Momentum twistors, special functions and symbols

Lecture 2

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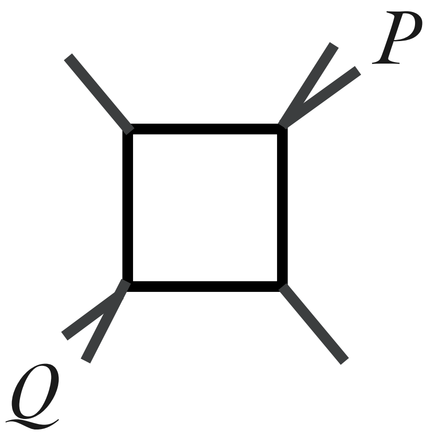
- 1st lecture: Kinematics
  - ➡ What are the arguments of the special functions?

- Today's lecture:
  - ➡ What are the kind of functions that can appear in loop computations?
  - ➡ Properties of some of these functions.
  - ➡ General theorems.
  - ➡ Numerical evaluation of some of these functions.

The two-mass easy box function



$$= \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2(k-p_4)^2}$$

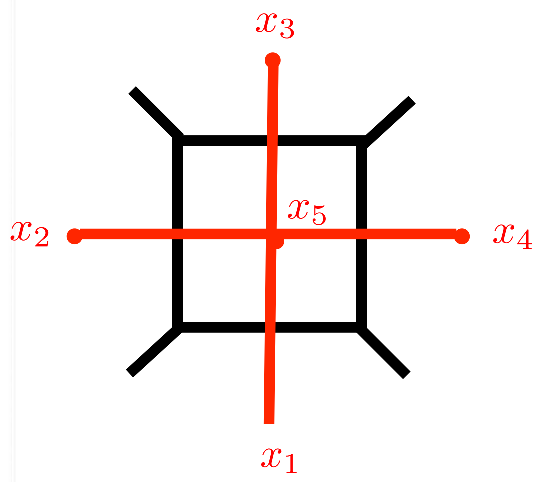
- Computation can be done via various methods.
- Analytic result (textbook): [See Smirnov's lecture]

$$\frac{1}{\epsilon}\ln\left(\frac{st}{P^2Q^2}\right) + \frac{1}{2}\left[\ln^2\left(\frac{-P^2}{\mu^2}\right) + \ln^2\left(\frac{-Q^2}{\mu^2}\right) - \ln^2\left(\frac{-s}{\mu^2}\right) - \ln^2\left(\frac{-t}{\mu^2}\right)\right]$$
$$+ \mathrm{Li}_2\left(1-\frac{P^2}{s}\right) + \mathrm{Li}_2\left(1-\frac{Q^2}{s}\right) + \mathrm{Li}_2\left(1-\frac{P^2}{t}\right) + \mathrm{Li}_2\left(1-\frac{Q^2}{t}\right)$$
$$- \mathrm{Li}_2\left(1-\frac{P^2Q^2}{st}\right) + \frac{1}{2}\ln^2\left(\frac{s}{t}\right) + \mathcal{O}(\epsilon)$$

- Not an elementary function. Needs the dilogarithm:

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{d}t}{t}\ln(1-t) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}$$

The four-mass box function



$$= \mathrm{Li}_2(1-\alpha^+) - \mathrm{Li}_2(1-\alpha^-) + 1/2 \ln v \ \ln \frac{\alpha^+}{\alpha^-}$$

$$\alpha_\pm \equiv \frac{2u}{1+u-v \pm \sqrt{(1-u-v)^2 - 4uv}}$$

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

- Again a dilogarithm, but this time with algebraic rather than rational argument (square root!)

One-loop Hexagon in 6 dimensions

$$\frac{1}{\sqrt{\Delta}}\left[-2\sum_{i=1}^{3}L_3(x_i^+,x_i^-)+\frac{1}{3}\left(\sum_{i=1}^{3}\ell_1(x_i^+)-\ell_1(x_i^-)\right)^3+\frac{\pi^2}{3}\chi\sum_{i=1}^{3}\left(\ell_1(x_i^+)-\ell_1(x_i^-)\right)\right],$$

$$x_i^\pm = u_i x^\pm, \quad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3},$$

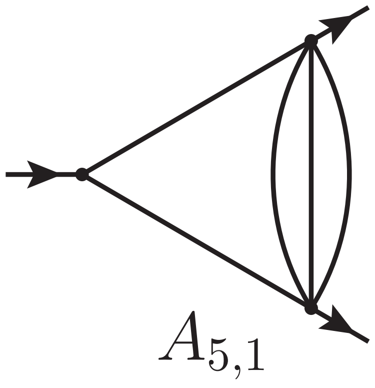
$$L_3(x^+,x^-) = \sum_{k=0}^{2}\frac{(-1)^k}{(2k)!!}\ln^k(x^+x^-)\left(\ell_{3-k}(x^+)-\ell_{3-k}(x^-)\right),$$

$$\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x)-(-1)^n\mathrm{Li}_n(1/x)\right),$$

- Dilogarithm no longer enough. Need trilogarithm!

$$\mathrm{Li}_3(z) = \int_0^z \frac{\mathrm{d}t}{t}\mathrm{Li}_2(t) = \sum_{n=1}^{\infty}\frac{z^n}{n^3} \qquad \mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{d}t}{t}\ln(1-t) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}$$

Three-loop form factor [Lee, Smirnov, Smirnov]



$$
= \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(2-3\epsilon)(1-2\epsilon)^2}\left\{-\frac{1}{12\epsilon^2}-\frac{\pi^2}{16}+\frac{23\epsilon\zeta_3}{12}-\frac{7\pi^4\epsilon^2}{1152}\right.
+\epsilon^3\left(\frac{23\pi^2\zeta_3}{16}+\frac{351\zeta_5}{20}\right)+\epsilon^4\left(\frac{65243\pi^6}{1451520}-\frac{529\zeta_3^2}{24}\right)+\epsilon^5\left(\frac{161\pi^4\zeta_3}{1152}\right.
\left.+\frac{1053\pi^2\zeta_5}{80}+\frac{5503\zeta_7}{28}\right)+\epsilon^6\left(-\frac{529}{32}\pi^2\zeta_3^2-\frac{8073\zeta_5\zeta_3}{20}+\frac{75527\pi^8}{860160}\right)
\left.+O\left(\epsilon^7\right)\right\}, \qquad (6)
$$

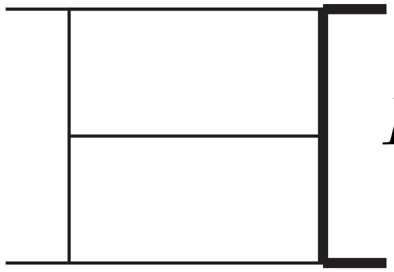
- No dilogarithms or trilogarithms, only zeta values (up to an overall scale):

$$\zeta_m = \sum_{n=1}^{\infty}\frac{1}{n^m}$$

➡ Link to dilogarithms and trilogarithms?

Massive double box

[Bonciani, Ferroglia,
Gehrmann, Studerus]



$$P_1 = \frac{1}{m^4} \sum_{i=-4}^{-1} A_i \varepsilon^i + \mathcal{O}(\varepsilon^0)\,,$$

$$
\begin{aligned}
A_{-4} &= \frac{1}{24(1+y)^2}\,,\\
A_{-3} &= \frac{1}{96(1+y)^2}\Big[-10G(-1;y) + 3G(0;x) - 6G(1;x)\Big]\,,\\
A_{-2} &= \frac{1}{192(1+y)^2}\Big[-47\zeta(2) - 24G(-1;y)G(0;x) + 48G(-1;y)G(1;x) + 32G(-1,-1;y)\\
&\quad -6G(0,-1;y)\Big]\,,
\end{aligned}
$$

- ???

Summary

- Loop integrals are in general not elementary functions (they are so-called transcendental functions, see next lecture)
- Functions we obtained form the previous examples:
  - ➡ Logarithms
  - ➡ Dilogarithms $\mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{d}t}{t} \ln(1-t) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$
  - ➡ Trilogarithms $\mathrm{Li}_3(z) = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_2(t) = \sum_{n=1}^{\infty} \frac{z^n}{n^3}$
  - ➡ Zeta Values $\zeta_m = \sum_{n=1}^{\infty} \frac{1}{n^m}$
  - ➡ Even other functions...
- In all cases: arguments are rational or algebraic.

Aim

- Can we classify the kind of functions that can appear?
- What are the properties of these functions?
- Is there some a priori knowledge about which functions / numbers can appear in a given Feynman integral, and which cannot?
- How can we evaluate these functions numerically?

Special functions

Polylogarithms

The dilogarithm

- Definition:

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\mathrm{d}t}{t} \ln(1-t) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$$

- The series is convergent for $|z| \leq 1$.
- The integral representation however allows to define the function outside the unit disc, but it then develops an imaginary part:

$$\mathrm{Li}_2(x) \; = -\mathrm{Li}_2(1/x) - \frac{1}{2}\ln^2(-x) - \frac{\pi^2}{6}$$

- The dilogarithm satisfies many other identities, e.g.,

$$\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) - \ln z \ln(1-z) + \frac{\pi^2}{6}$$

- How to obtain such identities will be the subject of lecture 4 & 5.

Classical Polylogarithms

- Definition:

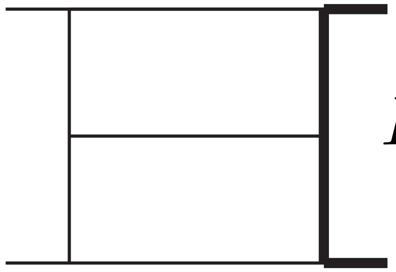
$$\mathrm{Li}_m(z) = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_{m-1}(t) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}$$

$$\mathrm{Li}_1(z) = -\ln(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

- m is called the weight.
- The series is convergent for $|z| \leq 1$.
- The integral representation however allows to define the function outside the unit disc, but it then develops an imaginary part.
- The trilogarithm also satisfies many other identities.
- These are all functions of only one scale... what if we have multiple scales?

Massive double box

[Bonciani, Ferroglia, Gehrmann, Studerus]



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&\quad -6G(0,-1;y)\Big]\,,
\end{aligned}$$

- Need to generalize the previous functions to more than one variable!

Multiple Polylogarithms

[Goncharov]

- Classical polylogarithm:

$$\mathrm{Li}_m(z) = \int_0^z \frac{\mathrm{d}t}{t}\mathrm{Li}_{m-1}(t) \qquad \mathrm{Li}_1(z) = -\ln(1-z)$$

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- Mutliple polylogarithms

$$G(a_1, \ldots, a_m; z) = \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_m; t)$$

$$G(a; z) = \ln\left(1 - \frac{z}{a}\right) \qquad G(\vec{0}_m; z) = \frac{1}{m!} \ln^m z$$

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- Multiple polylogarithms are a multivariable extension of classical ones, which they contain as special cases:

$$G(\vec{0}_{n-1}, a; z) = -\mathrm{Li}_n\left(\frac{z}{a}\right)$$

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[Goncharov]

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[Prove it!]

  - ➡ Hölder convolution:

$$G(w_1, \ldots, w_n; 1) = \sum_{k=0}^{n} (-1)^k \, G\left(1 - w_k, \ldots, 1 - w_1; 1 - \frac{1}{p}\right) \, G\left(w_{k+1}, \ldots, w_n; \frac{1}{p}\right)$$

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  - ➡ Reduces to classical polylogarithms in special cases, e.g.,

$$G(a,b;z) = \mathrm{Li}_2\left(\frac{b-z}{b-a}\right) - \mathrm{Li}_2\left(\frac{b}{b-a}\right) + \log\left(1-\frac{z}{b}\right)\log\left(\frac{z-a}{b-a}\right)$$

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  - ➡ etc.
- Many properties, and we need to be able to deal with them...
  - ➡ Look at math/0103059.

The shuffle algebra

- Let's multiply two mutliple polylogarithms of weight 1:

$$G(a;z)\, G(b;z) = ?$$

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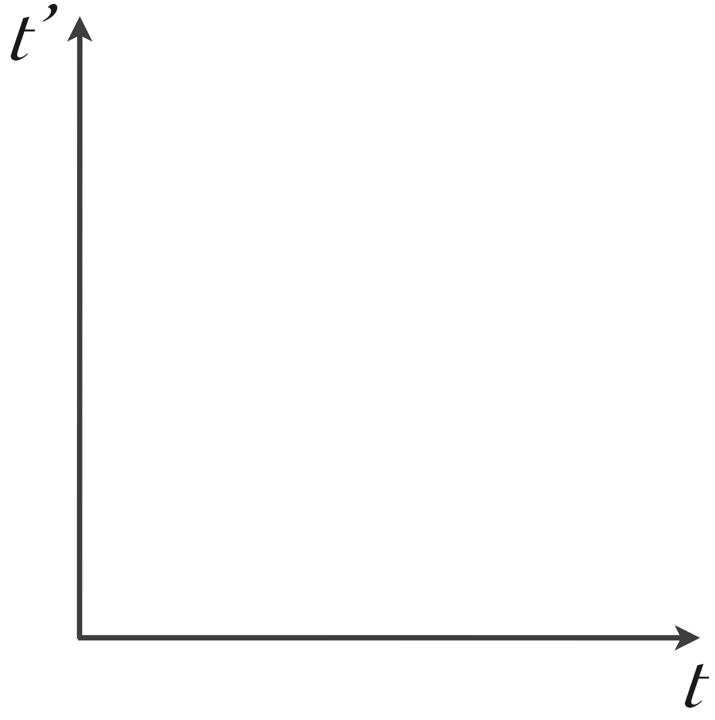

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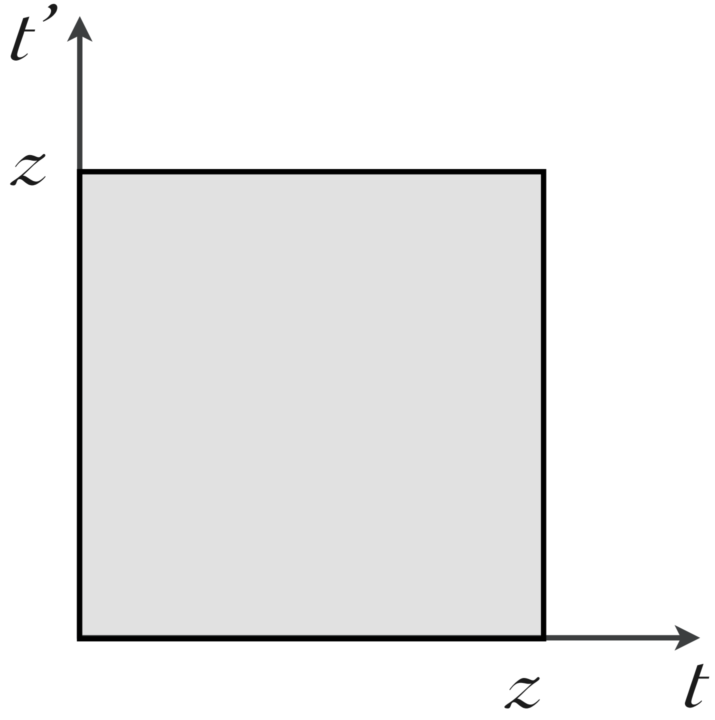

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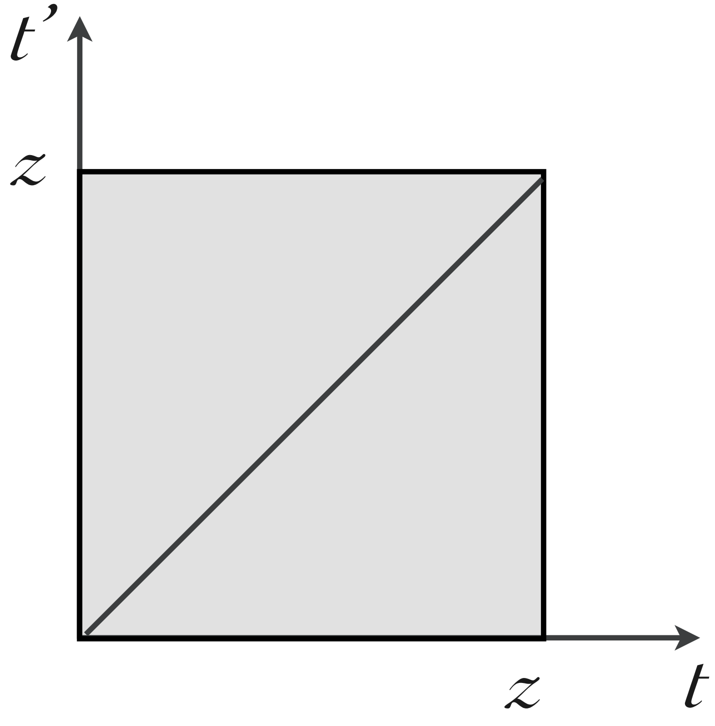

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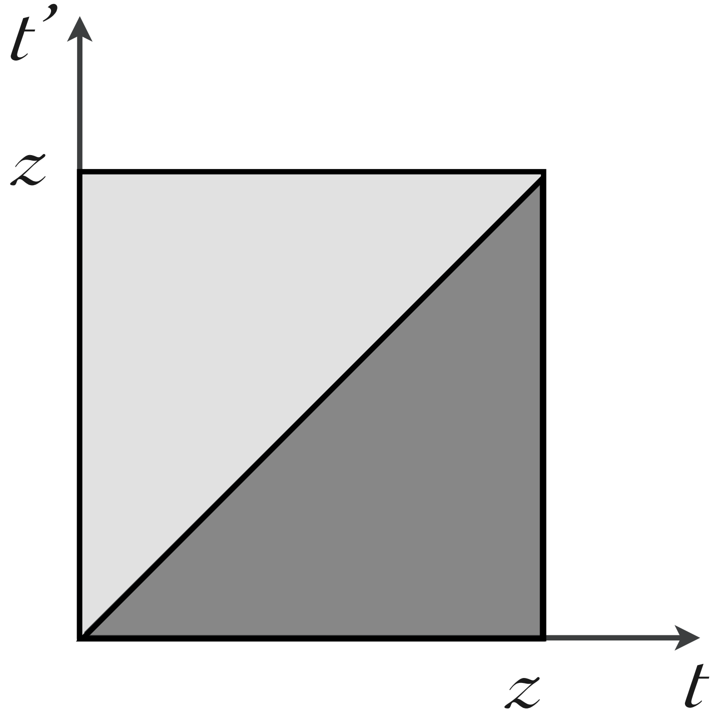

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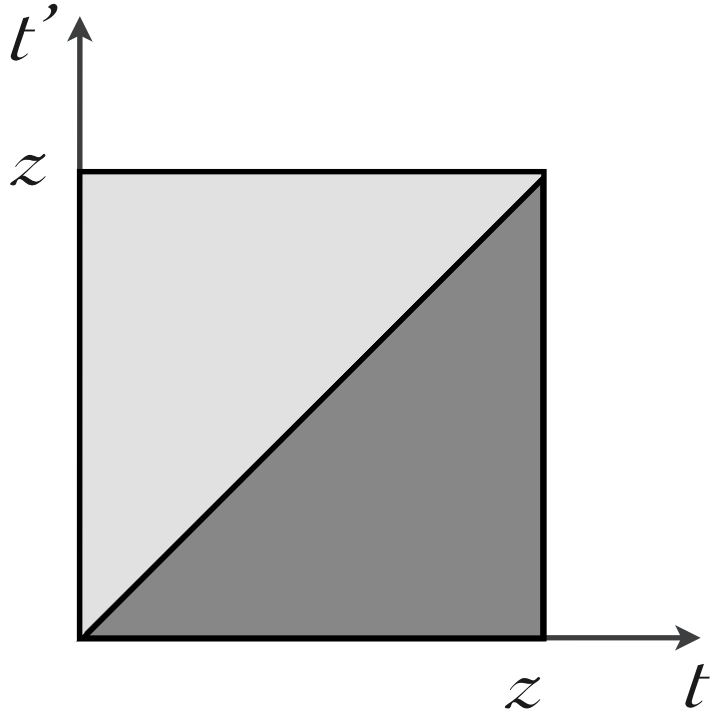

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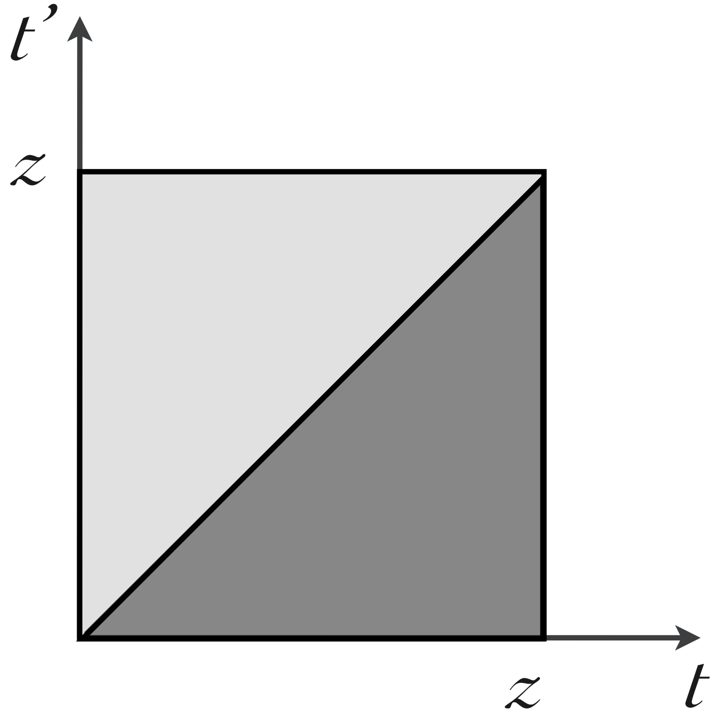

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$$= G(a,b;z) + G(b,a;z)$$

The shuffle algebra

- This is not a coincidence!
- Multiple polylogarithms form a so-called shuffle algebra:

$$G(a_1,\ldots,a_{n_1};x)\,G(a_{n_1+1},\ldots,a_{n_1+n_2};x) \;=\; \sum_{\sigma\in\Sigma(n_1,n_2)} G(a_{\sigma(1)},\ldots,a_{\sigma(n_1+n_2)};x)$$

$$\Sigma(n_1,n_2)=\{\sigma\in S_{n_1+n_2}|\,\sigma^{-1}(1)<\ldots<\sigma^{-1}(n_1)\;\;\text{and}\;\;\sigma^{-1}(n_1+1)<\ldots<\sigma^{-1}(n_1+n_2)\}$$

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The shuffle algebra

- This is not a coincidence!
- Multiple polylogarithms form a so-called shuffle algebra:

$$G(a_1,\ldots,a_{n_1};x)\,G(a_{n_1+1},\ldots,a_{n_1+n_2};x) \;=\; \sum_{\sigma\in\Sigma(n_1,n_2)} G(a_{\sigma(1)},\ldots,a_{\sigma(n_1+n_2)};x)$$

$$\Sigma(n_1,n_2)=\{\sigma\in S_{n_1+n_2}|\,\sigma^{-1}(1)<\ldots<\sigma^{-1}(n_1)\;\text{ and }\;\sigma^{-1}(n_1+1)<\ldots<\sigma^{-1}(n_1+n_2)\}$$

- Shuffles are best understood via examples:

G(a;z) G(b;z) = G(a,b;z) + G(b,a;z)

G(a;z) G(b,c;z) = G(a,b,c;z) + G(b,a,c;z) + G(b,c,a;z)

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- N.B.: Shuffles preserve the weight!

Harmonic polylogarithms

- Some special classes were (re)discovered independently by physicists, and go under the name *harmonic* polylogarithms.

- They are multiple polylogarithms with $a_i \in \{0, \pm 1\}$, but with a different sign convention:

$$H(\vec{w};x) = (-1)^p \, G(\vec{w};x)$$

where p is the number of indices equal to (+1).

- There are other special classes in two variables (re)discovered by physicists, called *two-dimensional* harmonic polylogarithms [Gehrmann, Remiddi].

$$\begin{aligned} A_{-4} &= \frac{1}{24(1+y)^2}\,, \\ A_{-3} &= \frac{1}{96(1+y)^2}\Big[-10G(-1;y)+3G(0;x)-6G(1;x)\Big]\,, \\ A_{-2} &= \frac{1}{192(1+y)^2}\Big[-47\zeta(2)-24G(-1;y)G(0;x)+48G(-1;y)G(1;x)+32G(-1,-1;y) \\ &\quad -6G(0,-1;y)\Big]\,, \end{aligned}$$

Series representation

- So far we have only looked at the integral representation.
- What about series representations?

$$\mathrm{Li}_m(z) = \int_0^z \frac{\mathrm{d}t}{t} \mathrm{Li}_{m-1}(t) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}$$

- The series representation is nice, because it is closer to Mellin-Barnes methods.
- The multi-dimensional generalization of the series representation is

$$\mathrm{Li}_{m_k,\ldots,m_1}(x_k,\ldots,x_1) = \sum_{n_1=1}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \sum_{n_2=1}^{n_1-1} \ldots \sum_{n_k=1}^{n_{k-1}-1} \frac{x_k^{n_k}}{n_k^{m_k}}$$

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- They also contain the iterated integrals we defined before:

$$Li_{m_k,\dots,m_1}(x_k,\dots,x_1)=(-1)^k\,G\left(\underbrace{0,\dots,0}_{m_1-1},\frac{1}{x_1},\dots,\underbrace{0,\dots,0}_{m_k-1},\frac{1}{x_1\dots x_k};1\right)$$

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- Again, these functions satisfy various relations, e.g.,

$$Li_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = d^{m_1+\ldots+m_k-k} \sum_{y_j^d=x_j,\, 1\leq j\leq k} Li_{m_1,\ldots,m_k}(y_1,\ldots,y_k)$$

The Stuffle algebra

- Let us now multiply two of these functions

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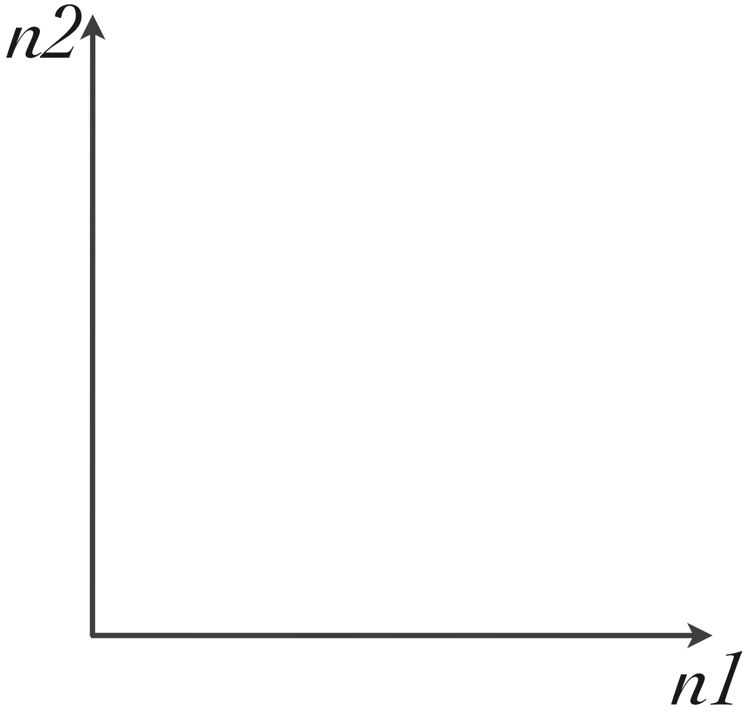

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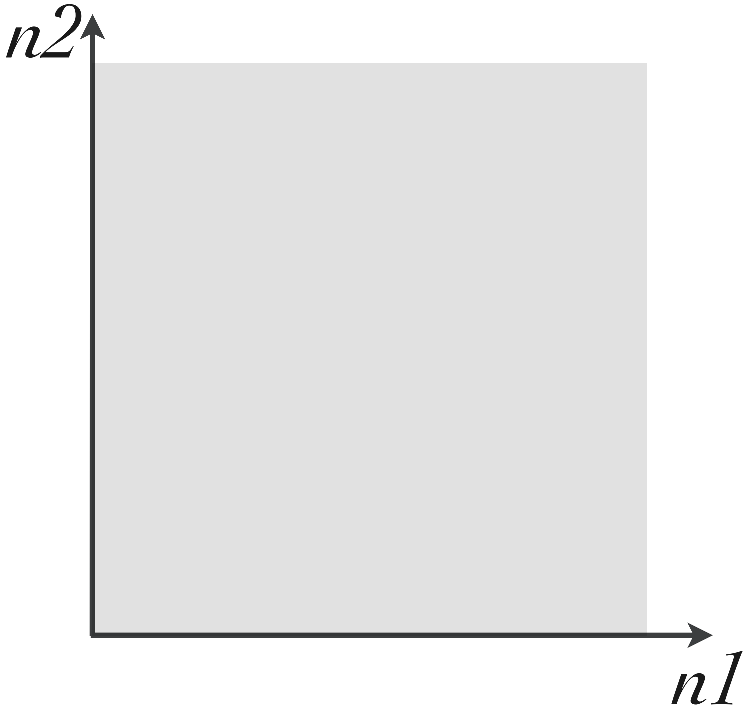

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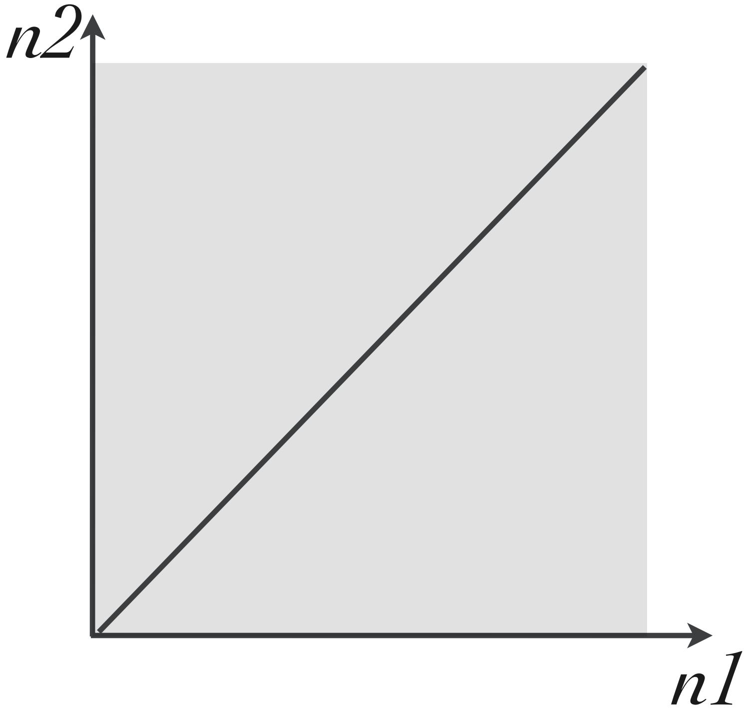

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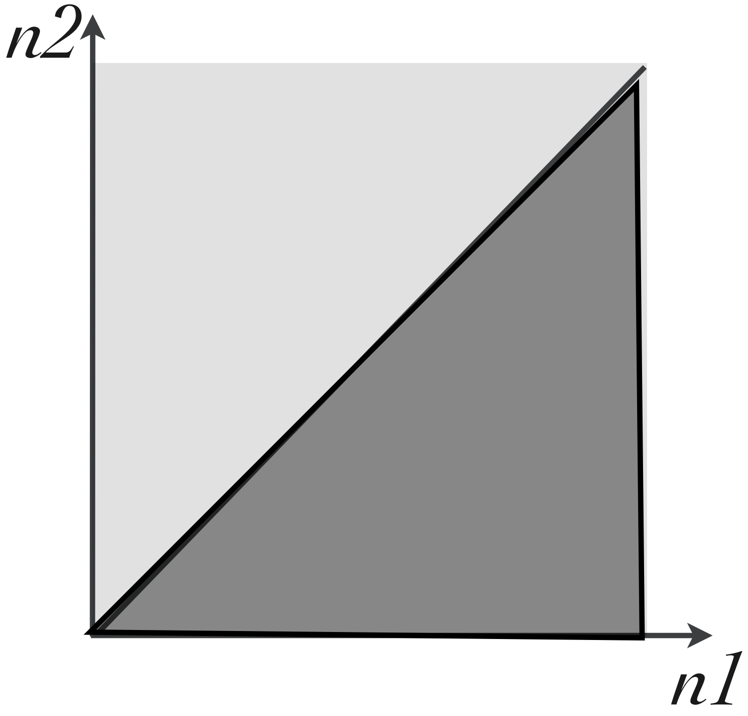

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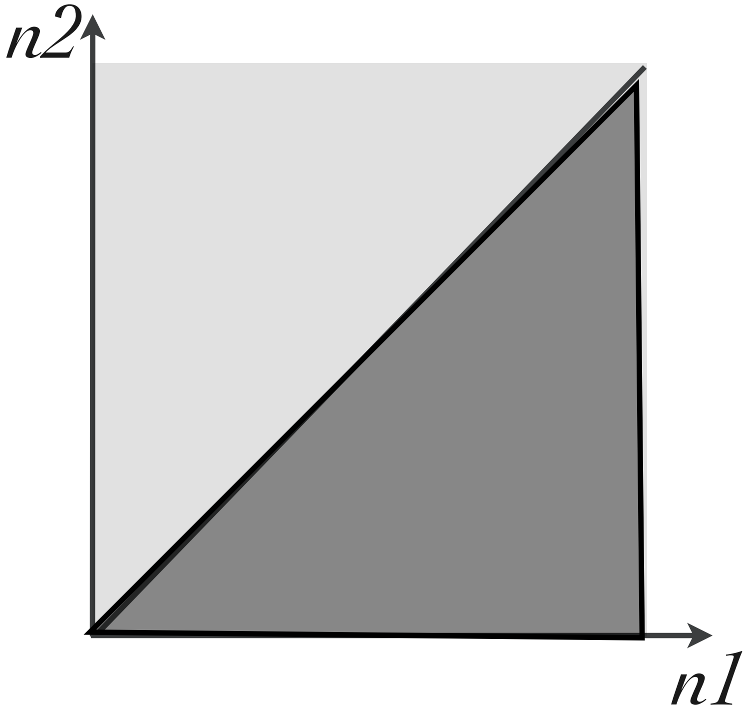

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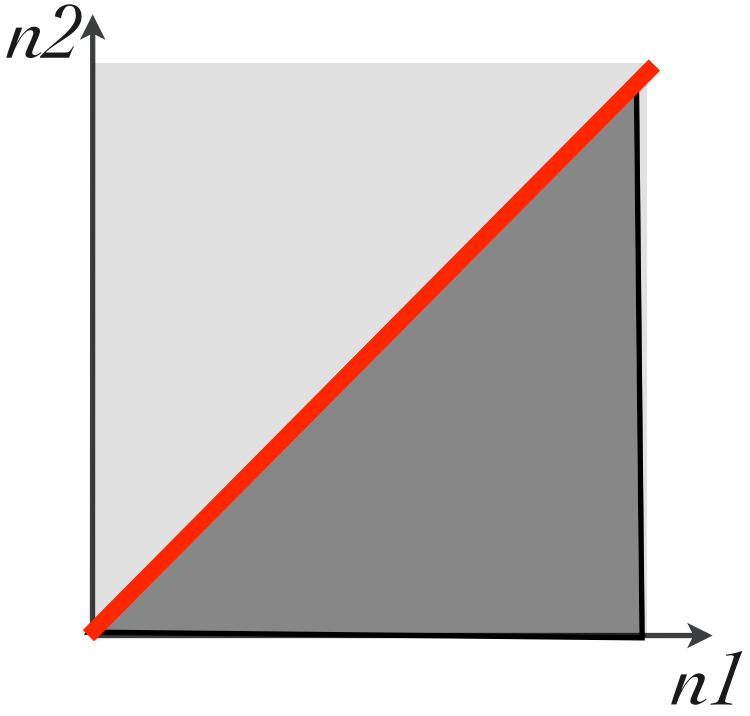

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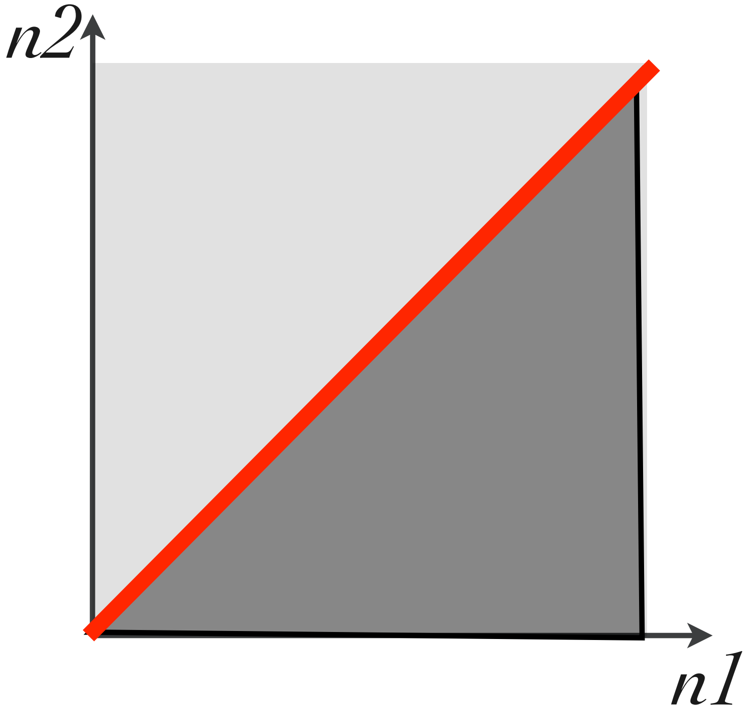

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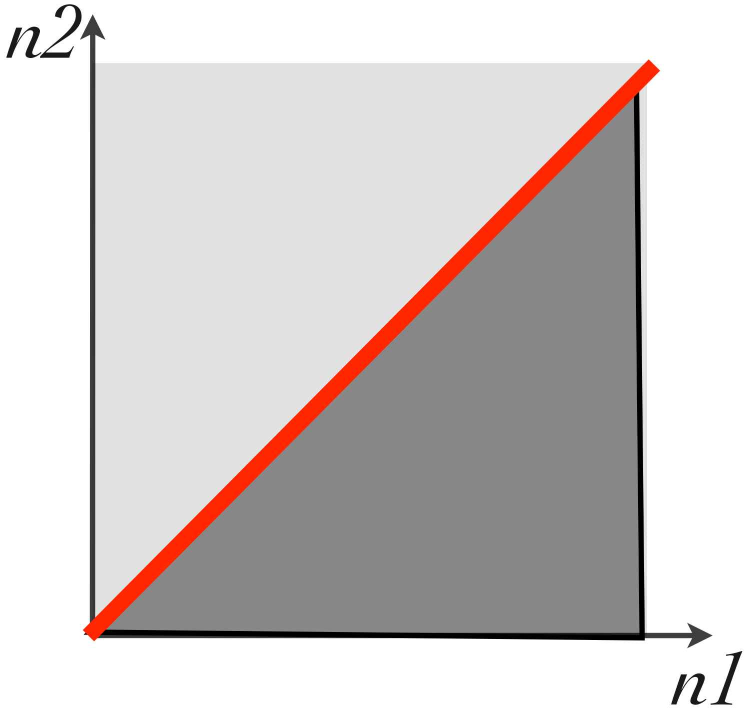

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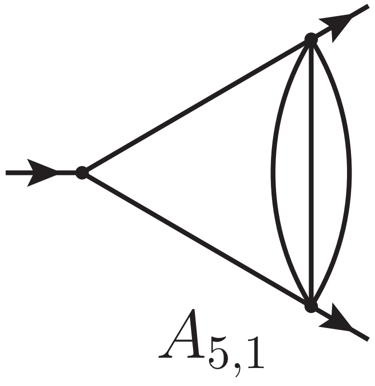
    - ➡ ALL these relations preserve the weight!

Special functions

Zeta Values

Three-loop form factor [Lee, Smirnov, Smirnov]



$$= \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(2-3\epsilon)(1-2\epsilon)^2}\left\{-\frac{1}{12\epsilon^2}-\frac{\pi^2}{16}+\frac{23\epsilon\zeta_3}{12}-\frac{7\pi^4\epsilon^2}{1152}\right.$$
$$+\epsilon^3\left(\frac{23\pi^2\zeta_3}{16}+\frac{351\zeta_5}{20}\right)+\epsilon^4\left(\frac{65243\pi^6}{1451520}-\frac{529\zeta_3^2}{24}\right)+\epsilon^5\left(\frac{161\pi^4\zeta_3}{1152}\right.$$
$$\left.+\frac{1053\pi^2\zeta_5}{80}+\frac{5503\zeta_7}{28}\right)+\epsilon^6\left(-\frac{529}{32}\pi^2\zeta_3^2-\frac{8073\zeta_5\zeta_3}{20}+\frac{75527\pi^8}{860160}\right)$$
$$\left.+O\left(\epsilon^7\right)\right\}, \qquad (6)$$

- No dilogarithms or trilogarithms, only zeta values (up to an overall scale):

$$\zeta_m = \sum_{n=1}^{\infty}\frac{1}{n^m}$$

Zeta values

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- As zeta values are closely related to polylogarithms, can we generalize..?

Multiple zeta values

[Zagier]

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- Conjecture:

  All the relations among MZV's are generated by the shuffle and stuffle relations.

Caveat!!!

- We have analyzed polylogarithms and MZV's as functions/ numbers that appear in loop integrals.
- This does NOT mean that this is ALWAYS the case!
- In general for example a Mellin-Barnes integral will give rise to sums that are not easily doable, and where it is not clear whether it will be multiple polylogarithms.
- More general theorems about which functions/numbers can appear in the next lectures.
- There are however theories in which it is expected that only multiple polylogarithms and MZV's appear (e.g., N=4 SYM).

Special functions

Transcentality and periods

Example:

- Can the following be Feynman integrals?

$$A = 2365$$

$$B = \pi^2 + \zeta_3$$

$$C = e^3\pi^2 - \ln 2$$

$$D = \ln^4 2 + \mathrm{Li}_4\left(\frac{1}{2}\right)$$

$$E = \frac{\ln^2 2}{\pi^2}$$

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- What drives this..?

Example:

- Why is this knowledge useful?
  - ➡ For checking your computations!
  - ➡ Make educated guesses for loop integrals.
  - ➡ This can for example be useful when using the PSLQ algorithm.

Transcendentality

- **Definition:** A complex number is said to be algebraic iff it is the root of a polynomial with rational coefficients. Otherwise the number is called transcendental.
- **Examples:**

$2/3$

$2 + i\sqrt[3]{5}$

$\ln 2$

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$\sqrt{2}$ $\quad \zeta_3 ?$

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- Algebraic numbers form a field, i.e., we can add, multiply, invert, etc.

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Feynman integrals are not algebraic!

➡ Can they be generic transcendental numbers?

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- Periods do not form a field, but only a ring (i.e., the inverse of a period is not necessarily a period).

Periods

- **Theorem** [Bogner, Weinzierl]: If all kinematic invariants and masses are non-positive algebraic numbers, then the coefficients of the Laurent expansion of a Feynman integral are periods.

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$$A = 2365$$

$$B = \pi^2 + \zeta_3$$

~~$C = e^3 \pi^2 - \ln 2$~~

$$D = \ln^4 2 + \text{Li}_4\left(\frac{1}{2}\right)$$

~~$E = \frac{\ln^2 2}{\pi^2}$~~

$$F = \zeta_2 - 24\text{Li}_{2,2}\left(e^{i\pi/3}, e^{-2i\pi/3}\right)$$

Special functions

Numerical evaluation of polylogarithms

Tools for multiple polylogarithms

- There is a variety of tools to compute multiple polylogarithms numerically:
  - ➡ HPL (Mathematica) [Maitre]
  - ➡ hplog (Fortran, HPL's up to weight 4, real arguments) [Gehrmann, Remiddi]
  - ➡ Chaplin(Fortran, HPL's up to weight 4, complex arguments) [Buehler, CD]
  - ➡ GiNaC (C++, generic multiple polylogarithms) [Vollinga, Weinzierl]

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- Why is it so difficult? Why not just use the series expansion?

$$\mathrm{Li}_m(z) = \int_0^z \frac{\mathrm{d}t}{t}\mathrm{Li}_{m-1}(t) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}$$

Numerics from series expansion

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- Truncated series:

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- Truncated series:
  - ➡ 100 terms: 1.63493203114955540992

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$$\mathrm{Li}_2(0.99999) = 1.644808936992927295 2$$

- Truncated series:
  - ➡ 100 terms: 1.6349320311495540992
  - ➡ 1000 terms: 1.6438597615158994092

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  - ➡ 10.000 terms: 1.6447366871058790583
  - ➡ 100.000 terms: 1.6448074520672384402

Numerical evaluation

- Different codes use different solutions
  - ➡ Functional equations to map the region close to the circle to a more stable region.
  - ➡ Better expansions than the Taylor expansion.
  - ➡ Reduction to 'basis functions' that can be computed in a fact and accurate way.
  - ➡ Mixtures thereof.

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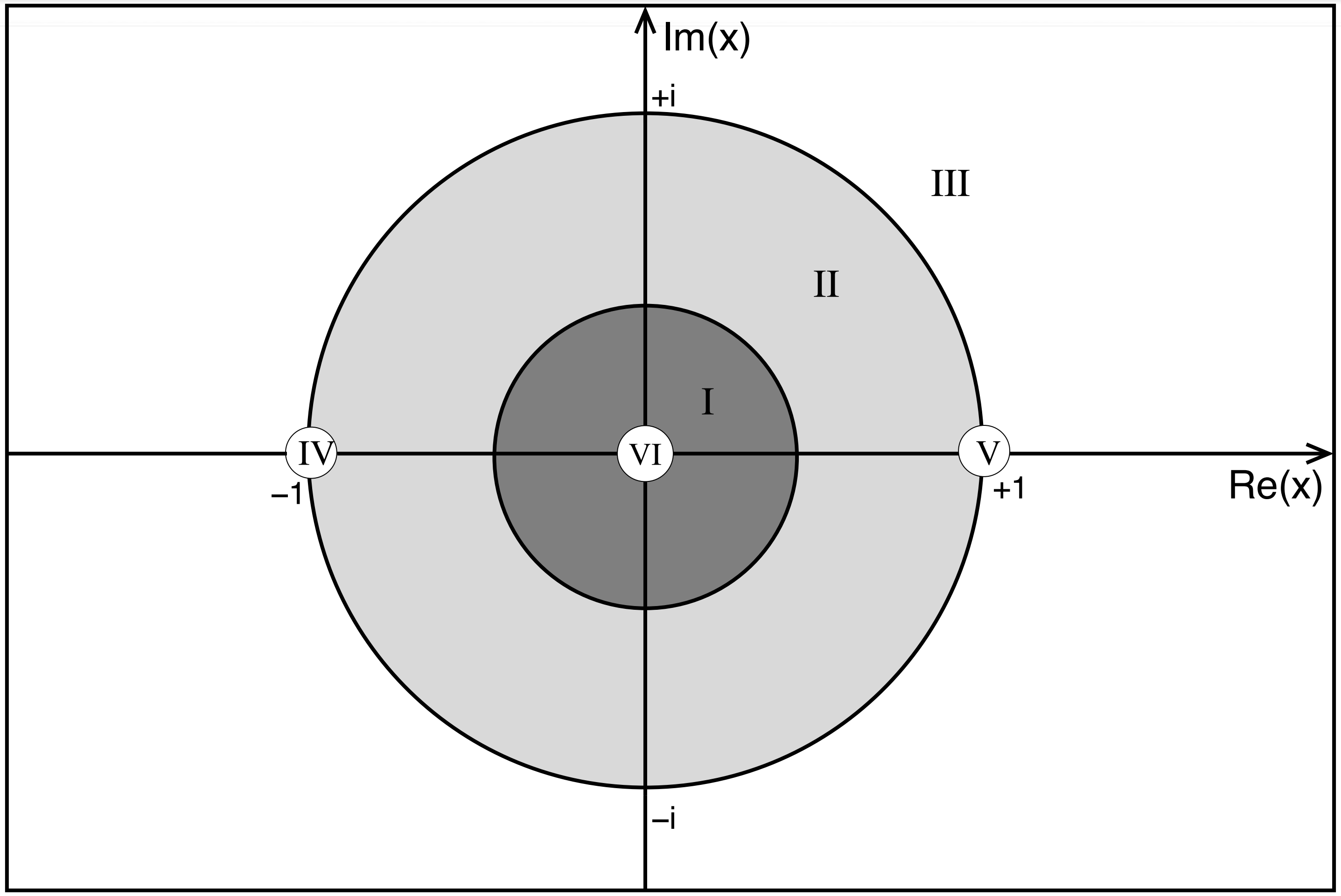
$$\mathrm{Li}_2(z) = \sum_{k=0}^{\infty} \frac{B_k}{(k+1)!} (-\log(1-z))^{k+1}$$

Chaplin



I. Expansion in log(1-z).
II. Expansion in log(z).
III. Inversion back to the unit disc.
IV.Taylor expansion around z=-1.
V.Taylor expansion around z=+1.
VI.Taylor expansion around z=0.

Summary of lecture 2

- Loop integrals are often expressed in terms of (multiple) polylogarithms.
- Multiple polylogarithms satisfy many identities.
- They form both a shuffle and stuffle algebra.
  - ➡ Need a way to deal with these relations!
- Next lecture:
  - ➡ More general and formal considerations about the analytic structure of loop integrals.