

Method of Dimensional Recurrence Relations

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Introduction

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- Evaluating master integrals for the three-loop form factors

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- Conclusion

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The missing finite parts of $A_{9,2}$ and $A_{9,4}$

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The evaluation of terms of transcendentality weight 8 characteristic for four loops: π^8 , $\zeta(5)\zeta(3)$, $\zeta(3)^2\pi^2$, $\zeta(-6, -2)$, i.e. up to ϵ^2 for $A_{9,2}$ and $A_{9,4}$ and ϵ^3 for $A_{9,1}$.

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One-scale Feynman integrals, $p_1^2 = p_2^2 = 0$, $q^2 = (p_1 + p_2)^2$

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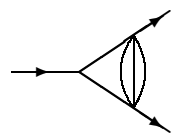
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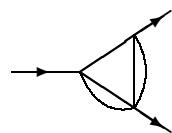
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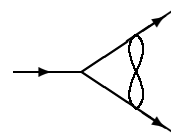
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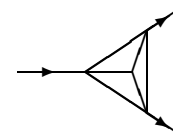
$A_{5,1}$



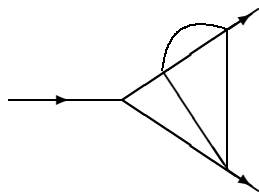
$A_{5,2}$



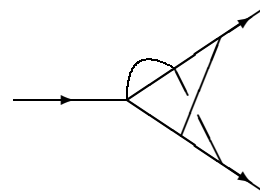
$A_{6,1}$



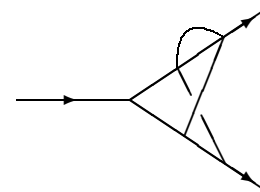
$A_{6,2}$



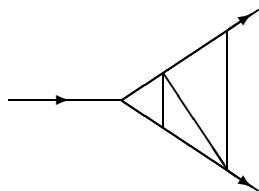
$A_{6,3}$



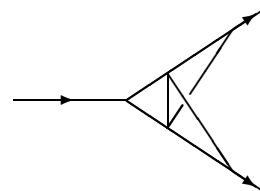
$A_{7,1}$



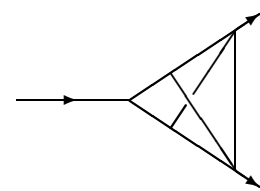
$A_{7,2}$



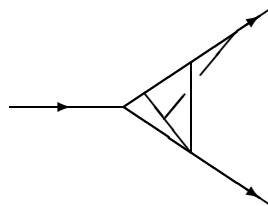
$A_{7,3}$



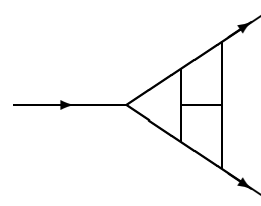
$A_{7,4}$



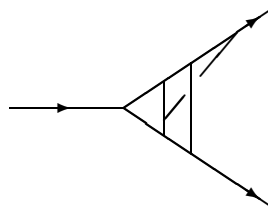
$A_{7,5}$



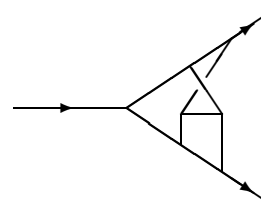
A_8



$A_{9,1}$

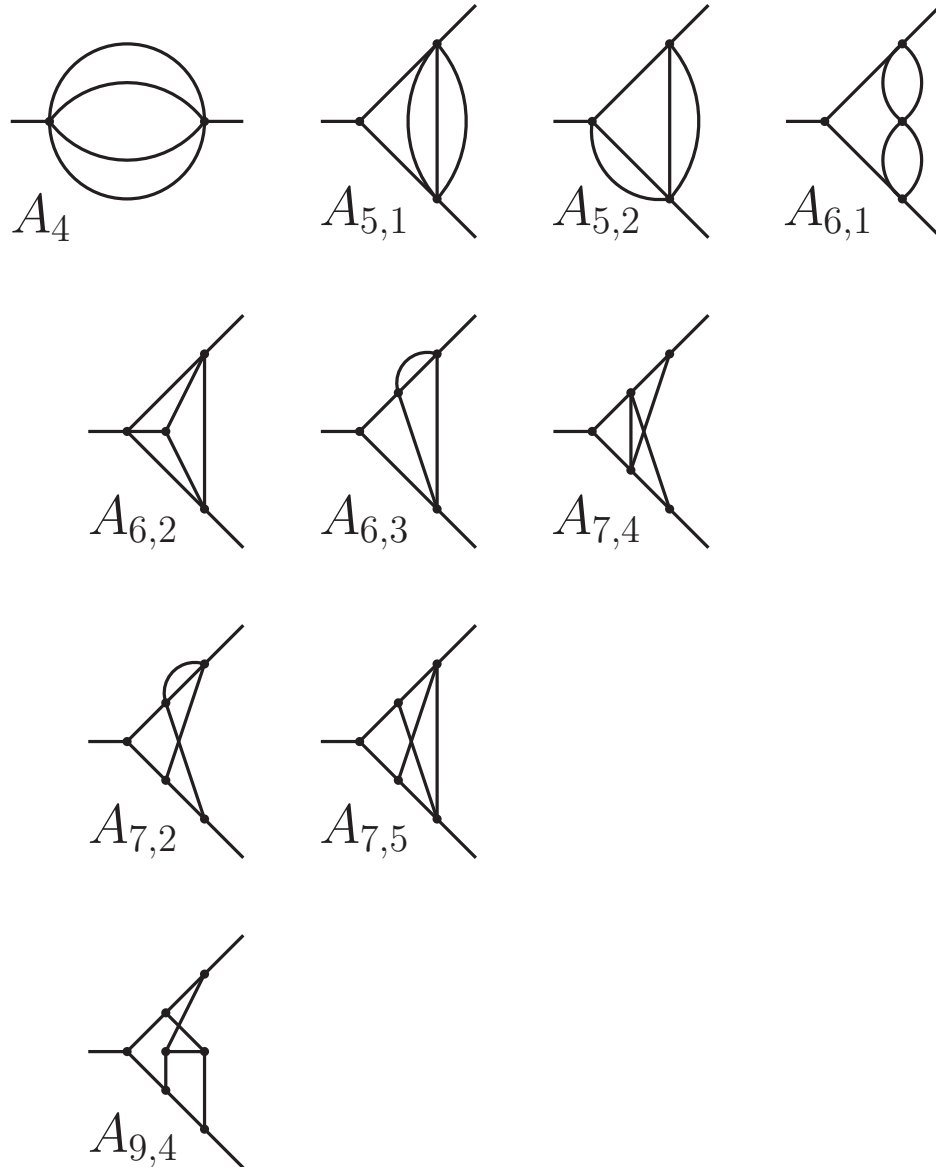


$A_{9,2}$



$A_{9,4}$

$A_{9,4}$ and lower master integrals



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For $A_{6,3}$, there is only one lower master integral, A_4 .

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- Obtain analytic results using PSLQ

Graph $\Gamma \rightarrow$ dimensionally regularized Feynman integral

$$F_{\Gamma}(a_1, \dots, a_L; d) = \frac{e^{i\pi(a+h(1-d/2))/2} \pi^{hd/2}}{\prod_l \Gamma(a_l)} \\ \times \int_0^{\infty} d\alpha_1 \dots \int_0^{\infty} d\alpha_L \prod_l \alpha_l^{a_l-1} \mathcal{U}^{-d/2} e^{i\mathcal{V}/\mathcal{U} - i \sum m_i^2 \alpha_i},$$

where $a = \sum a_i$

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For a Feynman integral with $1/(m^2 - k^2 - i0)^{a_l}$ propagators,

$$\mathcal{U} = \sum_{\text{trees } T} \prod_{l \notin T} \alpha_l,$$

$$\mathcal{V} = \sum_{\text{2-trees } T} \prod_{l \notin T} \alpha_l (q^T)^2.$$

Dimensional recurrence relation

[O. Tarasov'96]

$$d \rightarrow d - 2$$

$$\mathcal{U}^{-d/2} \rightarrow \mathcal{U}^{-(d-2)/2} = \mathcal{U}\mathcal{U}^{-d/2}$$

Inserting $\alpha_l \rightarrow (-ia_l)\mathbf{1}^+$

$$F_{\Gamma}(a_1 \dots, a_L; d - 2) = \frac{1}{\pi} \sum_T \prod_{l \notin T} a_l \mathbf{1}^+ F_{\Gamma}(a_1 \dots, a_L; d)$$

For $F = A_{6,3}$:

$$\begin{aligned} F(1, 1, 1, 1, 1, 1; d - 2) &= \frac{1}{\pi} \{ F(1, 1, 1, 2, 2, 2; d) + F(1, 1, 2, 1, 2, 2; d) \\ &+ F(1, 1, 2, 2, 1, 2; d) + F(1, 2, 1, 2, 1, 2; d) + F(1, 2, 1, 2, 2, 1; d) \\ &+ F(1, 2, 2, 1, 1, 2; d) + F(1, 2, 2, 1, 2, 1; d) + F(1, 2, 2, 2, 1, 1; d) \\ &+ F(2, 1, 1, 2, 1, 2; d) + F(2, 1, 1, 2, 2, 1; d) + F(2, 1, 2, 1, 1, 2; d) \\ &+ F(2, 1, 2, 1, 2, 1; d) + F(2, 1, 2, 2, 1, 1; d) \} \end{aligned}$$

The integrals with raised indices are reduced to master integrals by IBP relations

FIRE [A. Smirnov] and R.Lee's code

$A_{6,3}$ is of complexity level 1.

$$\begin{aligned} A_{6,3}(d-2) &= \frac{8(-3+d)(-9+2d)(-7+2d)(-10+3d)}{-16+3d} A_{6,3}(d) \\ &+ \frac{32(-3+d)(-9+2d)(-7+2d)(-5+2d)(-10+3d)}{(-5+d)(-4+d)^2(-16+3d)} \\ &\times (-8+3d)(-32+7d) A_4(d) \end{aligned}$$

To solve this difference relation turn to $g(d) = \Sigma(d)A_{6,3}(d)$,
where

$$\Sigma(d) = \frac{1}{\sqrt{\pi}} 32^{4d - \frac{5}{2}} \left(\frac{d}{2} - \frac{5}{3}\right) \sin\left(\frac{1}{2}\pi(d - 5)\right) \sin\left(\frac{1}{2}\pi\left(d - \frac{14}{3}\right)\right) \\ \times \sin^2\left(\frac{\pi d}{2}\right) \Gamma\left(\frac{d}{2} - \frac{5}{4}\right) \Gamma\left(\frac{d}{2} - \frac{3}{4}\right) \Gamma\left(\frac{d}{2} - \frac{1}{2}\right)$$

Equation for g :

$$g(d - 2) = g(d) + h(d)$$

with

$$h(d) = \frac{1}{3(d - 5)} 1024\pi(32 - 7d) \sin\left(\frac{\pi d}{2}\right) \sin(\pi d) \\ \times \cos\left(\frac{1}{6}(3\pi d + \pi)\right) \Gamma\left(7 - \frac{3d}{2}\right) \Gamma\left(\frac{d}{2} - 2\right)^3$$

$$\begin{aligned}
g(d) &= g(d-2) - r(d) \\
&= g(d-4) - r(d) - r(d-2) \\
&= \dots \\
&= g(-\infty) - \sum_{k=0}^{\infty} h(d-2k) \\
&= g(+\infty) + \sum_{k=1}^{\infty} h(d+2k)
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$r_+(d) = -\sum_{k=0}^{\infty} h(d-2k)$ and $r_-(d) = +\sum_{k=1}^{\infty} h(d+2k)$ are solutions.

$$\xi r_+(d) + (1 - \xi)r_-(d)$$

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However, $r_-(d)$ is a divergent series (for $A_{6,3}$).

$r_+(d)$ converges very well because

$$h(d + 2(k + 1))/h(d + 2k) \rightarrow -\frac{1}{27}, \quad k \rightarrow \infty$$

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The general solution:

$$g(d) = r_+(d) + \omega(d)$$

where $\omega(d) = \Omega(e^{i\pi d})$ is a periodic function of period 2.

The analytic structure of $g(d)$ follows from the analytic properties of $A_{6,3}(d)$. They can be obtained by

FIESTA

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`SDAnalyze [U, F, h, degrees, dmin, dmax]`

where U and F are the basic functions, h is the number of loops, `degrees` are the indices, and `dmin` and `dmax` are values of the real part of d that determine the basic stripe.

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For $A_{6,3}$, **FIESTA** says that in the band $(3, 5]$ there can be simple poles at $d = 10/3, 4, 14/3, 5$.

Analyzing behaviour at infinity ($\text{Im}d \rightarrow \pm\infty$) with the help of the alpha representation provides the following Ansatz for $\Omega(z)$, with $z = e^{i\pi d}$:

$$a_0 + \frac{a_1}{z - e^{-2i\pi/3}} + \frac{a_2}{z - 1}$$

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which corresponds to the following Ansatz for $\omega(d)$:

$$b_0 + b_1 \cot \left(\frac{1}{2} \pi (d - 4) \right) + b_2 \cot \left(\frac{1}{2} \pi \left(d - \frac{10}{3} \right) \right)$$

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The constants b_i are fixed by using an MB representation for $A_{6,3}$

$$512\pi^4 3^{-1/2} \cot \left(\frac{1}{2}\pi \left(d - \frac{10}{3} \right) \right) - 512\pi^4 3^{-1/2} \cot \left(\frac{1}{2}\pi(d - 4) \right)$$

Result:

$$A_{6,3}(d) = S(d) \sum_{k=0}^{\infty} h(d + 2k) + g(d),$$

$$S(d) = -\sin(\pi d)g(d) = \frac{\pi^4 2^{11-3d} \operatorname{csc}\left(\frac{3\pi d}{2}\right) \operatorname{csc}\left(\frac{\pi d}{2}\right)}{(3d-10)\Gamma\left(d-\frac{5}{2}\right)\Gamma\left(\frac{d-1}{2}\right)},$$

$$h(d) = \frac{(7d-18)\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)^3}{3\pi^2(d-3)\Gamma\left(\frac{3d}{2}-3\right)}.$$

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[H.R.P. Ferguson & D.H. Bailey'91]

is an algorithm that provides the best approximation of a given numerical number as a linear combination, with rational coefficients, of a given set of transcendental numbers.

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Available in `Mathematica 8`

`FindIntegerNullVector[]`

PSLQ gives

$$\begin{aligned}
 A_{6,3}(4-2\epsilon) = & \frac{e^{-3\gamma\epsilon}}{(1-4\epsilon)(1-3\epsilon)(1-2\epsilon)} \left\{ \frac{1}{6\epsilon^3} + \frac{\pi^2}{8\epsilon} - \frac{35\zeta_3}{6} \right. \\
 & - \frac{77\pi^4\epsilon}{2880} - \epsilon^2 \left(\frac{49\pi^2\zeta_3}{24} + \frac{651\zeta_5}{10} \right) + \epsilon^3 \left(\frac{1141\zeta_3^2}{12} - \frac{93451\pi^6}{725760} \right) \\
 & - \epsilon^4 \left(\frac{713\pi^2\zeta_5}{40} - \frac{511}{320}\pi^4\zeta_3 + \frac{9017\zeta_7}{14} \right) + \epsilon^5 \left(\frac{623}{48}\pi^2\zeta_3^2 \right. \\
 & \left. - \frac{544}{9}\zeta_{-6,-2} + \frac{11195\zeta_5\zeta_3}{6} - \frac{2022493\pi^8}{11612160} \right) + O(\epsilon^6) \left. \right\},
 \end{aligned}$$

After $A_{6,3}$ turn to $A_{7,2}$.

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The dimensional recurrence relation gives

$$A_{7,2}(d+2) = c_{7,2}(d)A_{7,2}(d) \\ + c_{6,3}(d)A_{6,3}(d) + c_{5,2}(d)A_{5,2}(d) + c_{5,1}(d)A_{5,1}(d) + c_4(d)A_4(d)$$

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where c_n are rational functions.

Turn to $\tilde{A}_{7,2}(d) = \Sigma(d)A_{7,2}(d)$

where the summing factor $\Sigma(d)$ is chosen as

$$\Sigma(d) = \frac{(d-3) \cos\left(\frac{\pi d}{2}\right) \cos\left(\frac{\pi}{6} - \frac{\pi d}{2}\right) \cos\left(\frac{\pi d}{2} + \frac{\pi}{6}\right) \Gamma\left(\frac{5d}{2} - 9\right)}{\Gamma\left(\frac{d}{2} - 2\right)^2}.$$

For $\tilde{A}_{7,2}(d)$, the recurrence relation is simpler:

$$\tilde{A}_{7,2}(d+2) = \tilde{A}_{7,2}(d) + \tilde{A}_{6,3}(d) + \tilde{A}_{5,2}(d) + \tilde{A}_{5,1}(d) + \tilde{A}_4(d),$$

where $\tilde{A}_n(d) = \sum (d+2)c_n(d)A_n(d)$.

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where $\tilde{A}_n(d) = \Sigma(d+2)c_n(d)A_n(d)$.

The general solution:

$$\begin{aligned} \tilde{A}_{7,2}(d) = & \sum_{l=0}^{\infty} \left[\tilde{A}_{5,2}(d-2-2l) + \tilde{A}_{5,1}(d-2-2l) + \tilde{A}_{6,3}^2(d-2-2l) \right] \\ - & \sum_{l=0}^{\infty} \tilde{A}_{6,3}^{1,1}(d+2l) \sum_{k=0}^{\infty} A_{6,3}^{1,2}(d+2l+2k) - \sum_{l=0}^{\infty} \tilde{A}_4(d+2l) + \omega(z), \end{aligned}$$

where $z = \exp[i\pi d]$.

To fix $\omega(z)$, we need information about analytical properties in the basic stripe chosen as $S = \{d \mid \operatorname{Re} d \in (4, 6]\}$.

For $A_{7,2}$, **FIESTA** says that there can be simple poles at $d = 14/3, 5, 16/3, 6$

$\omega(z)$ is fixed up to the function

$$a_1 + a_2 \cot \left(\frac{\pi}{2}(d - 6) \right)$$

To fix the two constants, an MB representation can be used

$$\begin{aligned}
 A_{7,2}(d) &= \frac{\Gamma\left(\frac{d}{2} - 2\right) \Gamma\left(\frac{d}{2} - 1\right)^2 \Gamma(d - 3)}{\Gamma(d - 2) \Gamma\left(\frac{3d}{2} - 5\right) \Gamma(2d - 7)} \frac{1}{(2\pi i)^2} \int \int \frac{\Gamma(-z_1) \Gamma(-z_2)}{\Gamma(d - z_1 - 4)} \\
 &\times \frac{\Gamma\left(\frac{d}{2} - z_1 - 2\right)}{\Gamma\left(\frac{3d}{2} - z_1 - 5\right)} \Gamma\left(\frac{3d}{2} - z_2 - 6\right) \Gamma(z_1 + z_2 + 1) \Gamma(d - z_1 - z_2 - 5) \\
 &\times \Gamma(z_2 + 1)^2 \Gamma\left(\frac{3d}{2} - z_1 - z_2 - 6\right) \Gamma\left(-\frac{3d}{2} + z_1 + z_2 + 7\right) dz_1 dz_2 .
 \end{aligned}$$

To fix the two constants, an MB representation can be used

$$\begin{aligned}
 A_{7,2}(d) &= \frac{\Gamma\left(\frac{d}{2} - 2\right) \Gamma\left(\frac{d}{2} - 1\right)^2 \Gamma(d - 3)}{\Gamma(d - 2) \Gamma\left(\frac{3d}{2} - 5\right) \Gamma(2d - 7)} \frac{1}{(2\pi i)^2} \int \int \frac{\Gamma(-z_1) \Gamma(-z_2)}{\Gamma(d - z_1 - 4)} \\
 &\quad \times \frac{\Gamma\left(\frac{d}{2} - z_1 - 2\right)}{\Gamma\left(\frac{3d}{2} - z_1 - 5\right)} \Gamma\left(\frac{3d}{2} - z_2 - 6\right) \Gamma(z_1 + z_2 + 1) \Gamma(d - z_1 - z_2 - 5) \\
 &\quad \times \Gamma(z_2 + 1)^2 \Gamma\left(\frac{3d}{2} - z_1 - z_2 - 6\right) \Gamma\left(-\frac{3d}{2} + z_1 + z_2 + 7\right) \mathbf{d}z_1 \mathbf{d}z_2 .
 \end{aligned}$$

$$A_{7,2}(6 - 2\epsilon) = -\frac{41}{15552\epsilon} + O(\epsilon^0), \quad A_{7,2}(5 - 2\epsilon) = -\frac{\pi^{5/2}}{24\epsilon} + O(\epsilon^0)$$

$$\begin{aligned}
\omega(z) = & \frac{\pi^3}{20\sqrt{5}} \tan\left(\frac{\pi}{10} - \frac{\pi d}{2}\right) - \frac{\pi^3}{36} \tan\left(\frac{\pi}{6} - \frac{\pi d}{2}\right) \\
& - \frac{\pi^3}{20\sqrt{5}} \tan\left(\frac{\pi d}{2} + \frac{\pi}{10}\right) + \frac{\pi^3}{36} \tan\left(\frac{\pi d}{2} + \frac{\pi}{6}\right) \\
& + \frac{\pi^3}{60} \cot^3\left(\frac{\pi d}{2}\right) + \frac{13\pi^3}{180} \cot\left(\frac{\pi d}{2}\right) \\
& + \frac{\pi^3}{20\sqrt{5}} \cot\left(\frac{\pi}{5} - \frac{\pi d}{2}\right) - \frac{\pi^3}{20\sqrt{5}} \cot\left(\frac{\pi d}{2} + \frac{\pi}{5}\right) .
\end{aligned}$$

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\omega(z) = & \frac{\pi^3}{20\sqrt{5}} \tan\left(\frac{\pi}{10} - \frac{\pi d}{2}\right) - \frac{\pi^3}{36} \tan\left(\frac{\pi}{6} - \frac{\pi d}{2}\right) \\
& - \frac{\pi^3}{20\sqrt{5}} \tan\left(\frac{\pi d}{2} + \frac{\pi}{10}\right) + \frac{\pi^3}{36} \tan\left(\frac{\pi d}{2} + \frac{\pi}{6}\right) \\
& + \frac{\pi^3}{60} \cot^3\left(\frac{\pi d}{2}\right) + \frac{13\pi^3}{180} \cot\left(\frac{\pi d}{2}\right) \\
& + \frac{\pi^3}{20\sqrt{5}} \cot\left(\frac{\pi}{5} - \frac{\pi d}{2}\right) - \frac{\pi^3}{20\sqrt{5}} \cot\left(\frac{\pi d}{2} + \frac{\pi}{5}\right).
\end{aligned}$$

A result is then expressed in terms of a double series with factorized expressions. Calculating it with very high precision and using PSLQ [H.R.P. Ferguson & D.H. Bailey'91]

$$\begin{aligned}
A_{9,4}(4-2\epsilon) &= e^{-3\gamma_E\epsilon} \left\{ -\frac{1}{9\epsilon^6} - \frac{8}{9\epsilon^5} + \left[1 + \frac{43\pi^2}{108} \right] \frac{1}{\epsilon^4} \right. \\
&+ \left[\frac{109\zeta(3)}{9} + \frac{14}{9} + \frac{53\pi^2}{27} \right] \frac{1}{\epsilon^3} \\
&+ \left[\frac{608\zeta(3)}{9} - 17 - \frac{311\pi^2}{108} - \frac{481\pi^4}{12960} \right] \frac{1}{\epsilon^2} \\
&+ \left. \left[-\frac{949\zeta(3)}{9} - \frac{2975\pi^2\zeta(3)}{108} + \frac{3463\zeta(5)}{45} + 84 + \frac{11\pi^2}{18} + \frac{85\pi^4}{108} \right] \frac{1}{\epsilon} \right.
\end{aligned}$$

[P. Baikov, K. Chetyrkin, A. and V. Smirnovs, & M. Steinhauser'09]

[G. Heinrich, T. Huber, D. Kosower and V. Smirnov'09]

$$\begin{aligned}
&+ \left[\frac{434\zeta(3)}{9} - \frac{299\pi^2\zeta(3)}{3} - \frac{3115\zeta_3^2}{6} + \frac{7868\zeta(5)}{15} - 339 \right. \\
&+ \left. \frac{77\pi^2}{4} - \frac{2539\pi^4}{2592} - \frac{247613\pi^6}{466560} \right] \text{ [R. Lee, A. and V. Smirnovs'10]}
\end{aligned}$$

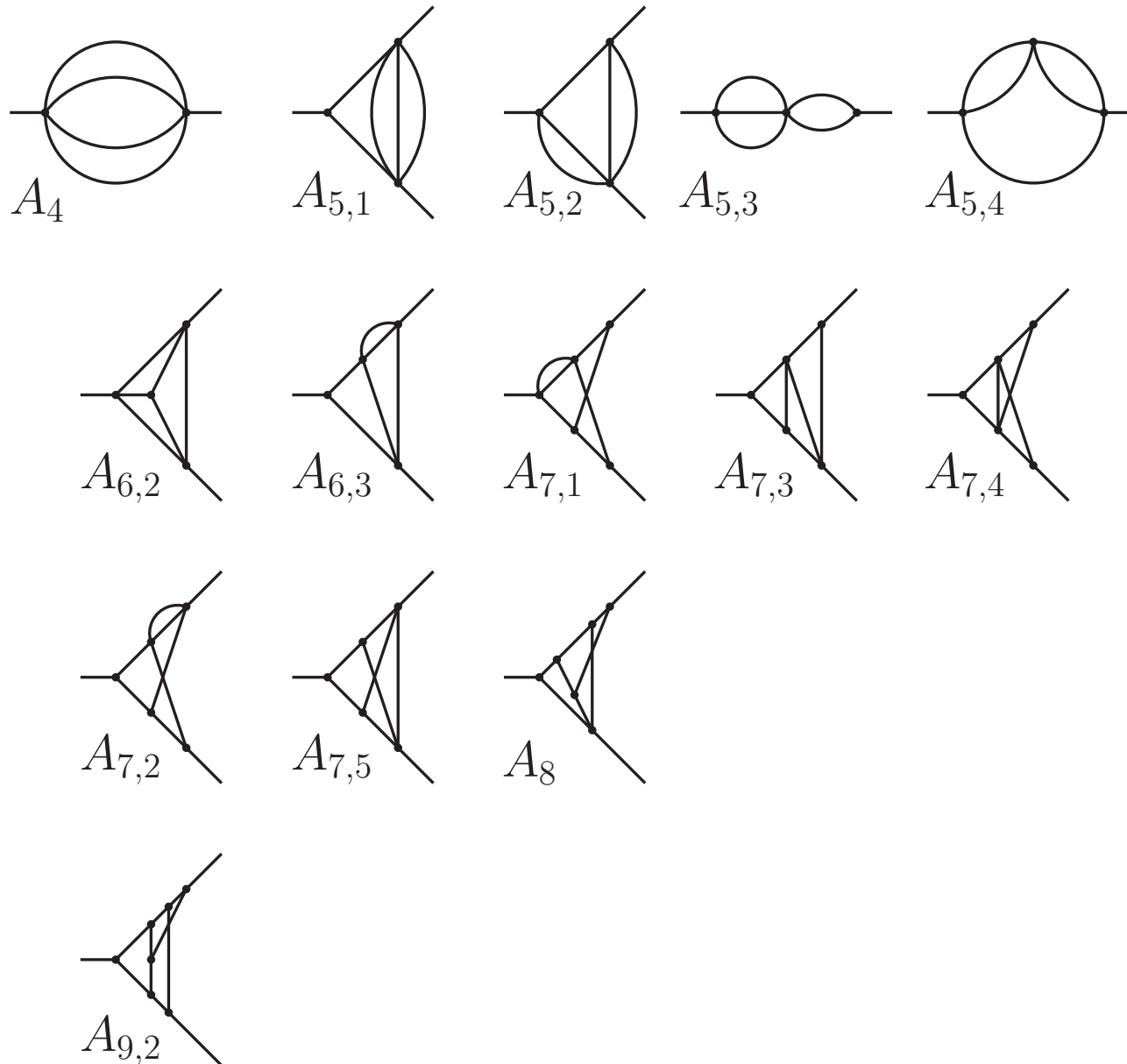
$$\begin{aligned}
& + \left[-1242 + 112\pi^2 - 589\zeta_3 + \frac{487\pi^4}{432} - \frac{19499\pi^2\zeta_3}{108} + \frac{30067\zeta_5}{45} + \frac{25567\pi^6}{30240} \right. \\
& + \left. \frac{18512\zeta_3^2}{9} + \frac{38903\pi^4\zeta_3}{2592} + \frac{113629\pi^2\zeta_5}{540} - \frac{8564\zeta_7}{63} \right] \epsilon \\
& + \left[4293 - \frac{1887\pi^2}{4} + 3756\zeta_3 - \frac{491\pi^4}{32} + \frac{4019\pi^2\zeta_3}{18} + \frac{7874\zeta_5}{15} \right. \\
& - \frac{9901847\pi^6}{3265920} - \frac{26291\zeta_3^2}{6} + \frac{9037\pi^4\zeta_3}{135} + \frac{35728\pi^2\zeta_5}{45} - \frac{72537\zeta_7}{14} \\
& \left. + \frac{30535087\pi^8}{31352832} - \frac{152299}{216}\pi^2\zeta_3^2 + \frac{730841\zeta_3\zeta_5}{135} - \frac{76288}{81}\zeta_{-6,-2} \right] \epsilon^2 + O(\epsilon^3) \Big\}
\end{aligned}$$

[R. Lee and V. Smirnov'10]

$$\zeta_m = \zeta(m), \quad \zeta_{m_1, m_2} = \zeta(m_1, m_2)$$

$$\zeta(m_1, \dots, m_k) = \sum_{i_1=1}^{\infty} \sum_1^{i_1-1} \cdots \sum_1^{i_{k-1}-1} \prod_{j=1}^k \frac{\text{sgn}(m_j)^{i_j}}{i_j^{|m_j|}}.$$

$A_{9,2}$ and lower master integrals



$$\begin{aligned}
A_{9,2}(4-2\epsilon) = & e^{-3\gamma_E\epsilon} \left\{ -\frac{2}{9\epsilon^6} - \frac{5}{6\epsilon^5} + \left[\frac{20}{9} + \frac{17\pi^2}{54} \right] \frac{1}{\epsilon^4} \right. \\
& + \left[\frac{31\zeta(3)}{3} - \frac{50}{9} + \frac{181\pi^2}{216} \right] \frac{1}{\epsilon^3} \\
& + \left[\frac{347\zeta(3)}{18} + \frac{110}{9} - \frac{17\pi^2}{9} + \frac{119\pi^4}{432} \right] \frac{1}{\epsilon^2} \\
& \left. + \left[-\frac{514\zeta(3)}{9} - \frac{341\pi^2\zeta(3)}{36} + \frac{2507\zeta_5}{15} - \frac{170}{9} + \frac{19\pi^2}{6} + \frac{163\pi^4}{960} \right] \frac{1}{\epsilon} \right.
\end{aligned}$$

[P. Baikov, K. Chetyrkin, A. and V. Smirnovs, & M. Steinhauser'09]

[G. Heinrich, T. Huber, D. Kosower and V. Smirnov'09]

$$\begin{aligned}
& + \left[\frac{1516\zeta(3)}{9} - \frac{737\pi^2\zeta(3)}{24} - 29\zeta(3)^2 + \frac{2783\zeta(5)}{6} - \frac{130}{9} \right. \\
& \left. + \frac{\pi^2}{2} - \frac{943\pi^4}{1080} + \frac{195551\pi^6}{544320} \right]
\end{aligned}$$

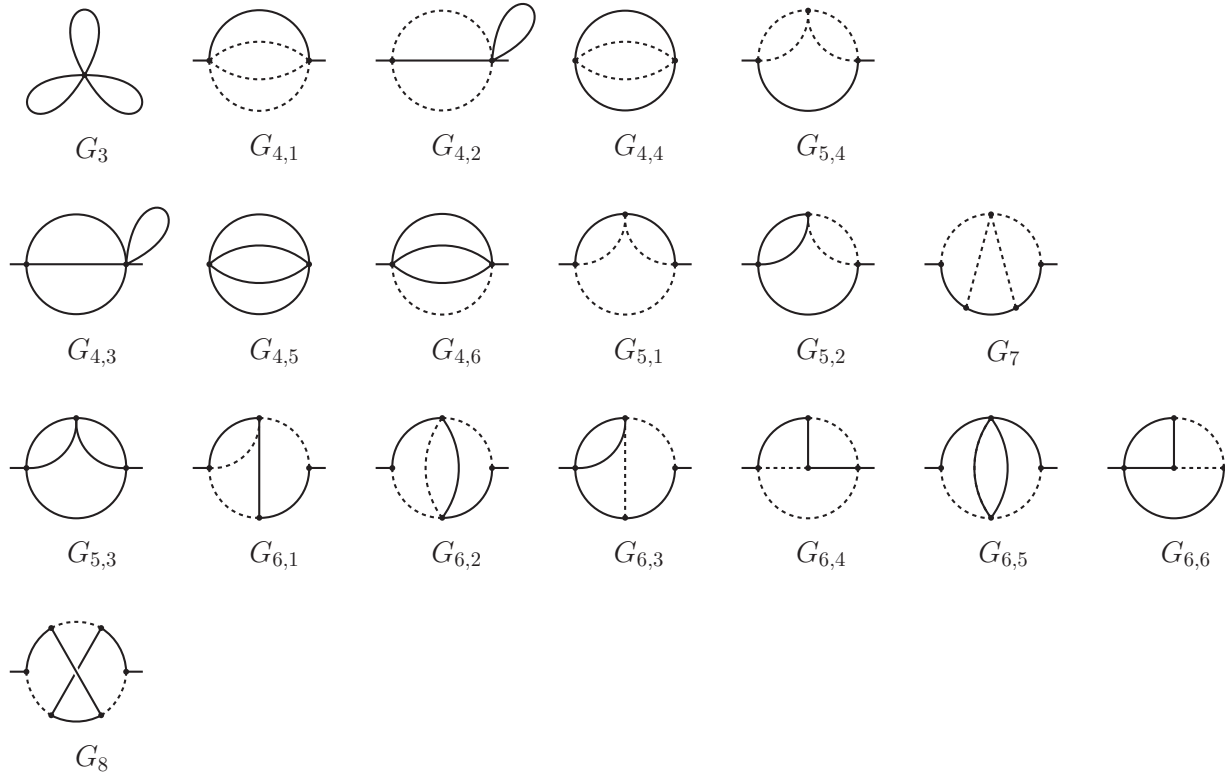
[R. Lee, A. and V. Smirnovs'10]

$$\begin{aligned}
& + \left[\frac{2950}{9} - \frac{83\pi^2}{2} - \frac{4444\zeta_3}{9} + \frac{8801\pi^4}{2160} + \frac{1357\pi^2\zeta_3}{18} - \frac{2830\zeta_5}{3} \right. \\
& + \left. \frac{2416889\pi^6}{2177280} + \frac{19169\zeta_3^2}{36} + \frac{5963\pi^4\zeta_3}{4320} - \frac{8183\pi^2\zeta_5}{60} + \frac{43329\zeta_7}{14} \right] \epsilon \\
& + \left[-\frac{19090}{9} + \frac{569\pi^2}{2} + \frac{12916\zeta_3}{9} - \frac{7795\pi^4}{432} - \frac{1433\pi^2\zeta_3}{9} + \frac{3112\zeta_5}{3} \right. \\
& - \frac{64733\pi^6}{54432} - \frac{1214\zeta_3^2}{9} + \frac{58517\pi^4\zeta_3}{1728} - \frac{146521\pi^2\zeta_5}{360} + \frac{580805\zeta_7}{42} \\
& \left. + \frac{24178127\pi^8}{14515200} + \frac{6419}{36}\pi^2\zeta_3^2 - \frac{101288\zeta_3\zeta_5}{15} - \frac{20752}{9}\zeta_{-6,-2} \right] \epsilon^2 + O(\epsilon^3) \Big\}
\end{aligned}$$

$$\begin{aligned}
A_{9,1}(4-2\epsilon) = e^{-3\gamma_E\epsilon} & \left\{ \frac{1}{18\epsilon^5} - \frac{1}{2\epsilon^4} + \frac{1}{\epsilon^3} \left(\frac{53}{18} + \frac{29\pi^2}{216} \right) \right. \\
& + \frac{1}{\epsilon^2} \left(\frac{35\zeta_3}{18} - \frac{29}{2} - \frac{149\pi^2}{216} \right) + \frac{1}{\epsilon} \left(-\frac{307\zeta_3}{18} + \frac{129}{2} + \frac{139\pi^2}{72} + \frac{5473\pi^4}{25920} \right) \\
& + \left(\frac{793\zeta(5)}{10} + \frac{871\pi^2\zeta(3)}{216} + \frac{1153\zeta(3)}{18} - \frac{3125\pi^4}{5184} - \frac{19\pi^2}{8} - \frac{537}{2} \right) \\
& + \epsilon \left(-\frac{287\zeta(3)}{2} + \frac{2969\pi^2\zeta_3}{216} + \frac{5521\zeta(3)^2}{36} - \frac{8251\zeta_5}{30} \right. \\
& \left. + \frac{2133}{2} - \frac{97\pi^2}{8} + \frac{4717\pi^4}{28115} + \frac{761151\pi^6}{186624} \right) \\
& + \epsilon^2 \left(\frac{195\zeta(3)}{2} - \frac{5887\pi^2\zeta(3)}{72} + \frac{138403\pi^4\zeta(3)}{25920} + \frac{799\zeta(3)^2}{4} + \frac{22487\zeta_5}{30} \right. \\
& \left. - \frac{11987\pi^2\zeta(5)}{10115} + \frac{228799\zeta_7}{126} - \frac{8181}{2} + \frac{969\pi^2}{8} - \frac{1333\pi^4}{320} - \frac{4286603\pi^6}{6531840} \right) + \dots \left. \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left[\frac{30537}{2} - \frac{5589\pi^2}{8} + \frac{2685\zeta_3}{2} + \frac{9163\pi^4}{960} + \frac{2047\pi^2\zeta_3}{8} - \frac{14613\zeta_5}{10} \right. \\
& + \frac{9379\pi^6}{16128} - \frac{73819\zeta_3^2}{36} + \frac{125549\pi^4\zeta_3}{25920} + \frac{295667\pi^2\zeta_5}{1080} - \frac{542683\zeta_7}{126} \\
& \left. + \frac{325463531\pi^8}{313528320} - \frac{70259}{432}\pi^2\zeta_3^2 + \frac{38845\zeta_3\zeta_5}{54} - \frac{78272}{81}\zeta_{-6,-2} \right] \epsilon^3 + O(\epsilon^4) \Big\}
\end{aligned}$$

master integrals for 3-loop $g = 2$



[S. Laporta and E. Remiddi'96-97]

[K. Melnikov and T. van Ritbergen'00]

A pure numerical approach to evaluate 4-loop $g - 2$:

[T. Kinoshita et al.'06-10]

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IBP reduction to master integrals and numerical evaluation
of the master integrals:

[S. Laporta and E. Remiddi'08]

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A warm-up before 4-loop calculation: numerical evaluation of 3-loop master integrals to higher orders in ϵ :

[S. Laporta'01]

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[S. Laporta and E. Remiddi'08]

A warm-up before 4-loop calculation: numerical evaluation of 3-loop master integrals to higher orders in ϵ :

[S. Laporta'01]

Analytic evaluation of 3-loop master integrals up to transcendentality weight 7:

[R. Lee and V. Smirnov'10]

$$\begin{aligned}
G_8(4 - 2\epsilon) = & \Gamma(1 + \epsilon)^3 \left\{ \left(-\frac{\pi^4}{6} + 4\pi^2 \ln^2 2 \right) \right. \\
& + \epsilon \left(\frac{2\pi^4}{3} - 16\pi^2 \ln^2 2 + \frac{166}{45}\pi^4 \ln 2 - \frac{208}{9}\pi^2 \ln^3 2 - \frac{32 \ln^5 2}{15} \right. \\
& \left. \left. + 256a_5 + \frac{17\pi^2 \zeta_3}{6} - 291\zeta_5 \right) \right. \\
& + \epsilon^2 \left(-\frac{14\pi^4}{3} + 112\pi^2 \ln^2 2 - \frac{664}{45}\pi^4 \ln 2 + \frac{832}{9}\pi^2 \ln^3 2 + \frac{128 \ln^5 2}{15} \right. \\
& - 1024a_5 - \frac{34\pi^2 \zeta_3}{3} + 1164\zeta_5 - \frac{21743\pi^6}{11340} - \frac{197}{9}\pi^4 \ln^2 2 + \frac{713}{9}\pi^2 \ln^4 2 \\
& \left. \left. + \frac{64 \ln^6 2}{9} - 104\pi^2 a_4 + 5120a_6 + 2688s_6 - 51\pi^2 \ln 2 \zeta_3 - 953\zeta_3^2 \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \epsilon^3 \left(\frac{80\pi^4}{3} - 640\pi^2 \ln^2 2 + \frac{4648}{45}\pi^4 \ln 2 - \frac{5824}{9}\pi^2 \ln^3 2 - \frac{896 \ln^5 2}{15} + 7168a_5 \right. \\
& + \frac{238\pi^2 \zeta_3}{3} - 8148\zeta_5 + \frac{21743\pi^6}{2835} + \frac{788}{9}\pi^4 \ln^2 2 - \frac{2852}{9}\pi^2 \ln^4 2 \\
& - \frac{256 \ln^6 2}{9} + 416\pi^2 a_4 - 20480a_6 - 10752s_6 + 204\pi^2 \ln 2 \zeta_3 + 3812\zeta_3^2 \\
& + \frac{4868}{189}\pi^6 \ln 2 + \frac{8492}{135}\pi^4 \ln^3 2 - \frac{7288}{45}\pi^2 \ln^5 2 - \frac{4864 \ln^7 2}{315} \\
& + 1776\pi^2 \ln 2 a_4 + 1520\pi^2 a_5 + 77824a_7 - \frac{106880}{7} \ln 2 s_6 + \frac{4003\pi^4 \zeta_3}{21} \\
& + \frac{2167}{3}\pi^2 \ln^2 2 \zeta_3 - \frac{316}{3} \ln^4 2 \zeta_3 - 2528a_4 \zeta_3 + \frac{133600}{7} \ln 2 \zeta_3^2 \\
& + \frac{875561\pi^2 \zeta_5}{84} + 37200 \ln^2 2 \zeta_5 - \frac{1325727\zeta_7}{7} + \frac{106880s_{7,a}}{7} \\
& \left. - \frac{161920s_{7,b}}{7} \right) + O(\epsilon^4) \Big\}
\end{aligned}$$

$$a_i = \text{Li}_i \left(\frac{1}{2} \right) ,$$

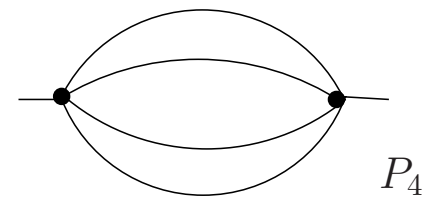
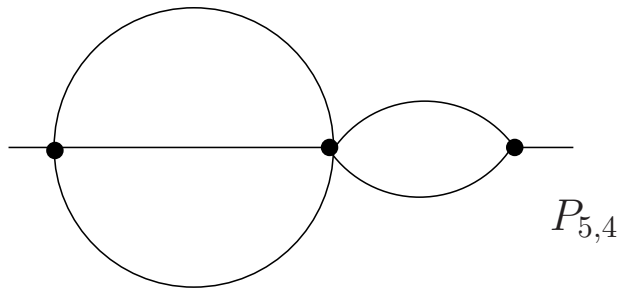
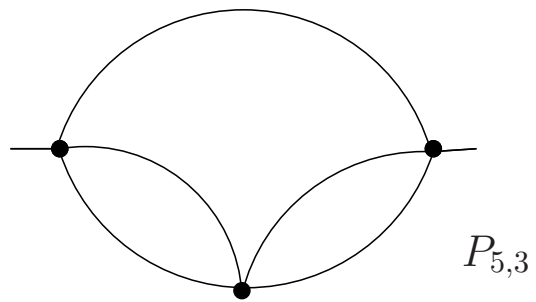
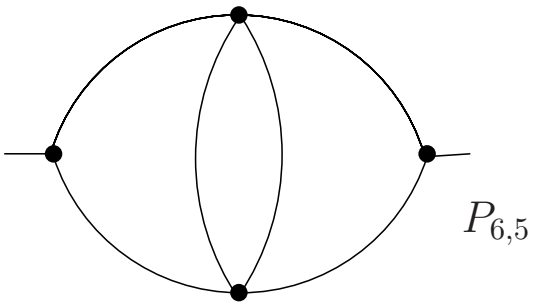
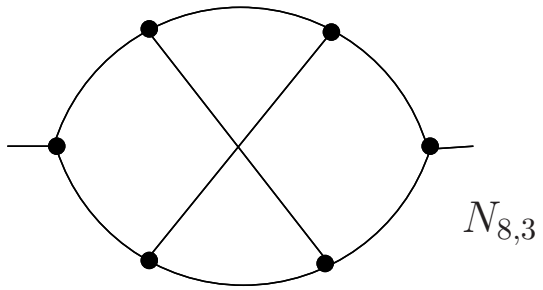
$$s_6 = \zeta(-5, -1) + \zeta(6) ,$$

$$s_{7a} = \zeta(-5, 1, 1) + \zeta(-6, 1) + \zeta(-5, 2) + \zeta(-7) ,$$

$$s_{7b} = \zeta(7) + \zeta(5, 2) + \zeta(-6, -1) + \zeta(5, -1, -1)$$

The coefficients in the ϵ -expansion of planar massless propagator diagrams up to five loops should be expressed in terms of multiple zeta values, while the non-planar graphs may contain, in addition, multiple sums with 6th roots of unity.

[Brown'08]



$$\begin{aligned}
N_{8,3}(d+2) &= \frac{(d-4)}{8(d-2)(d-1)(2d-7)(2d-5)} N_{8,3} \\
&+ \frac{4(5d^2 - 28d + 38)}{(d-4)^2(d-2)(d-1)(2d-5)} P_{5,3} \\
&[4(d-4)(d-2)(d-1)(2d-7)(2d-5)(3d-8)]^{-1} \\
&\times (37d^3 - 313d^2 + 858d - 752) P_{6,5} \\
&- [2(d-4)^2(d-3)(d-2)(d-1)(2d-7)(2d-5)]^{-1} \\
&\times (43d^4 - 478d^3 + 1963d^2 - 3530d + 2352) P_{5,4} \\
&- [(d-4)^3(d-3)^2(d-2)(d-1)(2d-7)(3d-8)]^{-1} \\
&\times (401d^6 - 7251d^5 + 54491d^4 - 217784d^3 \\
&+ 489064d^2 - 581248d + 287232) P_4
\end{aligned}$$

$$\begin{aligned}
& \frac{e^{-3\gamma_E \epsilon}}{1 - 2\epsilon} \left\{ 20\zeta(5) + \epsilon \left(68\zeta(3)^2 + \frac{10\pi^6}{189} \right) \text{ [Chetyrkin, Kataev & Tkachov'80, Kazakov'84]} \right. \\
& \quad + \epsilon^2 \left(\frac{34\pi^4\zeta(3)}{15} - 5\pi^2\zeta_5 + 450\zeta(7) \right) \text{ [Becavac'06]} \\
& \quad + \epsilon^3 \left(-\frac{9072}{5}\zeta(5,3) - 2588\zeta(3)\zeta(5) - 17\pi^2\zeta(3)^2 + \frac{6487\pi^8}{10500} \right) \\
& \quad + \epsilon^4 \left(-\frac{4897\pi^6\zeta_3}{630} - \frac{6068\zeta(3)^3}{3} + \frac{13063\pi^4\zeta_5}{120} - \frac{225\pi^2\zeta(7)}{2} + \frac{88036\zeta(9)}{9} \right) \\
& \quad + \epsilon^5 \left(\frac{2268}{5}\pi^2\zeta(5,3) + 42513\zeta(8,2) - 145328\zeta(3)\zeta(7) \right. \\
& \quad \left. - 73394\zeta(5)^2 + 647\pi^2\zeta(3)\zeta(5) - \frac{11813\pi^4\zeta(3)^2}{120} + \frac{28138577\pi^{10}}{9355500} \right) + \dots \left. \right\}
\end{aligned}$$

Pull out another factor to kill terms with pure π^2

[Chetyrkin, Kataev & Tkachov'80, D. Broadhurst'99]

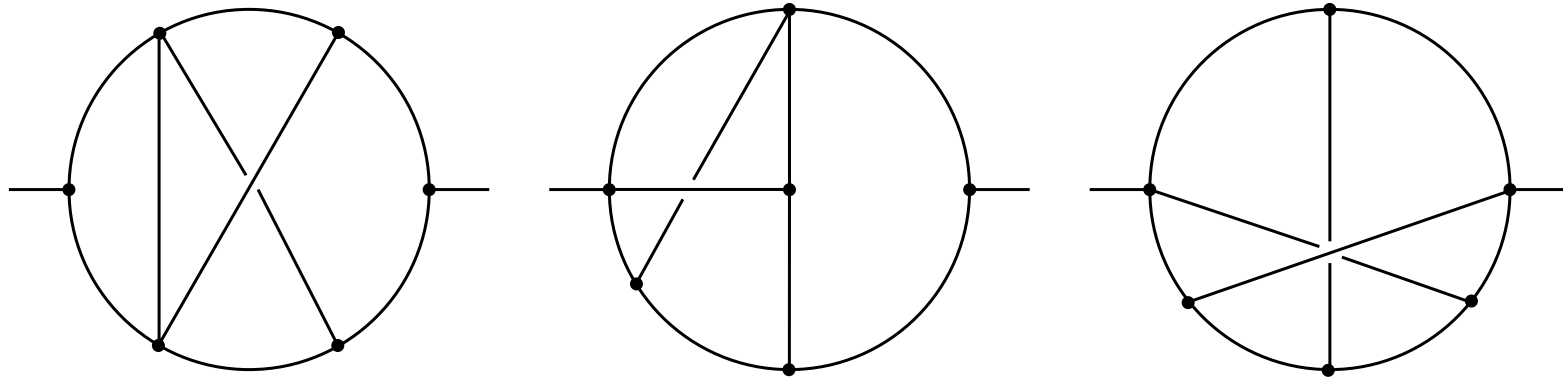
$$\begin{aligned}
 & (1 - 2\epsilon)^2 \left(\frac{\Gamma(1 - \epsilon)^2 \Gamma(1 + \epsilon)}{\Gamma(2 - 2\epsilon)} \right)^3 \left\{ 20\zeta(5) + \epsilon \left(68\zeta(3)^2 + \frac{10\pi^6}{189} \right) \right. \\
 & + \epsilon^2 \left(\frac{34\pi^4 \zeta(3)}{15} + 450\zeta_7 \right) \\
 & + \epsilon^3 \left(-\frac{12072}{5} \zeta(5, 3) - 2448\zeta(3)\zeta(5) + \frac{8519\pi^8}{13500} \right) \\
 & + \epsilon^4 \left(-\frac{1292\pi^6 \zeta_3}{189} - \frac{4640\zeta(3)^3}{3} + \frac{1202\pi^4 \zeta(5)}{5} + \frac{88036\zeta(9)}{9} \right) \\
 & + \epsilon^5 \left(42513\zeta(8, 2) - 142178\zeta(3)\zeta(7) - 73022\zeta_5^2 \right. \\
 & \quad \left. - \frac{232\pi^4 \zeta(3)^2}{3} + \frac{593053\pi^{10}}{187120} \right) + \dots \left. \right\}
 \end{aligned}$$

Pull out another factor to kill terms with pure π^2 [Kazakov'84]

$$\begin{aligned}
 & \frac{1}{(1-2\epsilon)\Gamma(1-\epsilon)^3} \left\{ 20\zeta(5) + \epsilon \left(68\zeta(3)^2 + \frac{10\pi^6}{189} \right) \right. \\
 & \quad + \epsilon^2 \left(\frac{34\pi^4\zeta(3)}{15} + 450\zeta_7 \right) \\
 & \quad + \epsilon^3 \left(-\frac{9072}{5}\zeta(5,3) - 2568\zeta(3)\zeta(5) + \frac{8519\pi^8}{13500} \right) \\
 & \quad + \epsilon^4 \left(-\frac{1352\pi^6\zeta(3)}{189} - \frac{5864\zeta(3)^3}{3} + \frac{542\pi^4\zeta(5)}{5} + \frac{88036\zeta(9)}{9} \right) \\
 & \quad + \epsilon^5 \left(42513\zeta(8,2) - 144878\zeta(3)\zeta(7) \right. \\
 & \quad \quad \left. - 73382\zeta(5)^2 - \frac{1466\pi^4\zeta(3)^2}{15} + \frac{592063\pi^{10}}{187110} \right) + \dots \left. \right\}
 \end{aligned}$$

F. Brown (2008):

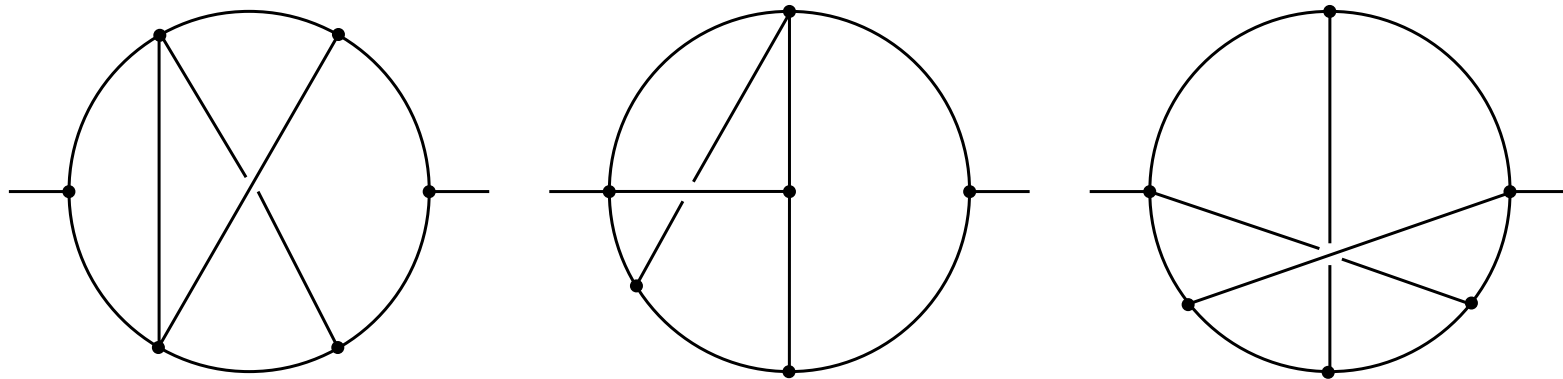
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$$\text{Li}_{n_1, \dots, n_r}(x_1, \dots, x_r) = \sum_{0 < k_1 < \dots < k_r} \frac{x_1^{k_1} \dots x_r^{k_r}}{k_1^{n_1} \dots k_r^{n_r}}$$

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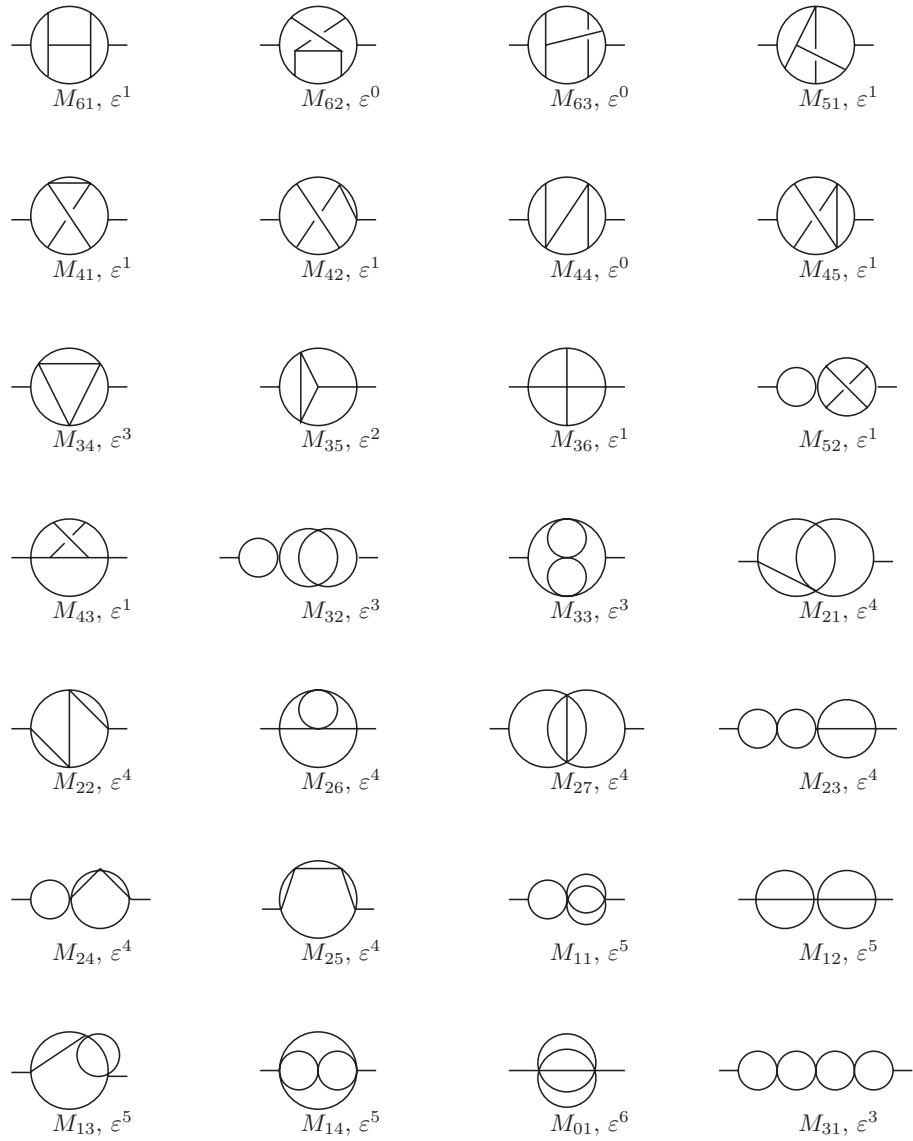
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ϵ -expansions of four-loop master integrals for massless propagators up to weight 7

[P.A. Baikov & K.G. Chetyrkin'10]



$$\begin{aligned}
M_{4,4} = & (1 - 2\epsilon)^3 \left(\frac{\Gamma(1 - \epsilon)^2 \Gamma(1 + \epsilon)}{\Gamma(2 - 2\epsilon)} \right)^3 \left\{ \frac{441\zeta_7}{8} + \epsilon \left(-216\zeta_3\zeta_5 + \frac{5733\zeta_8}{16} - \frac{81\zeta_{2,6}}{2} \right) \right. \\
& + \left(-267\zeta_3^3 - 81\zeta_4\zeta_5 - \frac{675\zeta_3\zeta_6}{2} + \frac{4583\zeta_9}{2} \right) \epsilon^2 + \left(-\frac{2403}{2}\zeta_3^2\zeta_4 - \frac{502287\zeta_5^2}{56} - \frac{7731\zeta_3\zeta_7}{56} \right. \\
& + \left. \frac{1324935\zeta_{10}}{112} + \frac{18441\zeta_{3,7}}{56} \right) \epsilon^3 + \left(-\frac{24315}{2}\zeta_3^2\zeta_5 - \frac{358023\zeta_5\zeta_6}{8} + \frac{139401\zeta_4\zeta_7}{8} \right. \\
& - \left. \frac{59895\zeta_3\zeta_8}{4} + \frac{232767\zeta_2\zeta_9}{4} - \frac{402081\zeta_{11}}{32} - \frac{621}{2}\zeta_3\zeta_{2,6} + \frac{6291}{2}\zeta_{2,1,8} \right) \epsilon^4 \\
& - \left(-6023\zeta_3^4 + 6660\zeta_3\zeta_4\zeta_5 - \frac{650997}{7}\zeta_2\zeta_5^2 + 40507\zeta_3^2\zeta_6 - \frac{1323426}{7}\zeta_2\zeta_3\zeta_7 \right. \\
& + \frac{1750957\zeta_5\zeta_7}{2} + \frac{964778\zeta_3\zeta_9}{3} - \frac{104287641323\zeta_{12}}{132672} - 2853\zeta_4\zeta_{2,6} + \frac{48222}{7}\zeta_2\zeta_{3,7} \\
& \left. - \frac{190175\zeta_{3,9}}{6} - 10716\zeta_{2,1,1,8} \right) \epsilon^5 + O(\epsilon^6) \left. \right\}
\end{aligned}$$

[R. Lee, A. and V. Smirnovs'11]

$$\begin{aligned}
M_{6,3} = & \left(\frac{\Gamma(1-\epsilon)^2 \Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)} \right)^3 \frac{(1-2\epsilon)^3}{(1+3\epsilon)(1+4\epsilon)} \left\{ -\frac{5\zeta_5}{\epsilon} - \left(20\zeta_5 + 41\zeta_3^2 + \frac{25\zeta_6}{2} - \frac{161\zeta_7}{2} \right) \right. \\
& + \left(-308\zeta_3^2 - 50\zeta_6 - 123\zeta_3\zeta_4 + 514\zeta_7 + 4862\zeta_3\zeta_5 - \frac{24451\zeta_8}{4} + 1566\zeta_{2,6} \right) \epsilon \\
& + \left(-924\zeta_3\zeta_4 - 1500\zeta_7 + 68636\zeta_3\zeta_5 - \frac{744639\zeta_8}{8} + 23220\zeta_{2,6} + \frac{1526\zeta_3^3}{3} \right. \\
& \left. - 2103\zeta_4\zeta_5 + 4325\zeta_3\zeta_6 + \frac{111709\zeta_9}{36} \right) \epsilon^2 + \left(235200\zeta_3\zeta_5 - \frac{710311\zeta_8}{2} \right. \\
& + 85536\zeta_{2,6} + \frac{22048\zeta_3^3}{3} - 36366\zeta_4\zeta_5 + 55695\zeta_3\zeta_6 + \frac{237103\zeta_9}{12} + 2289\zeta_3^2\zeta_4 \\
& + \frac{1341143\zeta_5^2}{56} + \frac{3816969\zeta_3\zeta_7}{56} - \frac{7815019\zeta_{10}}{112} - \frac{500565\zeta_{3,7}}{56} \left. \right) \epsilon^3 + \left(\frac{61040\zeta_3^3}{3} \right. \\
& - 160416\zeta_4\zeta_5 + 161860\zeta_3\zeta_6 - \frac{460411\zeta_9}{9} + 33072\zeta_3^2\zeta_4 + \frac{60035137\zeta_3\zeta_7}{56} \\
& + \frac{12859479\zeta_5^2}{56} - \frac{134815227\zeta_{10}}{112} - \frac{7724781\zeta_{3,7}}{56} - 453668\zeta_3^2\zeta_5 + \frac{280574047\zeta_{11}}{64} \\
& + \frac{1346777\zeta_5\zeta_6}{6} - \frac{4654793\zeta_4\zeta_7}{8} + 1309878\zeta_3\zeta_8 - 2749211\zeta_2\zeta_9 - 87752\zeta_3\zeta_{2,6} \\
& \left. - 148606\zeta_{2,1,8} \right) \epsilon^4
\end{aligned}$$

$$\begin{aligned}
& + \left(91560\zeta_3^2\zeta_4 + \frac{5319415\zeta_5^2}{14} + \frac{55472425\zeta_3\zeta_7}{14} - \frac{705626\zeta_3^4}{9} \right. \\
& - \frac{7239597\zeta_{3,7}}{14} - 5183674\zeta_3^2\zeta_5 + \frac{3338505\zeta_5\zeta_6}{2} - 1597650\zeta_{2,1,8} - \frac{1752859}{18}\zeta_3^2\zeta_6 \\
& + \frac{62741559\zeta_3\zeta_8}{4} - 29556525\zeta_2\zeta_9 + \frac{2990096591\zeta_{11}}{64} - \frac{29828681054659\zeta_{12}}{2388096} \\
& - \frac{53503881\zeta_4\zeta_7}{8} - \frac{142150835\zeta_{10}}{28} - 1729457\zeta_3\zeta_4\zeta_5 - \frac{25832277}{7}\zeta_2\zeta_5^2 \\
& - \frac{157544998}{21}\zeta_2\zeta_3\zeta_7 + \frac{311533051\zeta_5\zeta_7}{18} + \frac{1792012205\zeta_3\zeta_9}{108} - 1085340\zeta_3\zeta_{2,6} \\
& \left. - 227636\zeta_4\zeta_{2,6} + \frac{1913502}{7}\zeta_2\zeta_{3,7} - \frac{105698899\zeta_{3,9}}{108} - \frac{1275668}{3}\zeta_{2,1,1,8} \right) \epsilon^5 \\
& + O(\epsilon^6) \}
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extra slides

The full color dependence of the 4-loop 4-gluon amplitude in N=4 SUSY YM in terms of 50 4-loop 4-point integrals.

[Bern, Carrasco, Johansson & Roiban'10]

The critical dimension at which the amplitude first diverge. For 4 loops, this is $d=11/2$.

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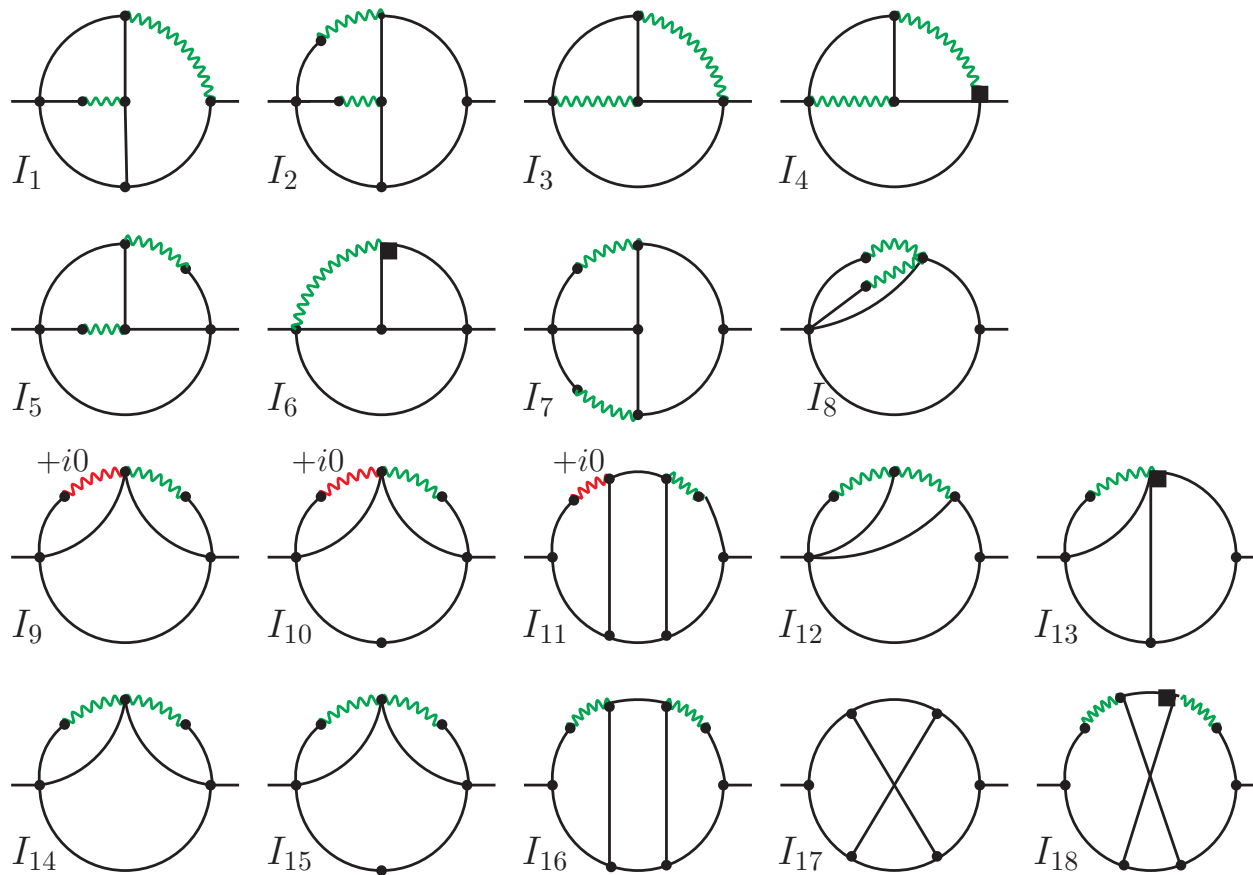
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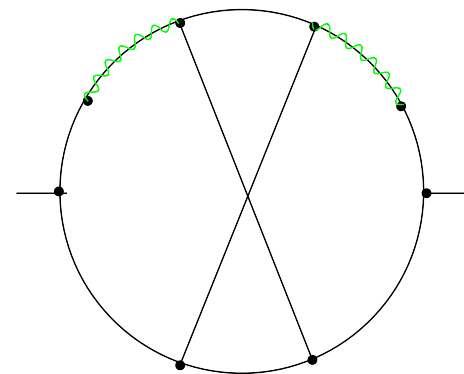
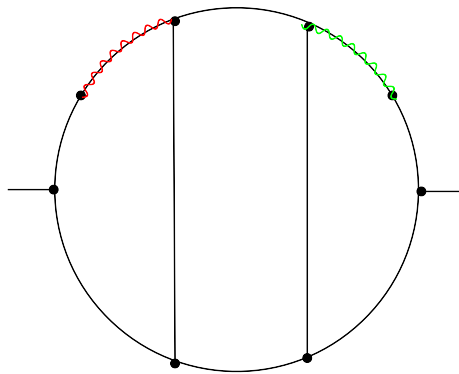
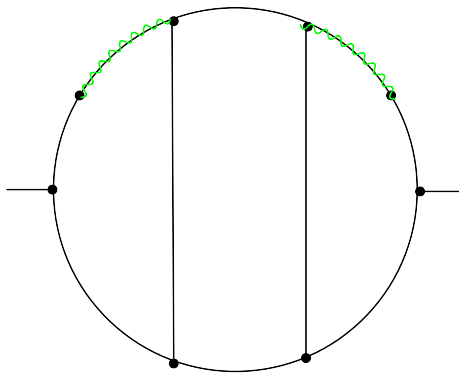
We have

−6.1983992267494959168200925479819368763478987989679152...

Most complicated master integrals for the three-loop static quark potential

[A. and V. Smirnovs & M. Steinhauser'09]





Method of differential equations

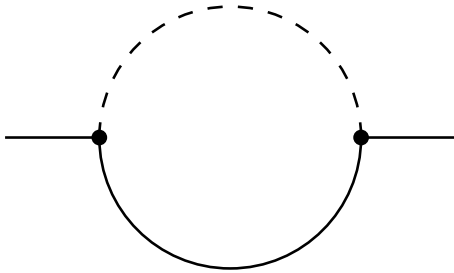
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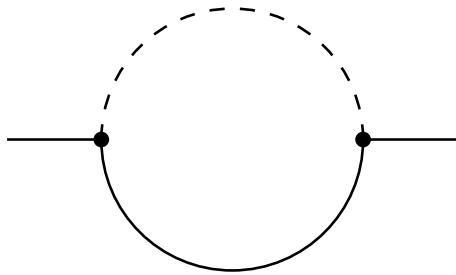


$$F_{\Gamma}(a_1, a_2) = \int \frac{d^d k}{(m^2 - k^2)^{a_1} (-(q - k)^2)^{a_2}}$$

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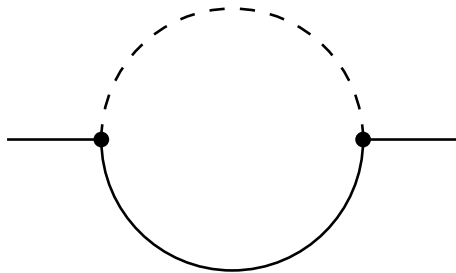
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Two master integrals, $I_1 = F(1, 1)$ and $I_2 = F(1, 0)$.

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Let us evaluate $I_1 = f(m^2)$.

Take a derivative in m^2

$$\frac{\partial}{\partial m^2} \frac{1}{k^2 - m^2} = \frac{1}{(k^2 - m^2)^2} \rightarrow \frac{\partial}{\partial m^2} F(1, 1) = F(2, 1)$$

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We obtain a differential equation

$$\frac{\partial}{\partial m^2} f(m^2) = \frac{1}{m^2 - q^2} \left[(1 - 2\epsilon)f(m^2) - \frac{1 - \epsilon}{m^2} I_2 \right]$$

For y defined by $f(m^2) = i\pi^{d/2}(m^2)^{-\epsilon}y(m^2)$, we have

$$y' - \frac{m^2(1 - \epsilon) - \epsilon q^2}{m^2(m^2 - q^2)}y = -\frac{\Gamma(\epsilon)}{m^2 - q^2}.$$

The solution to the corresponding homogeneous equation is

$$y(m^2) = C(m^2 - q^2)^{1-2\epsilon}(m^2)^{-\epsilon}.$$

The variation of the constant \rightarrow

$$f(m^2) = i\pi^{d/2}(m^2 - q^2)^{1-2\epsilon} \left[-\Gamma(\epsilon) \int_0^{m^2} \frac{dx x^{-\epsilon}}{(x - q^2)^{2-2\epsilon}} + C_1 \right],$$

The constant C_1 can be determined from the boundary value $f(0)$ which is a massless one-loop diagram. This gives

$$f(m^2) = -i\pi^{d/2}(m^2 - q^2)^{1-2\epsilon}\Gamma(\epsilon) \times \left[\int_0^{m^2} \frac{dx x^{-\epsilon}}{(x - q^2)^{2-2\epsilon}} - \frac{\Gamma(1 - \epsilon)^2}{\Gamma(2 - 2\epsilon)(-q^2)^{1-\epsilon}} \right].$$

Turn to an expansion in ϵ and integrate term by term.

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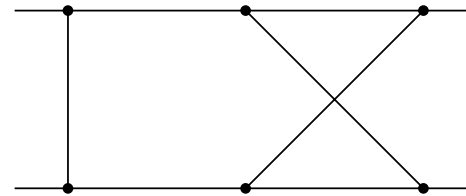
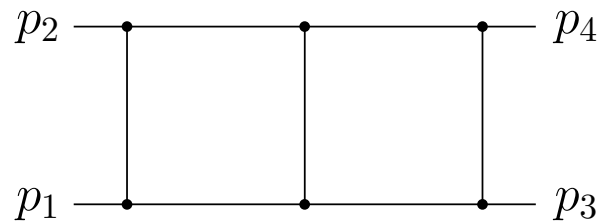
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- Solve DE

The first non-trivial application of the method of differential equations:

massless double boxes with one leg off-shell, $p_1^2 = q^2 \neq 0$,
 $p_i^2 = 0, i = 2, 3, 4$

[T. Gehrmann & E. Remiddi'01]



Systematic evaluation of master integrals by differential equations.

2dHPL