which is zero, if X^i, X^j share a common λ!
- Similar for generic polynomial. General picture:

$$p(z) = z^{n-2} + c_{n-4} z^{n-4} + \ldots + c_1 z + c_0$$

  - Degree of the polynomial $\rightarrow$ number of cusps.
  - Coefficients of the polynomial $\rightarrow$ shape of the polygon.

General strategy

Choose a polynomial $p(z)$:

- Solve the inverse problem and find the space-time cross-ratios for that polynomial.
- Find α with the correct boundary conditions and compute the regularized area.

All fine except two little obstacles! we need a better idea...

Integrable model

If one finds the appropriate trick, one is going to be able to perform computations.

Let us focus on the left problem.

In each sector, the small solution is well defined (up to a normalization constant). The large solution is not, as we can add to it a part of the small solution

- s_i^L: Small solution at the ith sector. So we have $s_0, s_1, ..., s_{n-1}$
- $SL(2)$ invariant product:

$$\psi_a^L \wedge \psi_b^L \equiv \epsilon^{\alpha\beta}\psi_{\alpha,a}^L \psi_{\beta,b}^L = \epsilon_{a,b}$$

- The small solutions can be used to extract the large components of the solution!

$$\psi_a^L \wedge s_i^L = \lambda_a^i$$

How do we construct cross-ratios?

- Remember that $X^i_{a,\dot{a}} = \lambda^i_a \tilde{\lambda}^i_{\dot{a}}$

$$(x^i - x^j)^2 \approx X^i.X^j = \langle \lambda^i \lambda^j \rangle \langle \tilde{\lambda}^i \tilde{\lambda}^j \rangle, \qquad \langle \lambda^i \lambda^j \rangle \equiv \epsilon^{ab} \lambda^i_a \lambda^j_b$$

Important identity: $\langle \lambda^i \lambda^j \rangle = (f_i f_j) s^L_i \wedge s^L_j$

- Cross-ratios can be constructed from products of the small solutions in the the corresponding sectors!

$$\frac{x^+_{12} x^+_{34}}{x^+_{13} x^+_{24}} = \frac{(s^L_1 \wedge s^L_2)(s^L_3 \wedge s^L_4)}{(s^L_1 \wedge s^L_3)(s^L_2 \wedge s^L_4)}$$

- The normalization constant of each s^L_i goes away!
- We can define the cross-ratios in terms of the small solutions ONLY, that were the ones well defined!

Let's try to use integrability!

- Introduce the spectral parameter θ ($\zeta = e^{\theta}$)

$$Y_s \to Y_s(\theta)$$

- Study the θ dependence of such deformed cross-ratios.

How do we do that?

- Study the small flat sections of the connection $B(\theta)$

$$(d + B(\theta))s_i = 0$$

$$\chi_{ijkl}(\theta) = \frac{(s_i \wedge s_j)(s_k \wedge s_l)}{(s_i \wedge s_k)(s_j \wedge s_l)}$$

- $\chi(\theta = 0) \to x^{+}$ cross-ratios.
- $\chi(\theta = i\pi/2) \to x^{-}$ cross-ratios.

Two more elements...

- $B(\theta+i\pi)=\sigma_3 B(\theta)\sigma_3 \rightarrow s_i \wedge s_j(\theta+i\pi)=s_{i+1}\wedge s_{j+1}(\theta)$
- Choose a normalization such that $s_i \wedge s_{i+1}=1$

Now the derivation is completely trivial!
Choose a basis s_0, s_1, with $s_0 \wedge s_1 = 1$ and write everything in terms of them:

$$\begin{aligned} s_k &= (s_k \wedge s_1)s_0 - (s_k \wedge s_0)s_1 \\ s_{k+1} &= (s_{k+1}\wedge s_1)s_0 - (s_{k+1}\wedge s_0)s_1 \end{aligned}$$

Introduce $f^{\pm}=f(\theta \pm i\pi/2)$, then

$$s_k \wedge s_{k+1} = 1 \rightarrow -(s_{k-1}\wedge s_0)^{++}(s_{k+1}\wedge s_0)+(s_k\wedge s_0)^{++}(s_k \wedge s_0)=1$$

Some technicalities, but bear with me...

- Call $T_k \equiv s_0 \wedge s_{k+1}(\theta - i(k+1)\pi/2)$ and
- Introduce the $Y-$functions $Y_s \equiv T_{s-1}T_{s+1}$.

These are actually cross-ratios!

$$Y_k(\theta) = \frac{(s_0 \wedge s_k)(s_{-1} \wedge s_{k+1})}{(s_{-1} \wedge s_0)(s_k \wedge s_{k+1})}(\theta - ik\pi/2)$$

But just the wedges in the denominator are equal to one.

What about the above equation in terms of the $Y's$??

Y-system equations!

$$Y_s(\theta + i\pi/2)Y_s(\theta - i\pi/2) = (1 + Y_{s+1}(\theta))(1 + Y_{s-1}(\theta))$$

- The Y_s are our deformed cross-ratios.
- Y_s non zero for $s = 1, ..., n-3$.
- Evaluating Y_s at $\theta = 0, i\pi/2$ we obtain the physical cross-ratios.
- These equations came from trivial identities (*e.g* no information about $p(z)$!)
- Need to supplement them with the analytic properties of $Y(\theta)$

- $Y_s(\theta)$ are analytic away from $\theta = \pm\infty$
- As $\theta \to \pm\infty$ the flat connection simplifies and we can use a WKB approximation!

$$\log Y_s \approx -m_s \cosh\theta + ..., \qquad \text{for large } \theta.$$

- m_s encode the information in the polynomial $p(z)$
- They are usually complex $\to 2(n-3)$ d.o.f!

Old integrability trick:

- Functional equations
- Boundary conditions

$\left.\begin{array}{l}\text{Functional equations}\\ \text{Boundary conditions}\end{array}\right\}$ Integral equations for the $Y's$!

TBA equations!

$$\log Y_s = -m_s \cosh\theta + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \frac{1}{\cosh(\theta-\theta')} \log(1+Y_{s-1}(\theta'))(1+Y_{s+1}(\theta'))$$

And the area is....

Free energy!

$$A_{min} = \sum_s \int \frac{d\theta}{2\pi} m_s \cosh\theta \log(1+Y_s(\theta))$$

General strategy:

- Choose some m_s and solve the integral equations.

- From $Y_s(\theta = 0, i\pi/2)$ we get the "physical" cross ratios.
- From the free energy we get the area.

} Area for these cross-ratios!

Scattering Amplitudes at Strong Coupling

Lecture 4: What else can we compute?! Correlation functions of Wilson loops with local operators.

Correlation functions in $\mathcal{N} = 4$ SYM (in the planar limit) and how to compute them at strong coupling.

- Natural generalization of two branches of AdS/CFT in which a lot of progress has been made:
  - Spectral problem.
  - Scattering amplitudes.
- Prove the theory at the non-planar level.

$$\langle \mathcal{O}\mathcal{O} \rangle \sim 1 \rightarrow \langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle \sim \frac{1}{N}$$

Spectral problem:

Two point functions of single trace local operators in $\mathcal{N}=4$ SYM.

$$\mathcal{O}_1 = \mathrm{tr}ZZXX - \mathrm{tr}ZXZX, \qquad \mathcal{O}_2 = \mathrm{tr}ZZXX + \mathrm{tr}ZXZX$$

Conformal symmetry: $\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_{12}|^{2\Delta_i}}$

Spectral problem: Compute Δ_i to all values of the coupling!

$$\Delta_1 = 4, \qquad \Delta_2 = 4 + \frac{3}{\pi^2}\lambda + \ldots$$

AdS/CFT

Δ at strong coupling: Energy of a particular string configuration.

$$\Delta_1 = 4, \qquad \Delta_2 = 2\lambda^{1/4} + \ldots$$

What about three-point functions?

Conformal symmetry

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{13}^{\Delta_1+\Delta_3-\Delta_2}x_{23}^{\Delta_2+\Delta_3-\Delta_1}}$$

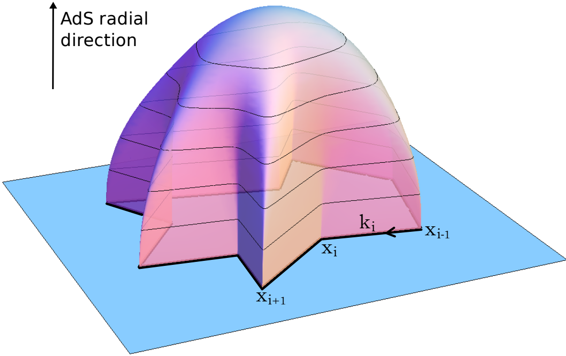
- We would like to compute $C_{123}(\lambda)$ to all values of the coupling constant.
- Knowing $\Delta_i(\lambda)$ plus $C(\lambda)$ we could compute any correlation function!

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle \to \sum_p \langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_p\rangle\langle \mathcal{O}_p\mathcal{O}_3\mathcal{O}_4\rangle$$

Scattering amplitudes:

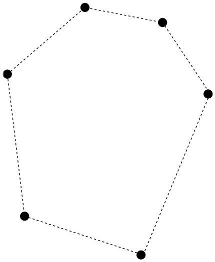
- Scattering of gluons $\mathcal{A}(k_1, ..., k_n, \lambda)$
- (Dual) Conformal symmetry $\rightarrow \mathcal{A}(\text{cross-ratios}, \lambda)$

Can be computed at strong coupling: Area of minimal surfaces



- Scattering amplitudes are naturally on-shell quantities.

- Off-shell analogous of scattering amplitude $\rightarrow$ correlation functions!
- Much richer objects (depend on many more cross-ratios).
- Scattering amplitudes are a particular case of them.



Different kind of operators

- V_L: light states, $\Delta \sim 1$ (*e.g.* sugra modes)
- V_H: semiclassical/heavy states $\Delta \sim \sqrt{\lambda}$ (harder to consider)

We are able to compute:

$$\langle V_H V_H V_L \rangle$$

Another interesting quantity: Correlation of a Wilson loop with a local operator!

$$\frac{\langle \mathcal{W} V_L^{\Delta}(x) \rangle}{\langle \mathcal{W} \rangle}$$

Computation at strong coupling: two ingredients:

- Classical solution (minimal surface) corresponding to $\langle \mathcal{W} \rangle$ (parametrized by X_{clas})
- A particular propagator $K^{\Delta}(x)$, which propagates from a point x in the boundary to the world-sheet of the classical solution.

$$\frac{\langle \mathcal{W} V_L^{\Delta}(x) \rangle}{\langle \mathcal{W} \rangle} = \int d^2\zeta K(x(\zeta)_{clas} - \hat{x}, z(\zeta)_{clas})$$

$$K^{\Delta}(x,z) = \left(\frac{z}{z^2 + x^2} \right)^{\Delta} \left(\frac{1}{z^2} ((\partial_i z)^2 + (\partial_i x)^2) \right)$$

- Interesting quantity to consider, our Wilson loops.
- Conformal invariance fix some dependence!

$$\frac{\langle \mathcal{W}_4 V_L^\Delta(x)\rangle}{\langle \mathcal{W}_\triangle\rangle} = \frac{(x_{13}^2 x_{24}^4)^{\Delta/2}}{\prod_i |x-x_i|^{\Delta/2}} F(\zeta,\lambda)$$

where $\zeta = \frac{(x-x_2)^2(x-x_4)^2(x_1-x_3)^2}{(x-x_1)^2(x-x_3)^2(x_2-x_4)^2}$

- Nice observable to try to extrapolate!
- Hard to compute for higher N!
- Can we use integrability?!