Wilson loop / gluon amplitude duality in $\mathcal{N} = 4$ SYM

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Based on work in collaboration with

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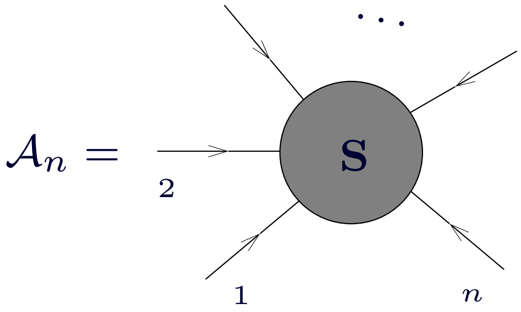
Outline

- ✔ Perturbative gluon scattering amplitudes in $\mathcal{N} = 4$ SYM
  - ✗ Bern-Dixon-Smirnov (BDS) conjecture
  - ✗ hidden conformal symmetry of planar gluon amplitudes
- ✔ Alday-Maldacena proposal for gluon scattering at strong coupling using AdS/CFT
- ✔ Wilson loops at weak coupling - a duality?
- ✔ Conformal Ward identities for Wilson loops
- ✔ Hexagonal Wilson loop and BDS ansatz

Gluon scattering amplitudes

- ✔ On-shell gluon scattering amplitudes



  - ✗ on-shell gluons characterised by momentum p_i^μ, helicity ± 1, and colour
  - ✗ amplitudes require infrared (IR) regularisation

- ✔ Colour-ordered planar partial amplitudes

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1}T^{a_2}\ldots T^{a_n}\right] A_n^{h_1,h_2,\ldots,h_n}(p_1,p_2,\ldots,p_n) + \text{[Bose symmetry]}$$

- ✔ Maximal helicity violating (MHV) amplitudes
  - ✗ Supersymmetry relations

$$A^{++\cdots+} = A^{-+\cdots+} = 0$$

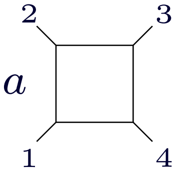
  - ✗ MHV amplitudes

$$A^{(\mathrm{MHV})} = A^{--+\cdots+}\,, \quad \ldots$$

  - ✗ NMHV amplitudes

$$A^{(\mathrm{next-MHV})} = A^{---+\cdots+}\,, \quad \ldots$$

Example: four-gluon amplitude at one loop

$$A_4 = A_4^{\rm tree}\left[1 + a \right.$$  $$\left. + O(a^2)\right]\,, \qquad a = \frac{g_{\rm YM}^2 N_c}{8\pi^2}$$

- ✔ Contains IR divergences $\rightarrow$ use dimensional regularisation
- ✔ amplitude can be factorised into an IR divergent and a finite part

$$A_4 = A_4^{\rm tree}\left[1 - \frac{a}{\epsilon_{\rm IR}^2}\left(\frac{\mu^2}{-s}\right)^{\epsilon_{\rm IR}}\right]\left[1 - \frac{a}{\epsilon_{\rm IR}^2}\left(\frac{\mu^2}{-t}\right)^{\epsilon_{\rm IR}}\right]\left[1 + a\left(\tfrac{1}{2}\ln^2\frac{s}{t} + 4\zeta_2\right)\right] + O(a^2, \epsilon_{\rm IR})$$

- ✗ IR divergences exponentiate

$$\text{div}(x) = \exp\left\{-\frac{1}{2}\sum_{l=1}^{\infty} a^l x^{l\epsilon_{\rm IR}}\left[\frac{\Gamma_{\rm cusp}^{(l)}}{(l\epsilon_{\rm IR})^2} + \frac{G^{(l)}}{l\epsilon_{\rm IR}}\right]\right\}$$

$\Gamma_{\rm cusp}(a) = \sum_l a^l \Gamma_{\rm cusp}^{(l)} =$ *cusp anomalous dimension (of Wilson loops)*

$G(a) = \sum_l a^l G_{\rm cusp}^{(l)} =$ *collinear anomalous dimension*

☞ *What about the finite part of the amplitudes? Does it have a simple structure?*

Bern-Dixon-Smirnov conjecture for MHV amplitudes

- ✔ the structure of the IR divergences is known

$$\ln\left[A_n/A_n^{\text{tree}}\right] = \text{div} + F_n(a, p_i \cdot p_j) + O(\epsilon_{\text{IR}})$$

- ✔ the BDS conjecture is a statement about the finite part:

$$F_n = \frac{1}{2}\Gamma_{\text{cusp}}(a) F_n^{(1)}$$

- ✔ For example, for four and five points

$$F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\left[\ln^2\frac{s}{t} + \text{const}\right]$$

$$F_5 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\left[\sum_{i=1}^{5} \ln\frac{s_{i,i+1}}{s_{i+1,i+2}} \ln\frac{s_{i+2,i+3}}{s_{i+3,i+4}} + \text{const}\right]$$

- ✔ the BDS conjecture has been confirmed so far
  - ✗ up to three loops for F_4 [Bern, Dixon, Smirnov]
  - ✗ up to two loops for F_5 [Cachazo, Spradlin, Volovich],[Bern, Czakon, Korower, Roiban, Smirnov]

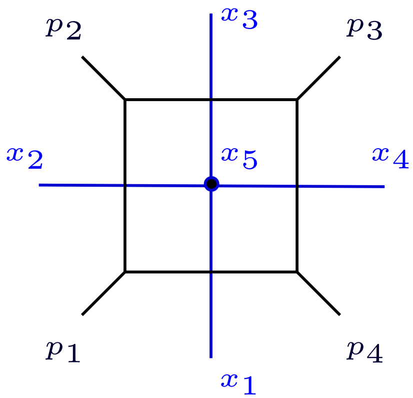
☞ *Where does the simplicity of the finite part come from?*

Dual conformal symmetry

One-loop: 'scalar box' integral

- ✔ Change variables to go to a *dual 'coordinate space'* picture (not a Fourier transform!)

$$p_1 = x_1 - x_2 \equiv x_{12}\,, \quad p_2 = x_{23}\,, \quad p_3 = x_{34}\,, \quad p_4 = x_{41}\,, \quad k = x_{15}$$



$$= \int \frac{d^D k}{k^2 (k-p_1)^2 (k-p_1-p_2)^2 (k+p_4)^2} = \int \frac{d^D x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2}$$

Conformal inversion: $x_i^\mu \to x_i^\mu / x_i^2\,, x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2 x_j^2}$

[Broadhurst],[Drummond,J.H.,Smirnov,Sokatchev]

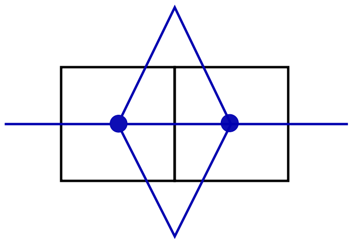
- ✔ Consider the integral off-shell and for $D=4$
  - ✗ The integral is conformal in the dual space
  - ✗ The symmetry *is not related* to the conformal symmetry of $\mathcal{N}=4$ SYM
- ✔ The dual conformal symmetry is broken by the infrared regulator, $D = 4 - 2\epsilon$.

☞ *Is this broken symmetry present at higher loops?*

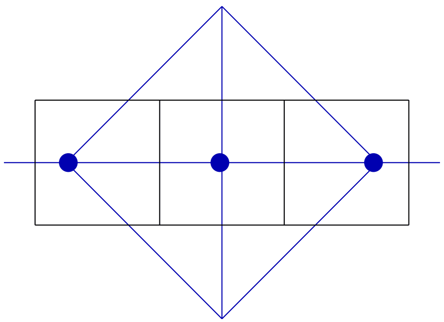
Dual conformal symmetry

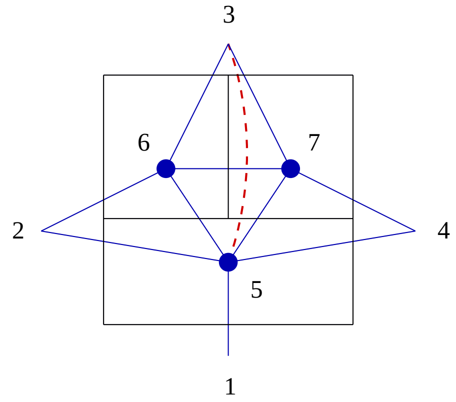
The dual conformal structure continues to higher loops [Broadhurst],[Drummond,J.H.,Smirnov,Sokatchev]

- ✔ Two loops [Bern, Rozowski, Yan]



- ✔ Three loops [Bern, Dixon, Smirnov]





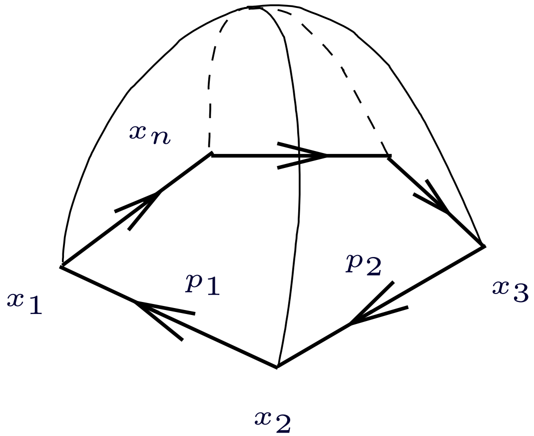
- ✔ Continues to four loops [Bern, Czakon, Dixon, Kosower, Smirnov]
- ✔ Even five? [Bern, Carrasco, Johannson, Kosower]

☞ *Where does the dual conformal symmetry come from?*

Gluon amplitudes from AdS/CFT I

Alday-Maldacena proposal:

- ✔ Gluon amplitude is described by scattering of strings in AdS$_5$
  - ✗ gluons $\longrightarrow$ open strings ending on a D-brane extended along z_{3+1} and located at large $r_{\rm IR}$
  - ✗ dominated by classical solution [Gross, Mende'87]
- ✔ Minimal surface in the 'T-dual coordinates' [Kallosh,Tseytlin'98]



- ✗ gluon momenta $p_1^\mu, \ldots, p_n^\mu$ define sequence of light-like segments on the boundary $r = 0$
- ✗ *dual coordinates* x_i^μ related to the on-shell gluon momenta by

$$x^\mu_{i,i+1} \equiv x^\mu_i - x^\mu_{i+1} := p^\mu_i$$

☞ *The same dual variables appeared in the dual conformal invariance of scalar integrals!*

- ✔ IR divergencies of $\mathcal{A}_4$ $\overset{\text{T-duality}}{\Longrightarrow}$ UV (cusp) divergences

Gluon amplitudes from AdS/CFT II

Strong coupling prediction [Alday,Maldacena]

$$\mathcal{A}_4 = \exp\left\{ -\frac{1}{\epsilon^2}\frac{\sqrt{\lambda}}{2\pi} - -\frac{1}{\epsilon}\frac{\sqrt{\lambda}}{4\pi}(1-\ln 2)\left[\left(\frac{\mu^2}{-s}\right)^{\epsilon} + \left(\frac{\mu^2}{-t}\right)^{\epsilon}\right]\right\} \exp\left\{\frac{\sqrt{\lambda}}{4\pi}\ln^2\frac{s}{t} + \text{const}\right\}$$

- ✔ Is in agreement with the Bern-Dixon-Smirnov (BDS) ansatz for $n = 4$ amplitudes
- ✔ Difficult to construct solutions for n-gluon amplitudes
- ✔ Disagreement with BDS ansatz is found for $n \to \infty$ [Alday,Maldacena]

- ✔ Calculation is “mathematically similar” to that of the expectation value of a *cusped* Wilson loop at strong coupling [Kruczenski]

☞ Are the gluon amplitudes at weak coupling also related to Wilson loops?

[Drummond, Korchemsky, Sokatchev]

Light-like Wilson loops

The expectation value of light-like Wilson loop in $\mathcal{N} = 4$ SYM

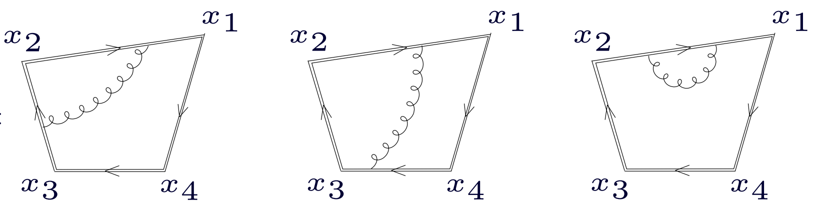
$$W(C_4) = \frac{1}{N_c} \langle 0 | \mathrm{Tr}\, \mathrm{P} \exp \left(ig \oint_{C_4} dx^\mu A_\mu(x) \right) | 0 \rangle$$

To lowest order in the coupling constant,

$$W(C_4) = 1 + \frac{1}{2} (ig)^2 C_F \sum_{1 \leq j,\, k \leq 4} \int_{\ell_j} dx^\mu \int_{\ell_k} dy^\nu \, G_{\mu\nu}(x-y) + O(g^4)$$

- Feynman diagram representation

$$\ln W(C_4) =$$



- The light-like Wilson loop is IR finite but has UV divergences due to cusps on contour C_4

$$\ln W(C_4) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon^2} \left(-x_{13}^2 \, \mu^2 \right)^\epsilon - \frac{1}{\epsilon^2} \left(-x_{24}^2 \, \mu^2 \right)^\epsilon + O(\epsilon^0) \right\} + O(g^4) \, .$$

Gluon scattering amplitudes / Wilson loops duality I

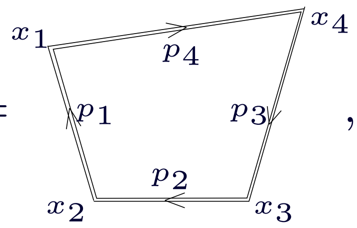
- ✔ One-loop light-like Wilson loop

$$\ln W(C_4) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm UV}{}^2} \left[(-x_{13}^2\mu^2)^{\epsilon_{\rm UV}} + (-x_{24}^2\mu^2)^{\epsilon_{\rm UV}} \right] + \frac{1}{2}\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{const} \right\} + O(g^4)$$

- ✔ One-loop gluon scattering amplitude

$$\ln \mathcal{A}_4(s,t) = \frac{g^2}{4\pi^2} C_F \left\{ -\frac{1}{\epsilon_{\rm IR}{}^2} \left[(-s/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} + (-t/\mu_{\rm IR}^2)^{\epsilon_{\rm IR}} \right] + \frac{1}{2}\ln^2\left(\frac{s}{t}\right) + \text{const} \right\} + O(g^4)$$

- ✔ Identity the light-like segments in Minkowski space-time with the on-shell gluon momenta:

$C_4 =$ 

$$, \qquad x^\mu_{i,i+1} \equiv x^\mu_i - x^\mu_{i+1} := p^\mu_i \,, \qquad p_i^2 = 0$$

- ✔ Identify the dimensional regularisation parameters:

$$x_{13}^2\,\mu^2 := s/\mu_{\rm IR}^2\,, \qquad x_{24}^2\,\mu^2 := t/\mu_{\rm IR}^2\,, \qquad x_{13}^2/x_{24}^2 := s/t$$

  - ☞ the UV divergences of the light-like Wilson loop match the IR divergences of the gluon amplitude
  - ☞ the finite terms $\sim \ln^2(s/t)$ coincide

[Drummond,Korchemsky,Sokatchev]

Gluon scattering amplitudes / Wilson loops duality II

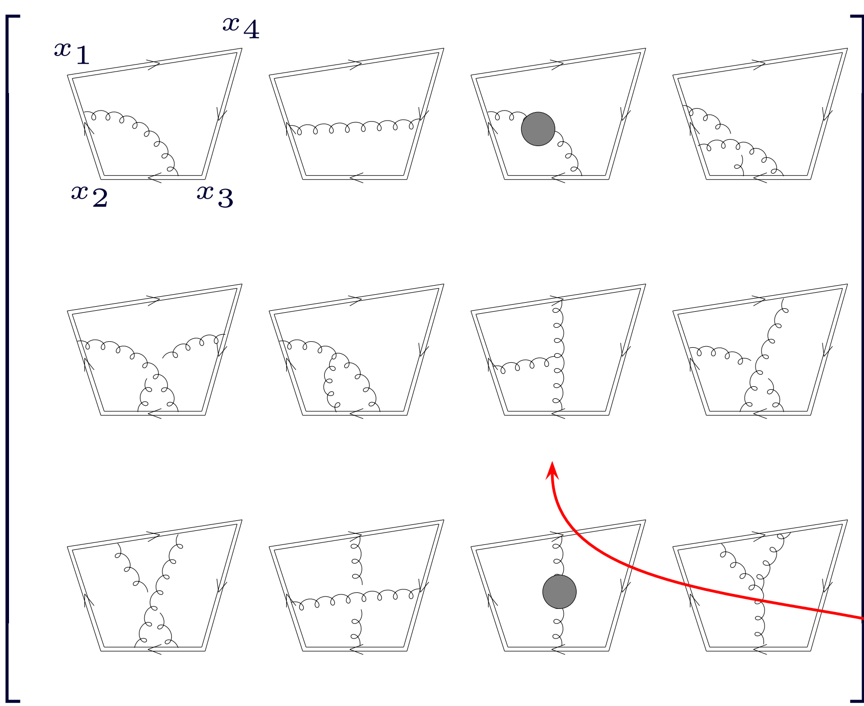
- ✔ Proposal : gluon amplitudes at weak coupling are dual to light-like Wilson loops

[Drummond,Korchemsky,Sokatchev]

$$\ln A_4 = \ln W(C_4) + O(1/N_c^2)$$

- ✔ At strong coupling, the relation holds up to corrections in $1/\sqrt{\lambda}$ [Alday,Maldacena]
- ✔ At weak coupling, the relation was verified to two loops [Drummond,J.H.,Korchemsky,Sokatchev]

$$\ln \mathcal{A}_4 = \ln W(C_4) =$$



$$= \frac{1}{4}\Gamma_{\rm cusp}(g)\,\ln^2\left(\frac{s}{t}\right) + \text{div}$$

The duality holds as a result of “massive cancellation” of various terms, e.g. $\frac{1}{24}M_1^2 - \frac{1}{4}M_3 + \ldots$

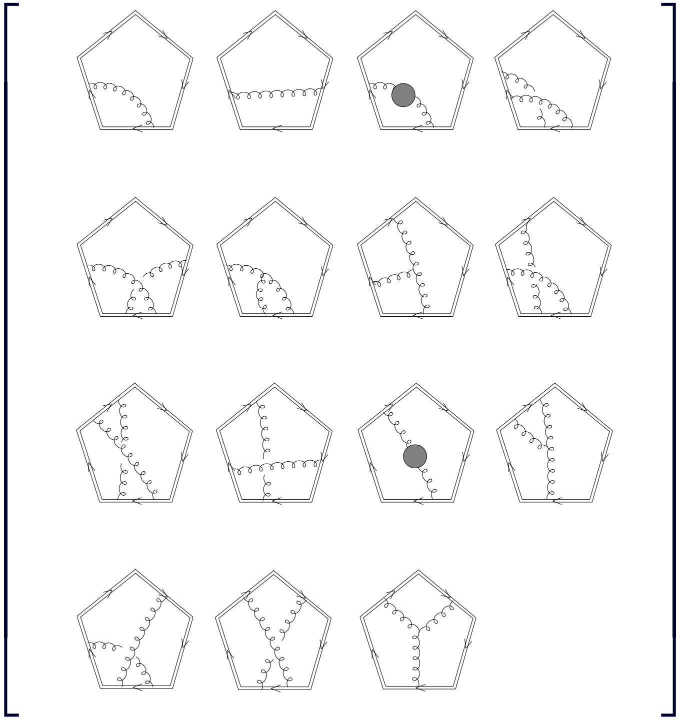
$$M_1 = \int_0^1 \frac{d\beta}{\beta-\bar{\alpha}} \ln\left(\frac{\bar{\alpha}\bar{\beta}}{\alpha\beta}\right), \quad M_3 = \int_0^1 \frac{d\beta}{\beta-\bar{\alpha}} \ln\left(\frac{\bar{\alpha}\bar{\beta}}{\alpha\beta}\right)\ln^2(\beta\bar{\beta}), \qquad (\alpha=\tfrac{s}{s+t}, \quad \bar{\alpha}=1-\alpha).$$

Generalisation to $n \geq 5$ gluon MHV amplitudes

$$\ln \mathcal{A}_n^{(\mathrm{MHV})} = \ln W(C_n) + O(1/N_c^2)\,,$$ C_n is made of n light-like segments

- ✔ At one loop, the n-point Wilson loops reproduce the BDS ansatz [Brandhuber,Heslop,Travaglini]
- ✔ The duality relation for $n = 5$ (pentagon) was verified to two loops [Drummond,J.H.,Korchemsky,Sokatchev]

$$\ln \mathcal{A}_5 = \ln W(C_5) =$$



$= $ BDS ansatz

☞ *Can we understand why the Wilson loop gives such simple results?*

Conformal Ward identities for light-like Wilson loop I

- Conformal transformations map a light-like polygon C_n into another light-like polygon C'_n
- If the Wilson loop $W(C_n)$ was finite in $D = 4$ dimensions then

$$W(C_n) = W(C'_n)$$

- Cusp divergences $\to$ dimensional regularisation breaks conformal invariance

$$W(C_n) = W(C'_n) \times \text{[cusp anomaly]}$$

- Anomalous conformal Ward identities for the *finite part* of the Wilson loop

[Drummond,J.H.,Korchemsky,Sokatchev]

$$W(C_n) = \exp(F_n) \times \text{[UV divergences]}$$

under dilatations, $\mathbb{D}$, and special conformal transformations, $\mathbb{K}^\mu$,

$$\mathbb{D}\, F_n \equiv \sum_{i=1}^{n} (x_i \cdot \partial_{x_i}) F_n = 0$$

$$\mathbb{K}^\mu\, F_n \equiv \sum_{i=1}^{n} \left[2x_i^\mu (x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu\right] F_n = \frac{1}{2}\Gamma_{\rm cusp}(a) \sum_{i=1}^{n} x_{i,i+1}^\mu \ln\left(\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}\right)$$

Conformal Ward identities for light-like Wilson loop II

- ✔ The Wilson loop in the dimensionally regularised $\mathcal{N} = 4$ SYM theory is

$$\langle W_n \rangle = \int \mathcal{D}A\,\mathcal{D}\lambda\,\mathcal{D}\phi\; \mathrm{e}^{iS_\epsilon[A,\,\lambda,\,\phi]}\; \mathrm{Tr\,P\,exp}\left(i \oint_{C_n} dx \cdot A(x) \right) ,$$

integration goes over gauge fields, A, gaugino, λ, and scalars, ϕ, with the action

$$S_\epsilon = \frac{1}{g^2 \mu^{2\epsilon}} \int d^D x\, \mathcal{L}(x)\,, \qquad \mathcal{L} = \mathrm{Tr}\left[-\tfrac{1}{4} F_{\mu\nu}^2 \right] + \text{gaugino} + \text{scalars} + \text{gauge fixing} + \text{ghosts}.$$

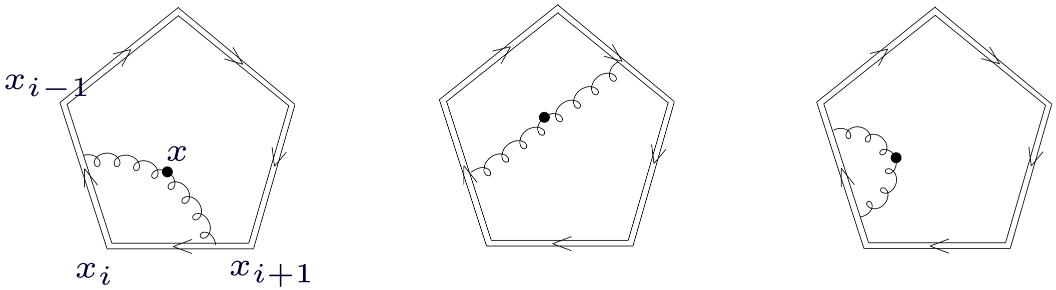
- ✔ The regularised action is no longer invariant under conformal transformations,

$$\mathbb{K}^\nu \langle W_n \rangle = \sum_{i=1}^{n} (2x_i^\nu x_i \cdot \partial_i - x_i^2 \partial_i^\nu) \langle W_n \rangle = \langle \delta_{\mathbb{K}^\nu} S_\epsilon\, W_n \rangle = \frac{4i\epsilon}{g^2 \mu^{2\epsilon}} \int d^D x\, x^\nu \,\langle \mathcal{L}(x) W_n \rangle$$

To make use of this relation we need only the divergent part of $\langle \mathcal{L}(x) W_n \rangle$

- ✔ one - loop example: $\mathcal{L}(x)$ is inserted into the gluon propagator [Belitsky,Müller]

$$\langle \mathcal{L}(x) W_n \rangle =$$



Conformal Ward identities for light-like Wilson loop III

✔ Divergent part to one loop

$$\frac{2i}{g^2\mu^{2\epsilon}}\frac{\langle \mathcal{L}(x) W_n\rangle}{\langle W_n\rangle} = -a\sum_{i=1}^{n}\left(-x_{i-1,i+1}^2\mu^2\right)^{\epsilon}\left\{\epsilon^{-2}\delta^{(D)}(x-x_i)+\epsilon^{-1}\Upsilon_i^{(1)}(x)+O(\epsilon^0)\right\},$$

where

$$\Upsilon_i^{(1)}(x) = \int_0^1\frac{ds}{s}\left[\delta^{(D)}(x-x_i-sx_{i-1,i})+\delta^{(D)}(x-x_i+sx_{i,i+1})-2\delta^{(D)}(x-x_i)\right].$$

☞ Divergences are localised

- ✗ at the cusp points x_i (leading ϵ^{-2} pole)
- ✗ on the light-like edges adjacent to one cusp (ϵ^{-1} pole).

☞ $\Upsilon_i^{(1)}(x)-$term does not contribute to Ward indentity by virtue of

$$\sum_{i=1}^{n}\int d^Dx\, x^\nu\,\Upsilon_i^{(1)}(x) = \sum_{i=1}^{n}(x_{i-1,i}-x_{i,i+1})^\nu = 0$$

✔ Generalisation to all loops $\frac{2i}{g^2\mu^{2\epsilon}}\frac{\langle \mathcal{L}(x) W_n\rangle}{\langle W_n\rangle} =$

$$= -\sum_{l\geq 1}a^l\sum_{i=1}^{n}\left(-x_{i-1,i+1}^2\mu^2\right)^{l\epsilon}\left\{\frac{1}{2}\left(\frac{\Gamma_{\rm cusp}^{(l)}}{l\epsilon^2}+\frac{\Gamma^{(l)}}{\epsilon}\right)\delta^{(D)}(x-x_i)+\epsilon^{-1}\Upsilon_i^{(l)}(x)+O(\epsilon^0)\right\}$$

Consequences of the conformal Ward identity

- Four and five points: the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(a)\sum_{i=1}^{5}\ln\left(\frac{x_{i,i+2}^2}{x_{i,i+3}^2}\right)\ln\left(\frac{x_{i+1,i+3}^2}{x_{i+2,i+4}^2}\right) + \text{ const}$$

  Exactly the functional form of the BDS ansatz for the 4- and 5-point gluon amplitudes!

- Starting from six points there are conformal invariants in the form of cross-ratios

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$

  Hence the general solution of the Ward identity contains *an arbitrary function* of the conformal cross-ratios.

- For *arbitrary n* the BDS ansatz is still a solution of the conformal Ward identity, but the solution is no longer unique.

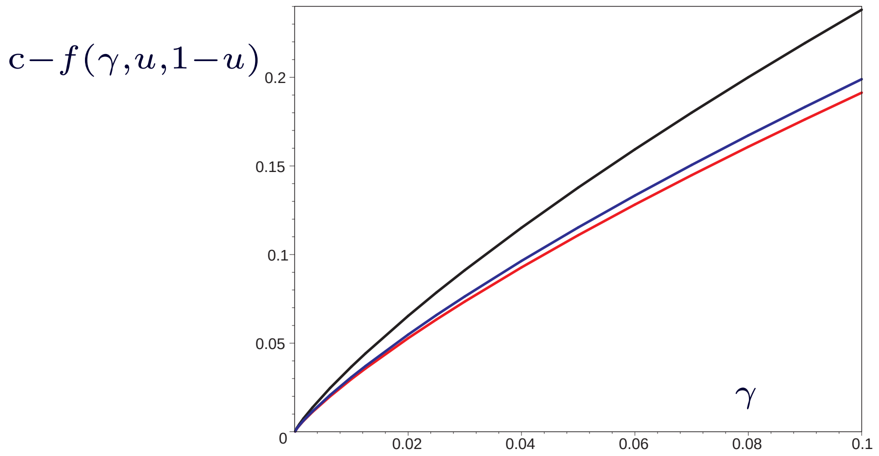
Hexagonal Wilson loop at two loops

Wilson loop at six points and two loops was computed recently

[Drummond, J.H., Korchemsky, Sokatchev]

$$F_6 = F_6^{(BDS)} + f(u_1, u_2, u_3)$$

- ✔ We evaluated (numerically) the function $f(u_1, u_2, u_3)$ not fixed by the conformal Ward identity.
- ✔ It is not a constant, i.e. at two loops and six points, the finite part of the Wilson loop is not given by the BDS ansatz for the gluon amplitude
- ✔ $f(u_1, u_2, u_3)$ goes to a constant in the collinear limit, so it could in principle appear in a gluon amplitude



- ☞ Is the BDS ansatz correct at two loops and for six gluons?
- ☞ Does the duality hold at this level?

Hexagon Wilson loop = six-gluon amplitude

[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich]

[Drummond, J.H., Korchemsky, Sokatchev]

✔ Numerical tests for different kinematical configurations $K^{(i)}$.

Kinematical point	(u_1, u_2, u_3)	$f_{\rm WL} - f_{\rm WL}^{(0)}$	$f_{\rm A} - f_{\rm A}^{(0)}$
$K^{(1)}$	$(1/4, 1/4, 1/4)$	$< 10^{-5}$	-0.018 ± 0.023
$K^{(2)}$	$(0.547253, 0.203822, 0.88127)$	-2.75533	-2.753 ± 0.015
$K^{(3)}$	$(28/17, 16/5, 112/85)$	-4.74460	-4.7445 ± 0.0075
$K^{(4)}$	$(1/9, 1/9, 1/9)$	4.09138	4.12 ± 0.10
$K^{(5)}$	$(4/81, 4/81, 4/81)$	9.72553	10.00 ± 0.50

☞ The BDS ansatz needs to be corrected at six gluons and two loops!

☞ The duality holds!

Conclusions and outlook

- ✔ Evidence of a (broken) dual conformal symmetry of planar MHV amplitudes
- ✔ This symmetry becomes manifest within the gluon amplitude / Wilson loop duality
- ✔ The duality relation was verified in several nontrivial cases
- ✔ If the gluon amplitude / Wilson loop duality is correct, it leads to the following consequences
  - ✗ the broken conformal Ward identities uniquely fix the form of the finite part of four and five gluon amplitudes, in complete agreement with the BDS conjecture.
  - ✗ starting from six-gluon amplitudes, conformal symmetry is *not* sufficient to fix the finite part.
  - ✗ it predicts the BDS ansatz for the six gluon amplitude to fail at two loops. *It did!*

- ☞ Can the duality be extended beyond the MHV case? E.g. NMHV amplitudes appear starting from six points.
- ☞ What is the origin of the dual conformal symmetry of gluon amplitudes? ... Related to integrability of planar $\mathcal{N}=4$ SYM?!