Aspects of scattering amplitudes

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Plan

- Box decomposion, infrared divergences
- Leading singularities, integral reduction
- Turning integrals into ‘symbols’

Integral reduction

- All one-loop amplitudes can be expressed in terms of boxes, triangles, bubbles, (tadpoles) integrals, and rational terms

(Passarino and Veltman)

- Consider some integral with many denominators

$$I = \int \frac{d^4x}{i\pi^2} \frac{x^\mu x^\nu}{((x-x_1)^2+m_1^2)\cdots((x-x_6)^2+m_6^2)}$$

- Basic idea is ‘completing the square’

- Lets introduce 6-vector objects

$$X_i \equiv \begin{pmatrix} x_i^\mu \\ 1 \\ x_i^2 + m_i^2 \end{pmatrix} \qquad X \equiv \begin{pmatrix} x^\mu \\ 1 \\ x^2 \end{pmatrix}$$

- with the (4,2) metric

$$\begin{aligned} Y \cdot Z &\equiv \begin{pmatrix} y^\mu \\ Y^- \\ Y^+ \end{pmatrix} \cdot \begin{pmatrix} z^\mu \\ Z^- \\ Z^+ \end{pmatrix} \\ &\equiv -2y_\mu z^\mu + Z^+ Y^- + Z^- Y^+ \end{aligned}$$

- Then, $X \cdot X_i = (x - x_i)^2 + m_i^2$

- If Y_1, Y_2 are unit vectors along μ, ν,

$$\frac{x^\mu x^\nu}{((x-x_1)^2+m_1^2)\cdots((x-x_6)^2+m_6^2)} = \frac{X\cdot Y_1 X\cdot Y_2}{X\cdot X_1\cdots X\cdot X_6}$$

- Since the Xs form a basis for the 6-dimensional space of numerators

$$Y_1 = a_1 X_1 + a_2 X_2 + \ldots + a_6 X_6$$

- we have that

$$I = a_1 \int \frac{d^4x}{i\pi^2}\frac{X\cdot Y_2}{X\cdot X_2\cdots X\cdot X_6} + \ldots$$

No hexagons!

- In general it pays to think 'conformally': the dot product is SO(6) invariant

- The measure is also:

$$\int \frac{d^4x}{i\pi^2} \to \int_X \equiv \int \frac{d^6X\,\delta(X^2)}{i\pi^2\mathrm{vol}(\mathrm{GL}(1))}$$

- All integrals should have weight -4 in X. By adding powers of X.I ($I=(0,0,1)^T$) this can always be achieved. e.g.:

$$\int \frac{d^4x}{i\pi^2}\frac{1}{(x-x_1)^2\cdots(x-x_3)^2} = \int_X \frac{1}{X{\cdot}X_1\cdots X{\cdot}X_3 X{\cdot}I}$$

Symmetrical integration

Basic integrals:

$$\int_X \frac{1}{(X{\cdot}Y)^4} = \frac{1}{(\frac{1}{2}Y^2)^2}$$

Exercise: prove this!

$$\int_X \frac{X \cdot Z}{(X{\cdot}Y)^5} = -2\frac{Y{\cdot}Z}{(\frac{1}{2}Y^2)^3}$$

Feynman parametrization:

$$\frac{1}{X{\cdot}X_1 \cdots X{\cdot}X_n} = \int_0^\infty da_1 \dots da_n \delta(a_n - 1) \frac{1}{(X{\cdot}\sum_i a_i X_i)^n}$$

Theorem:

$$\int_X \frac{\epsilon_{i_1 i_2 i_3 i_4 i_5 i_6} X_1^{i_1} \cdots X_5^{i_5} X^{i_6}}{X.X_1 \cdots X{\cdot}X_5} = 0$$

(van Neerven & Vermaseren)

Summary:

Any 1-loop amplitude will be a linear combination of:

- Boxes

$$\int_X \frac{1}{X\cdot X_1 \cdots X\cdot X_4}$$

- Triangles

$$\int_X \frac{1}{X\cdot X_1 \cdots X\cdot X_3 X\cdot I}$$

- Bubbles

$$\int_X \frac{1}{X\cdot X_1 X\cdot X_2 (X\cdot I)^2}$$

- Rational terms: $\frac{\epsilon}{\epsilon_{\mathrm{UV}}} \to \mathrm{const.}$ (come from doing this in D dimensions)

Lets do an integral: the box with massless internal states

$$I_4^{4m} = \int \frac{d^4x\,N}{(x-x_1)^2\cdots(x-x_4)^2} = \int_X \frac{N}{X\cdot X_1\cdots X\cdot X_4}$$

- Use Feynman parameters with c=1:

$$N\int_0^\infty \frac{da\,db\,dd\,dc\delta(c-1)}{(ab1\cdot2+ac1\cdot3+ad1\cdot4+bc2\cdot3+bd2\cdot4+cd3\cdot4)^2}.$$

- The d integral is elementary:

$$N\int_0^\infty \frac{da\,db\,dc\,\delta(c-1)}{(a1\cdot4+b2\cdot4+c3\cdot4)(ab1\cdot2+ac1\cdot3+bc2\cdot3)}$$

- The b integral is also elementary, of the form

$$\int_0^\infty \frac{db}{(b+y)(b+z)} = \frac{\log\frac{z}{y}}{z-y}$$

- So, without much work one finds the integral

$$\int_0^\infty \frac{da}{[\text{quadratic in a}]} \log \frac{(a+1)(au+1)}{vb}$$

- How do we do that? Basic trick:

$$\int_0^\infty da R(a) \log(\text{const.}) \quad = \quad \oint \frac{da}{2\pi i} R(a) \log(-a) \log(\text{const.})$$

$$\int_0^\infty da R(a) \log a \quad = \quad \oint \frac{da}{2\pi i} R(a) \frac{1}{2} \log^2(-a)$$

$$\boxed{\int_0^\infty da R(a) \log(1+a) \quad = \quad -\oint \frac{da}{2\pi i} R(a) \mathrm{Li}_2(1+a)}$$

Hodges (2010)

- Result:

$$\boxed{I_{4m} = 2\mathrm{Li}_2(1-\alpha_+) - 2\mathrm{Li}_2(1-\alpha_-) + \log v \log \frac{\alpha_+}{\alpha_-}}$$

where $\alpha_\pm = \dfrac{1+u-v \pm \sqrt{(1-u-v)^2 - 4uv}}{2}$

if $N = 1{\cdot}32{\cdot}4(\alpha_+ - \alpha_-), \quad u = \dfrac{1{\cdot}23{\cdot}4}{1{\cdot}32{\cdot}4}, \quad v = \dfrac{1{\cdot}42{\cdot}3}{1{\cdot}32{\cdot}4}.$

- Note: Coefficient of dilogs, 1/N, is a leading singularity ("turn all integrals into contour integrals")

Special cases tabulated in Bern,Dixon,Kosower,Dunbar (1995), Also Hodges (2010)

- Lets consider integrals with massless corners
- These are collinear and soft divergent
- Consider the simplest case, $(x_1 - x_2)^2 \to 0$ in the above
- We have to regulate the integral in the sensitive region

$$I^{\mathrm{reg}} = I_{4m} + \int \frac{d^4x}{i\pi^2(x-x_3)^2(x-x_4)^2} \times \left[\frac{1}{((x-x_1)^2+m^2)(x-x_2)^2+m^2)} - (m^2=0)\right]$$

- Put $x_1 = 0, x_2 = (p^+, -p^2/p^+, 0_\perp)$
  where p²->0

$$\int \frac{d\ell^+ d\ell^- d^2\ell_\perp}{i2\pi^2} \left[\frac{1}{\ell^+\ell^- - \ell_\perp^2 - m^2 + i0} \frac{1}{(\ell^+ - p^+)(\ell^- \frac{p^2}{p^+}) - p_\perp^2 - m^2 + i0} - (m^2 \to 0) \right]$$

- "Pinching pole" argument

$$= \int_0^1 dx \int \frac{d^2p_\perp}{\pi} \left[\frac{1}{\ell_\perp^2 + m^2 + x(1-x)p^2} - (m^2 \to 0) \right]$$

$$= \int_0^1 dx \log \frac{p^2 + \frac{m^2}{x(1-1)}}{p^2}$$

- Conclusion:

When $x^2_{1,2} \to 0$, take all $x^2_{1,2} \to \mu^2_{\mathrm{IR}}$,

and add dilogs: $\log \frac{x^2_{1i}}{x^2_{2i}} \log x^2_{12}$

$$\to \quad \log \frac{x^2_{1i}}{x^2_{2i}} \log \mu^2_{\mathrm{IR}} - 2\mathrm{Li}_2(1 - \frac{x^2_{1i}}{x^2_{2i}}) + \frac{1}{2} \log^2 \left(\frac{x^2_{1i}}{x^2_{2i}} \right)$$

Explicit examples:

$$\begin{aligned} I_{3m} &= \log \frac{\mu^2_{\mathrm{IR}} x^2_{34}}{x^2_{13} x^2_{24}} \log \frac{x^2_{41} x^2_{23}}{x^2_{13} x^2_{24}} + 2\mathrm{Li}_2(1 - \frac{x^2_{41} x^2_{23}}{x^2_{13} x^2_{24}}) \\ &\quad -2\mathrm{Li}_2(1 - \frac{x^2_{23}}{x^2_{24}}) - 2\mathrm{Li}_2(1 - \frac{x^2_{13}}{x^2_{14}}) - \frac{1}{2} \log^2 \frac{x^2_{23}}{x^2_{24}} - \frac{1}{2} \log^2 \frac{x^2_{13}}{x^2_{14}} \end{aligned}$$

$$I_{2mh} = 2\mathrm{Li}_2(1 - \frac{x^2_{24}}{x^2_{34}}) + 2\mathrm{Li}_2(1 - \frac{x^2_{24}}{x^2_{14}}) + \frac{1}{2} \log^2 \frac{\mu^2_{\mathrm{IR}} x^2_{41} x^2_{43}}{x^2_{13} x^4_{24}}$$

Summary: any 1-loop amplitude is a linear combination of:

- Boxes

$$\int_x \frac{1}{(x-x_1)^2 \cdots (x-x_4)^2} = \frac{2\mathrm{Li}_2(1-\alpha_+) - 2\mathrm{Li}_2(1-\alpha_-) + \log v \log \frac{\alpha_+}{\alpha_-}}{x_{13}^2 x_{24}^2 \sqrt{(1-u-v)^2 - 4uv}}$$

plus degenerate cases

- Triangles

$$\int_x \frac{1}{(x-x_1)^2 \cdots (x-x_3)^2} = \frac{2\mathrm{Li}_2(1-\alpha_+) - 2\mathrm{Li}_2(1-\alpha_-) + \log v \log \frac{\alpha_+}{\alpha_-}}{\sqrt{(x_{12}^2 + x_{13}^2 - x_{23}^2) - 4x_{12}^2 x_{13}^2}}$$

- Bubbles

$$\int_x \frac{1}{(x-x_1)^2 (x-x_2)^2} = \frac{1}{\epsilon} \left(\frac{x_{12}^2}{\mu_{\mathrm{UV}}^2} \right)^{-\epsilon} (+2)$$

- Rational $\quad \dfrac{\epsilon}{\epsilon_{\mathrm{UV}}} = \text{const.}$

- General structure of infrared divergences

  - We've seen some funny dilogs appear from collinear regions

$$\mathrm{Li}_2(1-\frac{x_{24}^2}{x_{14}^2})$$

  - These come from collinear regions that couples different particles together, contradicting collinear factorization; these cancel out in any gauge theory.

Exercise: in N=4 SYM, the 1-loop correction to MHV amplitudes takes the form

$$A_{\mathrm{MHV}}^{\text{1-loop}} = A_{\mathrm{MHV}}^{\mathrm{tree}} \times \sum_{i,j} I^{2me}(i-1, i, j-1, j)$$

Check that the above dilogs cancel out