Bootstrapping the three-loop hexagon

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based on

L. J. Dixon, J. M. Drummond, J. M. H., arXiv:1108.4461 [hep-th]

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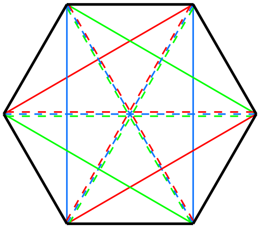
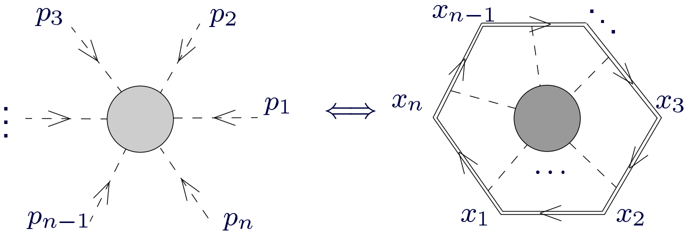
Outline

Bootstrapping the three-loop hexagon



- Cusped Wilson loops in $\mathcal{N} = 4$ super Yang-Mills
- Iterated integrals and symbols
- ansatz and constraints for the three-loop hexagon
- prediction for Regge limit

Duality between Wilson loops and scattering amplitudes



$$x^{\mu}_{i+1} - x^{\mu}_{i} = p^{\mu}_{i}$$

[Alday, Maldacena, 2007; Drummond, Korchemsky, Sokatchev, 2007; Brandhuber, Heslop, Travaglini, 2007]

- Checked by two-loop computations for $n \leq 6$ points

  [Drummond, J.M.H. ,Korchemsky,Sokatchev, 2007,2008]

  [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich, 2008]

- conformal symmetry of Wilson loops $\longrightarrow$ dual conformal symmetry of amplitudes

- all-order (dual) conformal Ward identities [Drummond, J.M.H., Korchemsky, Sokatchev, 2007]

  $\Rightarrow$ kinematical dependence of four-and five-point Wilson loops/scattering amplitudes fixed to all orders in the coupling!

Conformal Ward identity and six-point Wilson loop

- known structure of UV divergences [Korchemsky, Radyushkin (1987); Korchemskaya, Korchemsky (1992)]

$$\log W_n = [\mathrm{UV\ divergent}]_n + F_n^{\mathrm{WL}}\,.$$

- solution to Ward identity at $n = 6$ [Drummond, J.M.H., Korchemsky, Sokatchev, 2007]

$$F_6^{\mathrm{WL}} = \gamma_K(a)\, F_{6,1-\mathrm{loop}}^{\mathrm{WL}} + R_6(a; u, v, w)\,.$$

with $a \equiv g^2 N_c/(8\pi^2)$. $\gamma_K(a)$ cusp anomalous dimension.

- loop expansion of remainder function

$$R_6(a; u, v, w) \;=\; \sum_{L=2}^{\infty} a^L\, R_6^{(L)}(u, v, w)\,,\;.$$

Depends on three dual conformal cross ratios only,

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}\,, \qquad v = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}\,, \qquad w = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}\,.$$

- $R_6^{(L)}$ expected to be expressible as sum of $2L$-fold iterated integrals

Iterated integrals, pure functions and symbols

- we define a *pure* function of degree (or weight) k recursively,

$$d\,f^{(k)} = \sum_r f_r^{(k-1)} d \log \phi_r \,.$$

ϕ_r are algebraic functions. Degree zero functions are constants.
The definition includes logarithms and classical polylogarithms, as well as other iterated integrals, such as harmonic polylogarithms of one or more variables.

- *symbol* $\mathcal{S}(f)$ of a pure function f is defined recursively

$$\mathcal{S}(f^{(k)}) = \sum_r \mathcal{S}(f_r^{(k-1)}) \otimes \phi_r \,.$$

- $\mathcal{S}(f^{(k)})$ is an element of the k-fold tensor product of the space of algebraic functions,

$$\mathcal{S}(f^{(k)}) = \sum_{\vec{\alpha}} \phi_{\alpha_1} \otimes \ldots \otimes \phi_{\alpha_k} \,,$$

Iterated integrals, pure functions and symbols

Important properties of symbols:

- property derived from $\log(ab) = \log a + \log b$

$$\ldots \otimes \phi_1\phi_2 \otimes \ldots \quad = \quad \ldots \otimes \phi_1 \otimes \ldots \quad + \quad \ldots \otimes \phi_2 \otimes \ldots \,.$$

- integrability condition $d^2 f^{(k)} = 0$ for any function implies relations among the different elements.
- Branch cuts and discontinuities. Given

$$\mathcal{S}(f^{(k)}) = \sum_{\vec{\alpha}} \phi_{\alpha_1} \otimes \ldots \otimes \phi_{\alpha_k}\,,$$

the degree k function $f^{(k)}$ will have a branch cut starting at $\phi_{\alpha_1} = 0$. The discontinuity across this cut has the symbol

$$\mathcal{S}(\Delta_{\phi_{\alpha_1}} f^{(k)}) = \sum_{\vec{\alpha}} \phi_{\alpha_2} \otimes \ldots \otimes \phi_{\alpha_k}\,.$$

Iterated integrals, pure functions and symbols

- some examples:

$$\mathcal{S}(\log^2(u)) = 2u \otimes u$$

$$\mathcal{S}(-\mathrm{Li}_2(u)) = (1-u) \otimes u$$

$$\mathcal{S}(\log(u)\log(v)) = u \otimes v + v \otimes u$$

- Fine print:
  - Symbols do not know on which branch the functions are evaluated.
  - Related: symbols are defined only up to constants times lower degree functions.
- Nonetheless, the symbol is an extremely useful tool, especially in multi-variable cases. E.g., symbols were crucial to find a simple form for the two-loop remainder function $R_6^{(2)}$.

Symbol of the two-loop remainder function

- $R_6^{(2)}$ known analytically [Del Duca, Duhr, Smirnov (2009)]

  can be expressed in terms of classical polylogarithms [Goncharov, Spradlin, Vergu, Volovich (2010)]

  It has the symbol

$$
\begin{aligned}
\mathcal{S}(R_6^{(2)}) \;=\; & -\frac{1}{8}\Bigg\{\Bigg[u\otimes(1-u)\otimes\frac{u}{(1-u)^2}+2\left(u\otimes v+v\otimes u\right)\otimes\frac{w}{1-v} \\
& +2\,v\otimes\frac{w}{1-v}\otimes u\Bigg]\otimes\frac{u}{1-u} \\
& +\Big[u\otimes(1-u)\otimes y_u y_v y_w-2\,u\otimes v\otimes y_w\Big]\otimes y_u y_v y_w\Bigg\} \\
& +\text{ permutations}\,,
\end{aligned}
$$

  Here

$$
y_u=\frac{u-z_+}{u-z_-}\,,\qquad z_\pm=\frac{1}{2}\left(-1+u+v+w\pm\sqrt{(1-u-v-w)^2-4uvw}\right)\,.
$$

- What is the symbol of R_6 at higher loops?

Ansatz for $\mathcal{S}(R_6^{(3)})$

- Ansatz: symbol built from the following letters:

$$\mathcal{A} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}\,.$$

i.e.

$$\mathcal{S}(R_6^{(3)}) = \sum_{\vec{\alpha}} c_{\vec{\alpha}} \times a_{\alpha_1} \otimes a_{\alpha_2} \otimes a_{\alpha_3} \otimes a_{\alpha_4} \otimes a_{\alpha_5} \otimes a_{\alpha_6}\,,$$

with $a_{\alpha_i} \in \mathcal{A}$ and $c_{\vec{\alpha}} \in \mathbb{R}$.

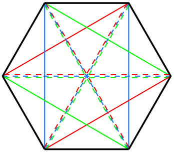
- Motivation:

  - explicit form of two-loop remainder function [Goncharov, Spradlin, Vergu, Volovich (2010)]

  - experience with loop integrals appearing in MHV scattering amplitudes

    [Drummond, J. .M. .H., Trnka (2010); Dixon, Drummond, J. .M. .H. (2011)]

- Is this ansatz consistent with all known constraints on $\mathcal{S}(R_6^{(3)})$?

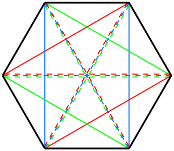


Constraints on $\mathcal{S}(R_6^{(3)})$

What constraints should the symbol of the remainder function obey?

- should be *integrable*, i.e. symbol of a function
- *discontinuities* of loop integrals at $x_{ij}^2 = 0 \longrightarrow$ first entry should be u, v or w
- it should be completely symmetric in u, v, w
- parity even $\longrightarrow$ even number of y_u, y_v, y_w variables
- collinear limit: $R(u, 1-u, 0) = 0$
- constraints in multi-Regge kinematics [Lipatov, Prygarin (2010), Bartels, Lipatov, Prygarin (2010)]

$$u \to 1\,, \qquad \frac{v}{1-u} \to x\,, \qquad \frac{w}{1-u} \to y\,.$$



OPE constraints

- OPE (operator product expansion) for cusped Wilson loops

[Alday, Gaiotto, Maldacena, Sever, Vieira (2010)]

[Gaiotto, Maldacena, Sever, Vieira (2011)]

predicts (multiple) discontinuity Δ_v^{L-1} at L loops from one-loop data

$$\Delta_v^{L-1}\mathcal{S}(R_6^{(L)}) \quad \propto \quad \mathcal{S}\Big(\int \frac{dp}{2\pi} e^{-ip\sigma}\Big(\sum_{m=1}^{\infty} \frac{[\gamma_{m+2}(p)]^{L-1}\cos(m\phi)}{p^2+m^2} + \sum_{m=2}^{\infty} \frac{[\gamma_{m-2}(p)]^{L-1}\cos((m-2)\phi)}{p^2+(m-2)^2}\Big)\mathcal{C}_m(p)\mathcal{F}_{m/2,p}(\tau)\Big) \,.$$

τ, σ, ϕ related to u, v, w

- we can test corollaries of this formula

$$\mathcal{D}_+\mathcal{D}_-\Delta_v\Delta_v\mathcal{S}(R_6^{(3)}) = 0$$

and

$$\Box\Delta_w\Delta_w\Delta_v\Delta_v\mathcal{S}(R_6^{(3)}) \propto \frac{w(1-u+v-w)}{(1-v)(1-w)}$$

Result

- we find a solution consistent with OPE constraints, with 26 parameters α_i

$$\mathcal{S}(R_6^{(3)}) = \mathcal{S}(X) + \sum_{i=1}^{26} \alpha_i \mathcal{S}(f_i)$$

highly nontrivial that ansatz is consistent with all constraints!

- Regge limit imposes three more constraints
- additional constraint: final entry drawn only from set

$$\frac{u}{1-u}\,,\frac{v}{1-v}\,,\frac{w}{1-w}\,,y_u\,,y_v\,,y_w$$

can be motivated by supersymmetry [Caron-Huot (2011)]
leads to

$$\mathcal{S}(R_6^{(3)}) = \mathcal{S}(X) + \alpha_1 \mathcal{S}(f_1) + \alpha_2 \mathcal{S}(f_2)$$

Prediction for multi-Regge limit

- multi-Regge limit

  Analytically continue to physical branch $u \to e^{-2\pi i}\, u$ and let

$$u \to 1\,, \qquad \frac{v}{1-u} \to x\,, \qquad \frac{w}{1-u} \to y\,.$$

  Expect

$$R_6^{(3)} \to (2\pi i) \sum_{r=0}^{2} \log^r(1-u) \left[g_r^{(3)}(x,y) + 2\pi i\, h_r^{(3)}(x,y) \right]$$

- we find agreement with prediction for leading terms $g_2^{(3)}$, $h_{2,1}^{(3)}$ [Lipatov, Prygarin (2010)]
- we have new predictions for $g_1^{(3)}$, $g_0^{(3)}$ and $h_0^{(3)}$

Explicit prediction for multi-Regge limit

- explicit formula for $g_1^{(3)}$

$$u \to 1\,, \qquad \frac{v}{1-u} \to \frac{1}{(1+w)(1+w^*)}\,, \qquad \frac{w}{1-u} \to \frac{ww^*}{(1+w)(1+w^*)}\,.$$

We find

$$\begin{aligned} g_1^{(3)}(w,w^*) \;=\; & \frac{1}{8}\Bigg\{\log|w|^2\left[\mathrm{Li}_3\left(\frac{w}{1+w}\right)+\mathrm{Li}_3\left(\frac{w^*}{1+w^*}\right)\right] \\ & +(5\,\log|1+w|^2-2\,\log|w|^2)\Big[\mathrm{Li}_3(-w)+\mathrm{Li}_3(-w^*)\Big] \\ & -\frac{3}{2}\log|w|^2\,\log\frac{|1+w|^4}{|w|^2}\Big[\mathrm{Li}_2(-w)+\mathrm{Li}_2(-w^*)\Big] \\ & -\frac{1}{12}\,\log^2|1+w|^2\left[\log|w|^2\,(\log|w|^2+2\,\log|1+w|^2)-10\,\log^2\frac{|1+w|^2}{|w|^2}\right] \\ & +\frac{1}{2}\,\log|w|^2\,\log\frac{|1+w|^2}{|w|^2}\,\log(1+w)\,\log(1+w^*)-2\,\zeta_3\,\log|1+w|^2\Bigg\} \\ & +\left(\frac{5}{2}+\gamma'\right)\zeta_2\,g_1^{(2)}(w,w^*)\,, \end{aligned}$$

Summary and outlook

- Summary
  - starting from our ansatz we fix the symbol of $\mathcal{R}_6^{(3)}$ up to two constants

$$\mathcal{S}(R_6^{(3)}) = \mathcal{S}(X) + \alpha_1 \mathcal{S}(f_1) + \alpha_2 \mathcal{S}(f_2)$$

    without evaluating any loop integrals!
  - prediction for multi-Regge limit, where α_1 and α_2 drop out
  - new functions other than classical polylogarithms needed

- Outlook
  - find functions X and f_2 (f_1 already determined)
  - constraints at function level are likely to fix α_1 and α_2.

$\longrightarrow$ analytical result for three-loop six-particle scattering process