Reduction to Master Integrals

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- IBP (integration by parts)

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- Solving IBP relations by hand: simple one-loop examples
- Laporta algorithm and its implementations
- Some other approaches to solve IBP relations

Evaluating a family of Feynman integrals associated with a given graph with general integer powers of the propagators (indices)

$$F_\Gamma(q_1,\ldots,q_n;d;a_1,\ldots,a_L)$$
$$= \int \ldots \int I(q_1,\ldots,q_n;k_1,\ldots,k_h;a_1,\ldots,a_L) \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots \mathrm{d}^d k_h$$

$$I(q_1,\ldots,q_n;k_1,\ldots,k_h;a_1,\ldots,a_L) = \frac{1}{(p_1^2-m_1^2)^{a_1}(p_2^2-m_2^2)^{a_2}\ldots}$$

An old straightforward analytical strategy:

to evaluate, by some methods, every scalar Feynman integral generated by the given graph.

The standard modern strategy:

to derive, without calculation, and then apply IBP identities between the given family of Feynman integrals as recurrence relations.

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The whole problem of evaluation$\rightarrow$

- constructing a reduction procedure
- evaluating master integrals

Integral calculus:

$$\int_a^b uv'\mathsf{d}x = uv|_a^b - \int_a^b u'v\mathsf{d}x$$

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$$\int_a^b u v' \mathrm{d}x = uv|_a^b - \int_a^b u' v \mathrm{d}x$$

Feynman integral calculus:
Use IBP and neglect surface terms

[Chetyrkin & Tkachov'81]

$$\int \ldots \int \left[\left(q_i \cdot \frac{\partial}{\partial k_j} \right) \frac{1}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots} \right] \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots = 0 \, ;$$

$$\int \ldots \int \left[\frac{\partial}{\partial k_j} \cdot k_i \frac{1}{(p_1^2 - m_1^2)^{a_1} (p_2^2 - m_2^2)^{a_2} \ldots} \right] \mathrm{d}^d k_1 \mathrm{d}^d k_2 \ldots = 0 \, .$$

An example

$$F(a) = \int \frac{\mathrm{d}^d k}{(k^2 - m^2)^a}$$

$F(a)$ for integer $a \leq 0$. We need $F(a)$ for positive integer a.

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Taking derivatives:

$$\frac{\partial}{\partial k} \cdot k = \frac{\partial}{\partial k_\mu} \cdot k_\mu = d$$

$$
\begin{aligned}
k \cdot \frac{\partial}{\partial k} \frac{1}{(k^2 - m^2)^a} &= -a \frac{2k^2}{(k^2 - m^2)^{a+1}} \\
&= -2a \left[\frac{1}{(k^2 - m^2)^a} + \frac{m^2}{(k^2 - m^2)^{a+1}} \right]
\end{aligned}
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IBP relation

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(d - 2a)F(a) - 2am^2 F(a+1) = 0
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IBP relation

$$(d - 2a)F(a) - 2am^2 F(a+1) = 0$$

Its solution

$$F(a) = \frac{d - 2a + 2}{2(a-1)m^2} F(a-1)$$

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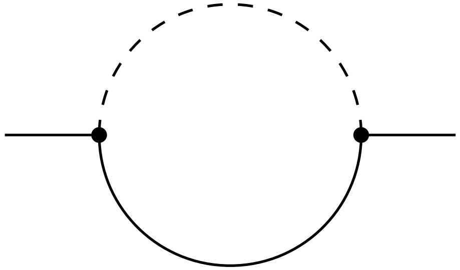
Explicitly,

$$F(a) = \frac{(-1)^a \, (1 - d/2)_{a-1}}{(a-1)!(m^2)^{a-1}} I_1 \ ,$$

where $(x)_a$ is the Pochhammer symbol

One more example



$$F_\Gamma(a_1,a_2) = \int \frac{\mathrm{d}^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

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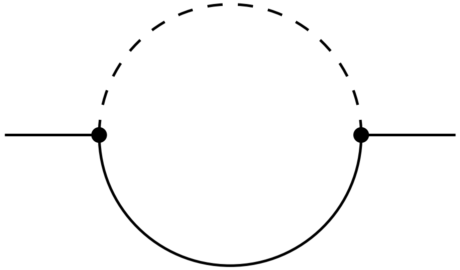

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Apply IBP

$$\int \frac{\partial}{\partial k}\cdot k \left(\frac{1}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}\right)\mathrm{d}^d k \;=\; 0\,,$$

$$\int q\cdot \frac{\partial}{\partial k} \left(\frac{1}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}\right)\mathrm{d}^d k \;=\; 0\,,$$

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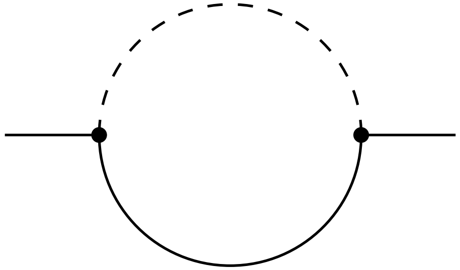

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$$\int q\cdot\frac{\partial}{\partial k} \left(\frac{1}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}} \right) \mathrm{d}^d k \;=\; 0\,,$$

use $2k\cdot(k-q) \to (k-q)^2 + (k^2-m^2) - q^2 + m^2$ to obtain

$$d - 2a_1 - a_2 - 2m^2 a_1 \mathbf{1}^+ - a_2 \mathbf{2}^+(\mathbf{1}^- - q^2 + m^2) = 0 \quad (A)$$

$$a_2 - a_1 - a_1 \mathbf{1}^+(q^2 + m^2 - \mathbf{2}^-) - a_2 \mathbf{2}^+(\mathbf{1}^- - q^2 + m^2) = 0 \quad (B)$$

where, e.g., $\mathbf{1}^+\mathbf{2}^- F(a_1, a_2) = F(a_1 + 1, a_2 - 1)$.

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$F_\Gamma(a_1, a_2) = 0$ for $a_1 \leq 0$

A manual solution.

1. Apply $(q^2+m^2)(A) - 2m^2(B)$,

$$(q^2-m^2)^2 a_2 \mathbf{2}^+ = (q^2-m^2) a_2 \mathbf{1}^- \mathbf{2}^+ \\ -(d-2a_1-a_2)q^2 - (d-3a_2)m^2 + 2m^2 a_1 \mathbf{1}^+ \mathbf{2}^-$$

to reduce a_2 to 1 or 0.

```
F[a1_, a2_ /; a2 > 1] :=
1/(a2 - 1)/(qq - mm)^2 (
(a2 - 1) (qq - mm) F[a1 - 1, a2]
- ((d - 2 a1 - a2 + 1) qq
+ (d - 3 a2 + 3) mm) F[a1, a2 - 1]
+ 2 mm a1 F[a1 + 1, a2 - 2]);
```

2. Suppose that $a_2 = 1$. Apply $(A) - (B)$, i.e.

$$(q^2 - m^2) a_1 \mathbf{1}^+ = a_1 + 2 - d + a_1 \mathbf{1}^+ \mathbf{2}^-$$

to reduce a_1 to 1 or a_2 to 0.

```
F[a1_ /; a1 > 1, 1] :=
1/(a1 - 1)/(qq - mm) ((a1 - 1) F[a1, 0]
-(d - a1 - 1)  F[a1 - 1, 1]);
```

Therefore, any $F(a_1, a_2)$ can be reduced to $I_1 = F(1,1)$ and integrals with $a_2 \leq 0$ (which can be evaluated in terms of gamma functions for general d).

3. Let $a_2 \leq 0$. Apply (A) to reduce a_1 to one.

```
F[a1_ /; a1 > 1, a2_ /; a2 <= 0] :=
1/(a1 - 1)/2/mm ((d - 2 a1 - a2 + 2) F[a1 - 1, a2]
-a2 F[a1 - 2, a2 + 1] + a2 (qq - mm) F[a1 - 1, a2 + 1]);
```

4. Let $a_1 = 1$. Apply the following corollary of (A) and (B)

$$(d - a_2 - 1)\mathbf{2}^- = (q^2 - m^2)^2 a_2 \mathbf{2}^+ + (q^2 + m^2)(d - 2a_2 - 1)$$

to increase a_2 to zero or one starting from negative values.

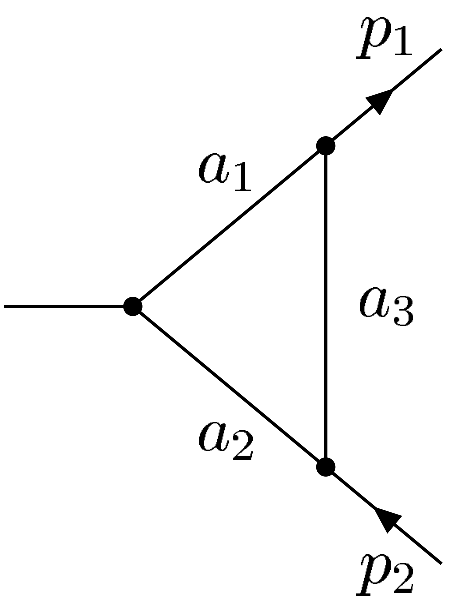
```
F[1, a2_ /; a2 < 0] := 1/(d - a2 - 2) (
(a2 + 1) (qq - mm)^2 F[1, a2 + 2] +
(qq + mm) (d - 2 a2 - 3) F[1, a2 + 1] );
```

Any $F(a_1, a_2)$ is a linear combination of the two master integrals $I_1 = F(1,1)$ and $I_2 = F(1,0)$.

For example,

```
F[3, 2]=
(-(((-5 + d) (-3 + d) (-4 mm + d mm - 8 qq + d qq))/(
2 (mm - qq)^4)) I1
+ ((-2 + d) (96 mm^2 - 39 d mm^2 + 4 d^2 mm^2
+ 28 mm qq - 6 d mm qq - 4 qq^2 + d qq^2))/
(8 mm^2 (mm - qq)^4) I2)
```

Triangle rule



$m_3 = 0$

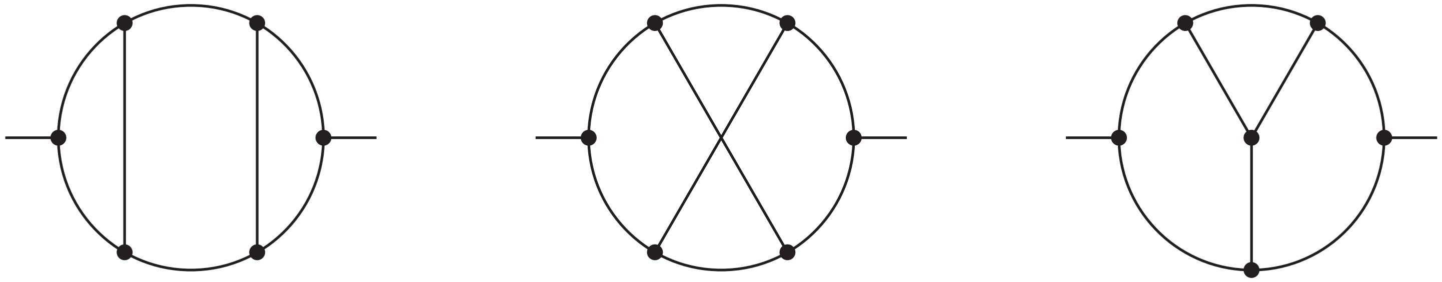
$$F(a_1, a_2, a_3) = \int \frac{\mathrm{d}^d k}{[(k+p_1)^2 - m_1^2]^{a_1}[(k+p_2)^2 - m_2^2]^{a_2}(k^2)^{a_3}}$$

The IBP identity with the operator $(\partial/\partial k)\cdot k \rightarrow$

$$
\begin{aligned}
1 \;\; = \;\; & \frac{1}{d-a_1-a_2-2a_3} \\
& \times \left[a_1 \mathbf{1}^+ \left(\mathbf{3}^- - (p_1^2 - m_1^2) \right) + a_2 \mathbf{2}^+ \left(\mathbf{3}^- - (p_2^2 - m_2^2) \right) \right]
\end{aligned}
$$

A manual solution of IBP relations for massless three-loop propagator diagrams

[K.G. Chetyrkin & F.V. Tkachov'81]



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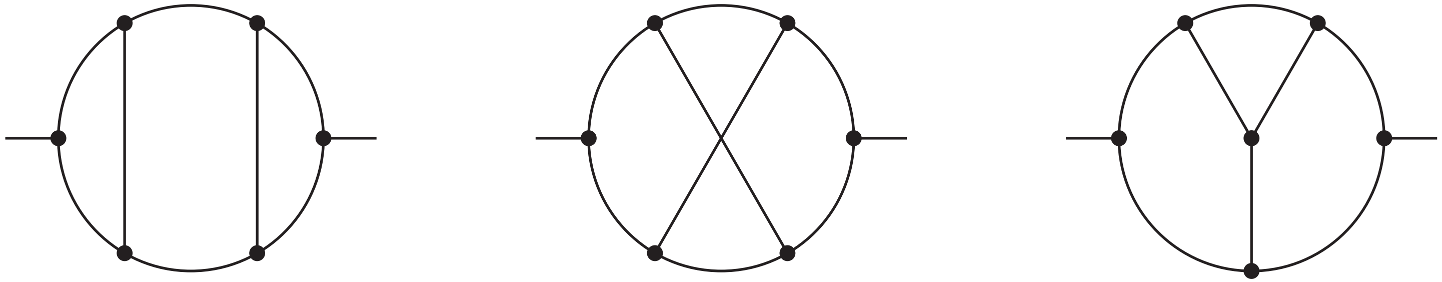

MINCER:

[S.G. Gorishny, S.A. Larin, L.R. Surguladze & F.V. Tkachov'89]

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(implemented in FORM)

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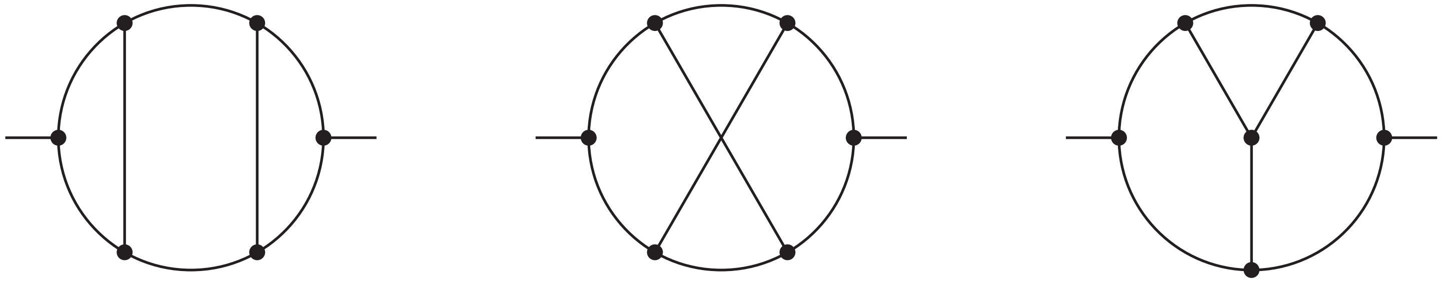

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Solving IBP relations algorithmically:

- Laporta's algorithm

  [Laporta & Remiddi'96; Laporta'00; Gehrmann & Remiddi'01]

  Use IBP relations written at points $(a_1, \ldots, a_L)$ with $\sum |a_i| \leq N$ and solve them for the Feynman integrals involved.

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  When N increases, the situation stabilizes, in the sense that the number of the master integrals becomes stable starting from sufficiently large N.

Experience tells us that the number of master integrals is always finite.

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Theorem [A. Smirnov & A. Petukhov'10]

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The number of master integrals is finite

(for Feynman integrals with usual propagators)

The same example

$$F_\Gamma(a_1,a_2) = \int \frac{\mathrm{d}^d k}{(m^2-k^2)^{a_1}(-(q-k)^2)^{a_2}}$$

The left-hand sides of the two primary IBP relations:

```
ibp1[a1_, a2_] := (d - 2 a1 - a2) F[a1, a2]
- 2 mm a1 F[a1 + 1, a2] - a2 (F[a1 - 1, a2 + 1]
+ (mm - qq)  F[a1, a2 + 1]);
ibp2[a1_, a2_] := (a2 - a1) F[a1, a2] -
a1 ((qq + mm) F[a1 + 1, a2] - F[a1 + 1, a2 - 1]) -
a2 (F[a1 - 1, a2 + 1] + (mm - qq)  F[a1, a2 + 1]);
```

Let us consider the sector $a_1 > 0, a_2 \leq 0$
Use IBP at various (a_1, a_2) with $a_1 + |a_2| \leq N$
Solve the corresponding linear system of equation with respect to $F(a_1, a_2)$ involved.
Increase N.

$N = 1$

```
Solve[{ibp1[1, 0] == 0, ibp2[1, 0] == 0},
{F[2, 0], F[2, -1]}]


{F[2, -1] -> ((-2 qq + d (mm + qq)) F[1, 0])/(2 mm),
 F[2, 0] -> ((-2 + d) F[1, 0])/(2 mm)}
```

$N = 2$

```
Solve[{ibp1[1, 0] == 0, ibp2[1, 0] == 0,
ibp1[2, 0] == 0, ibp2[2, 0] == 0,
ibp1[1, -1] == 0,  ibp2[1, -1] == 0 },
{F[2, 0], F[3, 0], F[1, -1],
F[2, -1], F[3, -1],  F[2, -2]}]

{F[2, -2] -> (((2 + d) mm^2 + 2 (2 + d) mm qq
+ (-2 + d) qq^2) F[1,0])/(2 mm),
F[3, -1] -> ((-2 + d) (-4 qq + d (mm + qq))
F[1, 0])/(8 mm^2),
F[3, 0] -> ((-4 + d) (-2 + d) F[1, 0])/(8 mm^2),
F[1, -1] -> (mm + qq) F[1, 0],
F[2, -1] -> ((-2 qq + d (mm + qq)) F[1, 0])/(2 mm),
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Private versions

[Gehrmann & Remiddi, Laporta, Czakon, Schröder, Pak, Sturm, Marquard & Seidel, Velizhanin, . . .]

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- Baikov's method [P.A. Baikov'96-. . .]

  [V.A. Smirnov & M. Steinhauser'03]

  An Ansatz for coefficient functions at master integrals

$$\int \ldots \int \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_N}{x_1^{a_1} \ldots x_N^{a_N}} \left[P(\underline{x'})\right]^{(d-h-1)/2} ,$$

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- Lee's approach (based on Lie algebras) [R.N. Lee'08]

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- R. Lee's method [R. Lee'09]
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