N=4 SYM and N=8 supergravity amplitudes

Lecture 1

Henrik Johansson

School of analytic computing

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Plan

Lecture 1:
- Unitarity method for loop amplitudes
- N=4 super-Yang Mills
- Example of amplitude construction
- Quadruple cut, hepta cut,
- On-shell superspace

Lecture 2:
- Non-planar amplitudes
- N=8 supergravity
- Kawai-Lewellen-Tye relations
- Calculation of UV divergences

Lecture 3:
- Color/Kinematics duality
- Open problems

‘Computing amplitudes’

Computing loop amplitudes, 2 main steps:

1. Express ampl’s in a compact form in terms of a few Feynman integrals

2. Integrate Feynman integrals [see Smirnov’s lectures]

‘Computing amplitudes’

How: express ampl’s in terms of Feynman integrals?

In principle: Use Feyman rules (simple ampl’s)

Practical way: Unitarity method (any theory, any ampl.)

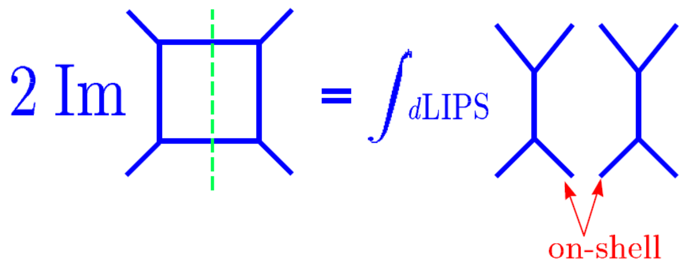
Slick way: ‘guess answer’ using symmetries, dualities, special properties etc. (special ampl’s/theories)

Unitarity Method

Optical theorem:

$$1 = S^\dagger S = (1 - iT^\dagger)(1 + iT)$$

$$2\,\mathrm{Im}\,T = T^\dagger T$$



- Old idea ('60): gives complicated dispersion integrals
- Message: ∃ relations between loop and tree-level ampl's
- New idea: Expand amplitude in terms of an integral basis,

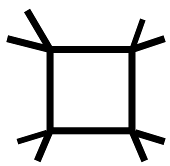
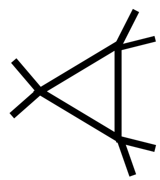
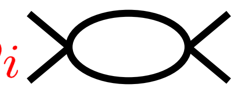
  use relations to fix coefficients [Bern, Dunbar, Dixon, Kosower]

$$A^{loop} = \sum_i c_i I_i$$

Integral basis

- Integral basis can be complete or over-complete
- A complete basis consists of only linearly independent master integrals [see Smirnov's lectures]
- At one loop D=4: scalar boxes, triangles, bubbles, (tadpoles)

$$A_n^{\text{1-loop}} = \sum_i d_i I_4^{(i)} + \sum_i c_i I_3^{(i)} + \sum_i b_i I_2^{(i)}$$

d_i  c_i  b_i 

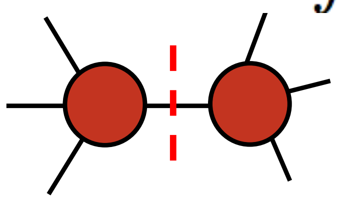
[see Caron-Huot's lectures]

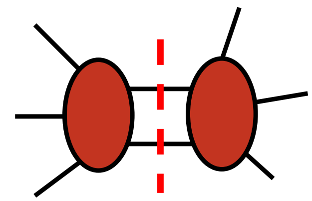
- At one loop in D dimensions: all scalar N-gons with N≤D
- Question: Name an over-complete basis to all loops?
- Answer: The set of all Feynman integrals! (this is a restricted set) [see Duhr's lectures]

Unitarity cuts

- Unitarity (optical theorem) is equivalent to factorization of the loop integrand on the poles of the integrand.

$$branch\ cuts \sim \int (poles)$$

factorization: 

unitarity: 

- ‘Completeness’ for on-shell propagators: $\frac{\eta^{\mu\nu}}{p^2} \rightarrow \frac{\sum_h \epsilon_h^{\mu} \epsilon_{-h}^{\nu}}{p^2}$

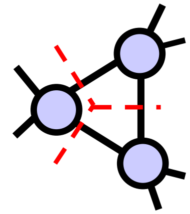
- Factorization: $\langle g_1 g_2 \cdots g_6 \rangle \rightarrow \sum_h \frac{\langle g_1 g_2 g_3 g^h(p) \rangle \langle g^{-h}(-p) g_4 g_5 g_6 \rangle}{p^2}$

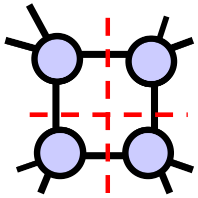
- Unitarity: $\langle g_1 g_2 \cdots g_6 \rangle \rightarrow \sum_{h_1 h_2} \int dLIPS \frac{\langle g_1 g_2 g_3 g^{h_1}(p_1) g^{h_2}(p_2) \rangle \langle g^{-h_1}(-p_1) g^{-h_2}(-p_2) g_4 g_5 g_6 \rangle}{p_1^2 p_2^2}$

Lorentz Invariant Phase Space: $dLIPS = d^4 p_1 \delta(p_1^2) \delta(p_2^2)$

Generalized Unitarity

- In the (old) literature: unitarity cuts ⇔ ampl branch cuts [Cutkosky]
- More general unitarity cuts that do not correspond to branch cuts can be considered





- Or overlapping cuts, or complex momenta
- Modern philosophy: if you see a propagator, you are allowed to 'cut it'
- In general:

$$cut = \int dLIPS \sum_{states} A_{(1)} A_{(2)} \cdots A_{(m)}$$

$$cut = A^{loop} \left[\frac{1}{p_i^2} \to 2\pi\delta(p_i^2) \right] = \int dLIPS \, \mathcal{I}^{loop} \prod_i p_i^2$$

Invisible terms?

• Do the generalized unitarity cuts 'see' all terms in the loop amplitude?

• Question: does factorization 'see' all terms in the tree amplitude?

• Answer: No, local terms without poles can exist!

• Similarly, unitarity cuts may miss local terms in the loop integrand. However, in dim. reg. these always integrate to zero.

$\Rightarrow$ Unitarity cuts 'see' all the nonvanishing terms in the loop ampl.

$\Rightarrow$ If an ansatz for a loop amplitude satisfies all unitarity cuts, then this proves that the ansatz is complete!

• Caveat: the cuts must be done in dimension D>4 if dim. reg. is used

Summary

- We can express any loop amplitude as a linear combination of basis integrals
  - scalar (N≤D)-gons at one loop
  - over-complete: all Feynman diagrams

- Unitarity cuts are sufficient to fix any loop amplitude in either basis using only tree-level input.

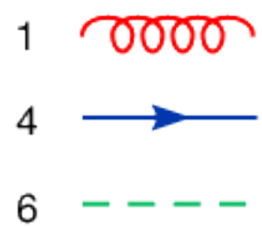
- D-dimensional cuts are required in general

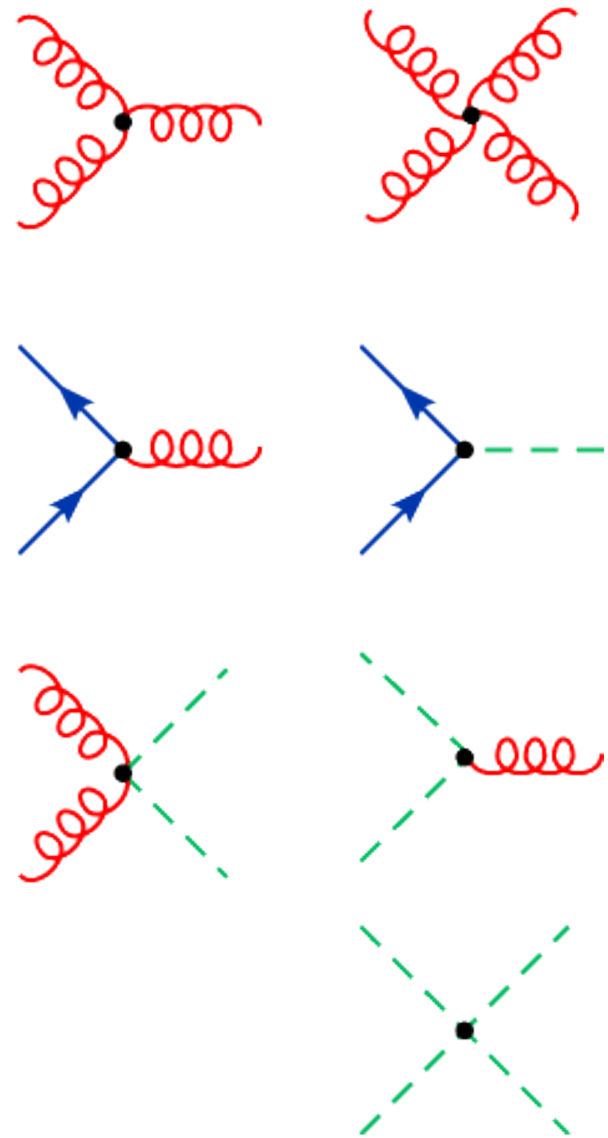
- Let’s work out some examples in N=4 SYM!

N=4 super-Yang-Mills

$$\mathcal{L}_{\rm YM} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\,\mu\nu}$$

+ fermions and scalars





Particles in adjoint group **SU(N_c)**

- Has maximum (four) supersymmetries for a gauge theory
- Same theory as N=1 SYM in D=10
- Conformal $\beta = 0$
- UV finite amplitudes
- Bubble and triangle loop integrals vanish
- Planar sector: dual conformal

N=4 and QCD

- N=4 SYM and QCD have the same tree-level gluon amplitudes
- At one loop QCD have a natural decomposition

$$A^{\mathcal{N}=0}_{one-loop} = A^{\mathcal{N}=4} - 4A^{\mathcal{N}=1\ chiral} + A^{scalar}$$

$$F_5^{\mathcal{N}=4} = A_5^{\rm tree} \sum_{j=1}^{5} \ln\left(\frac{-s_{j,j+1}}{-s_{j+1,j+2}}\right) \ln\left(\frac{-s_{j+2,j-2}}{-s_{j-2,j-1}}\right) + \frac{5}{6}\pi^2$$

$$F_5^{\mathcal{N}=1} = -\frac{\langle 1\,3\rangle^2 \langle 4\,1\rangle [2\,4]^2}{\langle 4\,5\rangle\langle 5\,1\rangle} \frac{\mathrm{Ls}_1\left(\frac{-s_{23}}{-s_{51}}, \frac{-s_{34}}{-s_{51}}\right)}{s_{51}^2} + \frac{\langle 1\,3\rangle^2 \langle 5\,3\rangle [2\,5]^2}{\langle 3\,4\rangle\langle 4\,5\rangle} \frac{\mathrm{Ls}_1\left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}}\right)}{s_{34}^2} - \frac{1}{2} \frac{\langle 1\,3\rangle^3 (\langle 1\,5\rangle [5\,2] \langle 2\,3\rangle - \langle 3\,4\rangle [4\,2] \langle 2\,1\rangle)}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,1\rangle} \frac{\mathrm{L}_0\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}}$$

$$\begin{aligned}
F^s = &-\frac{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle^2 [2\,4]^2}{\langle 4\,5\rangle\langle 5\,1\rangle\langle 2\,4\rangle^2} \frac{2\,\mathrm{Ls}_1\left(\frac{-s_{23}}{-s_{51}}, \frac{-s_{34}}{-s_{51}}\right) + \mathrm{L}_1\left(\frac{-s_{23}}{-s_{51}}\right) + \mathrm{L}_1\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}^2} \\
&+ \frac{\langle 3\,2\rangle\langle 2\,1\rangle\langle 1\,5\rangle\langle 5\,3\rangle^2 [2\,5]^2}{\langle 5\,4\rangle\langle 4\,3\rangle\langle 2\,5\rangle^2} \frac{2\,\mathrm{Ls}_1\left(\frac{-s_{12}}{-s_{34}}, \frac{-s_{51}}{-s_{34}}\right) + \mathrm{L}_1\left(\frac{-s_{12}}{-s_{34}}\right) + \mathrm{L}_1\left(\frac{-s_{51}}{-s_{34}}\right)}{s_{34}^2} \\
&+ \frac{2}{3} \frac{\langle 2\,3\rangle^2\langle 4\,1\rangle^3 [2\,4]^3}{\langle 4\,5\rangle\langle 5\,1\rangle\langle 2\,4\rangle} \frac{\mathrm{L}_2\left(\frac{-s_{23}}{-s_{51}}\right)}{s_{51}^3} - \frac{2}{3} \frac{\langle 2\,1\rangle^2\langle 5\,3\rangle^3 [2\,5]^3}{\langle 5\,4\rangle\langle 4\,3\rangle\langle 2\,5\rangle} \frac{\mathrm{L}_2\left(\frac{-s_{12}}{-s_{34}}\right)}{s_{34}^3} \\
&+ \frac{\mathrm{L}_2\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}^3} \left(\frac{1}{3} \frac{\langle 1\,3\rangle [2\,4] [2\,5] (\langle 1\,5\rangle [5\,2] \langle 2\,3\rangle - \langle 3\,4\rangle [4\,2] \langle 2\,1\rangle)}{\langle 4\,5\rangle} \right. \\
&\left. + \frac{2}{3} \frac{\langle 1\,2\rangle^2 \langle 3\,4\rangle^2 \langle 4\,1\rangle [2\,4]^3}{\langle 4\,5\rangle \langle 5\,1\rangle \langle 2\,4\rangle} - \frac{2}{3} \frac{\langle 3\,2\rangle^2 \langle 1\,5\rangle^2 \langle 5\,3\rangle [2\,5]^3}{\langle 5\,4\rangle \langle 4\,3\rangle \langle 2\,5\rangle} \right) \\
&+ \frac{1}{6} \frac{\langle 1\,3\rangle^3 (\langle 1\,5\rangle [5\,2] \langle 2\,3\rangle - \langle 3\,4\rangle [4\,2] \langle 2\,1\rangle)}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,5\rangle \langle 5\,1\rangle} \frac{\mathrm{L}_0\left(\frac{-s_{34}}{-s_{51}}\right)}{s_{51}} + \frac{1}{3} \frac{[2\,4]^2 [2\,5]^2}{[1\,2][2\,3][3\,4]\langle 4\,5\rangle[5\,1]} \\
&- \frac{1}{3} \frac{\langle 1\,2\rangle\langle 4\,1\rangle^2 [2\,4]^3}{\langle 4\,5\rangle\langle 5\,1\rangle\langle 2\,4\rangle[2\,3][3\,4] s_{51}} + \frac{1}{3} \frac{\langle 3\,2\rangle\langle 5\,3\rangle^2 [2\,5]^3}{\langle 5\,4\rangle\langle 4\,3\rangle\langle 2\,5\rangle[2\,1][1\,5] s_{34}} + \frac{1}{6} \frac{\langle 1\,3\rangle^2 [2\,4] [2\,5]}{s_{34} \langle 4\,5\rangle s_{51}} .
\end{aligned}$$

Simple tree amplitudes

Let's list the simplest tree amplitudes:

- Vanishes by SUSY: $A^{\text{tree}}(1^{\pm}2^{+}3^{+}\cdots n^{+}) = 0$

- MHV $A^{\text{tree}}(1^{+}2^{+}\cdots i^{-}\cdots j^{-}\cdots n^{+}) = i\dfrac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$

- e.g. 4-pt: $A^{\text{tree}}(1^{-}2^{-}3^{+}4^{+}) = i\dfrac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} = -i\dfrac{\langle 12\rangle^2[34]^2}{st}$

- SUSY Ward identities makes these simple!

SUSY Ward identities

Helicity raising operator:

$$[\bar{Q}, g^+] = 0$$

$$[\bar{Q}, g^-] = -\langle \epsilon p \rangle f^-$$

$$[\bar{Q}, f^+] = \langle \epsilon p \rangle g^+$$

Insert Q into amplitude:

$$0 = \langle [\bar{Q}, g_1^- g_2^- f_3^+ g_4^+ \cdots g_n^+] \rangle = -\langle \epsilon 1 \rangle \langle f_1^- g_2^- f_3^+ + \cdots g_n^+ \rangle$$
$$-\langle \epsilon 2 \rangle \langle g_1^- f_2^- f_3^+ + \cdots g_n^+ \rangle + \langle \epsilon 3 \rangle \langle g_1^- g_2^- g_3^+ + \cdots g_n^+ \rangle$$

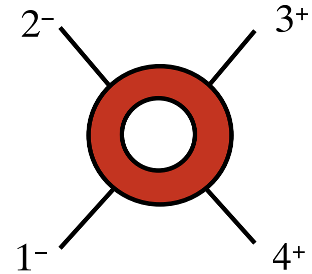
SWI valid to all loops: $\langle f_1^- g_2^- f_3^+ \cdots g_n^+ \rangle = \frac{\langle 21 \rangle}{\langle 23 \rangle} \langle g_1^- g_2^- g_3^+ \cdots g_n^+ \rangle$

Similarly using full N=4 SW:

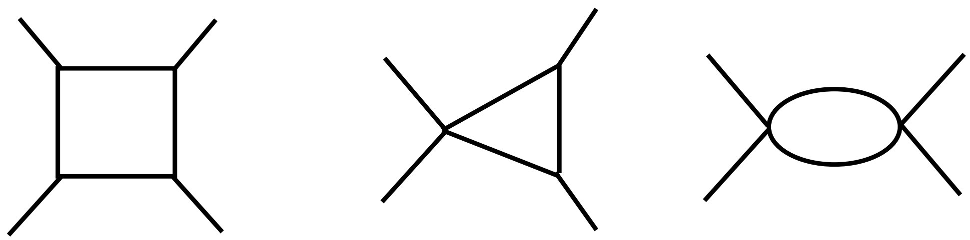
$$\frac{A^{\mathrm{MHV}}(1^+, \ldots, i^-, \ldots, j^-, \ldots, n^+)}{\langle ij \rangle^4} = \text{crossing symmetric function}$$

Work out 1-loop 4pt

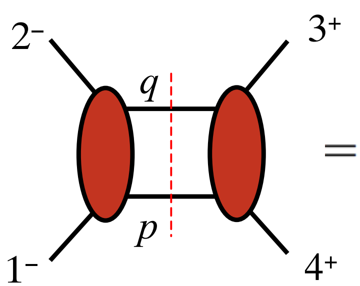
$$A^{1-loop}(1^-,2^-,3^+,4^+) =$$



What kind of integrals are expected?

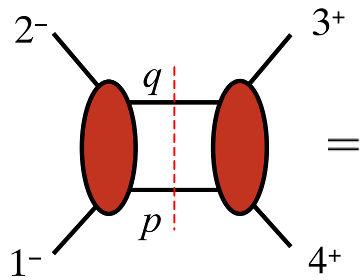


Compute the s-channel cut



$$= \int dLIPS \sum_{states} A^{tree}(1^-,2^-,q,p)A^{tree}(-p,-q,3^+,4^+)$$

s-channel cut



$$= \int dLIPS \sum_{states} A^{tree}(1^-, 2^-, q, p) A^{tree}(-p, -q, 3^+, 4^+)$$

- What are the states in the sum? g^+, f^+, s, f^-, g^-

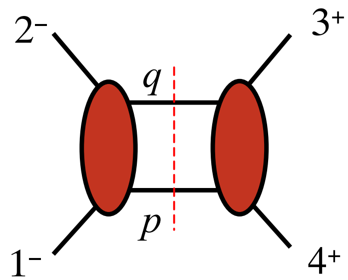
⇒Naively 16^2 combinations $1+4+6+4+1=16$

- SWI make all but one of them vanish!

$$\int dLIPS\ A^{tree}(1^-, 2^-, q^+, p^+) A^{tree}(p^-, q^-, 3^+, 4^+)$$

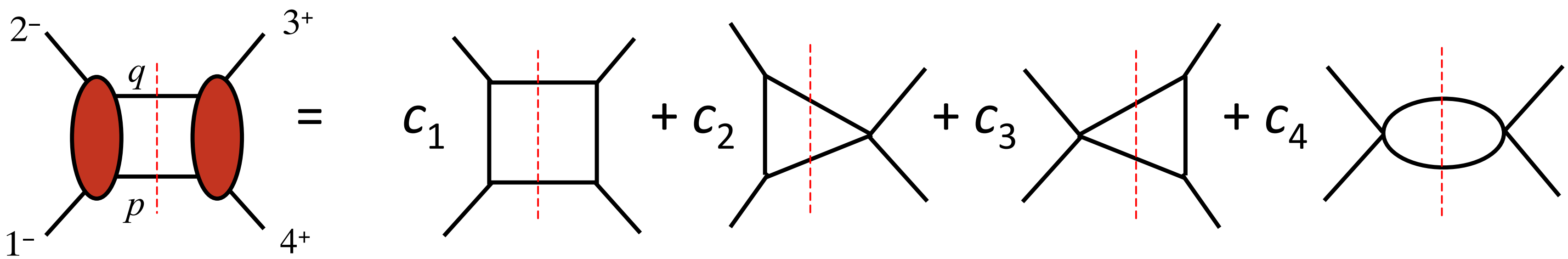
$$= \frac{\langle 12\rangle^2 [qp]^2}{s(k_1+p)^2} \times \frac{\langle pq\rangle^2 [34]^2}{s(k_4-p)^2} = \underbrace{\langle 12\rangle^2 [34]^2}_{stA^{tree}} \frac{1}{(k_1+p)^2 (k_4-p)^2}$$

Interpreting the cut



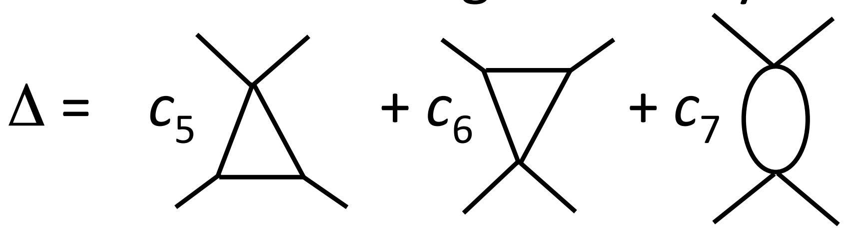
$$= stA^{tree}\int dLIPS\,\frac{1}{(k_1+p)^2(k_4-p)^2}$$

- Interpret the cut as a sum over basis integrals



$c_1 = stA^{tree}$, $c_2 = c_3 = c_4 = 0$

- Are we done?
- No there are 3 diagrams not yet constrained

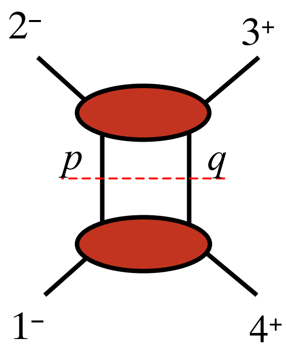


→ t-channel cut, or

→ crossing symmetry

The result

- State sum has 16 contributions, but result is as simple as before

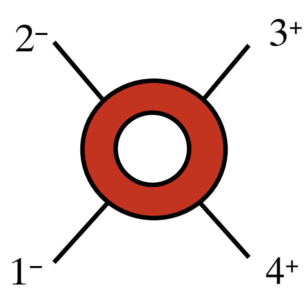


$$= stA^{tree} \int dLIPS \frac{1}{(k_2+p)^2(k_1-p)^2}$$

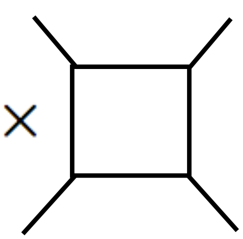
- Homework: check this! 1) do the cut, or 2) use SWI argument
- All coefficients are now known:

$$c_1 = stA^{tree} \qquad , c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0$$

- Amplitude:



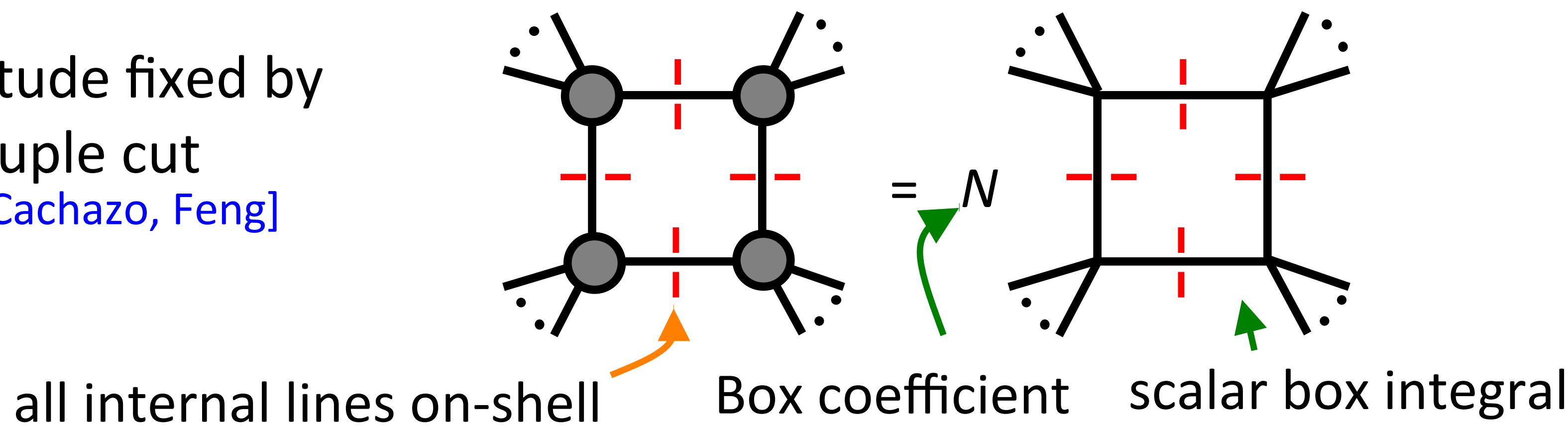
$$= stA^{tree} \times$$



Boxes & quadruple cuts

"no-triangle property": in N=4 SYM the set of box integrals is a sufficient one-loop basis

Amplitude fixed by quadruple cut
[Britto, Cachazo, Feng]



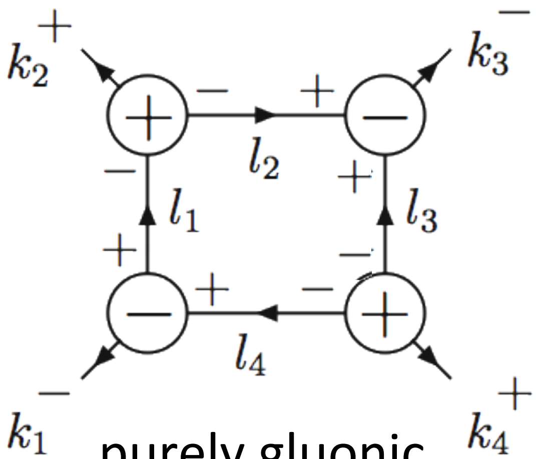
$$N = A^{\rm tree}_{(1)} A^{\rm tree}_{(2)} A^{\rm tree}_{(3)} A^{\rm tree}_{(4)}$$

On-shell conditions freezes momenta in *D*=4,

requires complex momenta

Homework: Redo the 4pt calculation using a quadruple cut

Quad and Hepta cuts details



purely gluonic
N=4 cut

two types of on-shell
3-pt amplitudes

$$\oplus = A(+--)$$

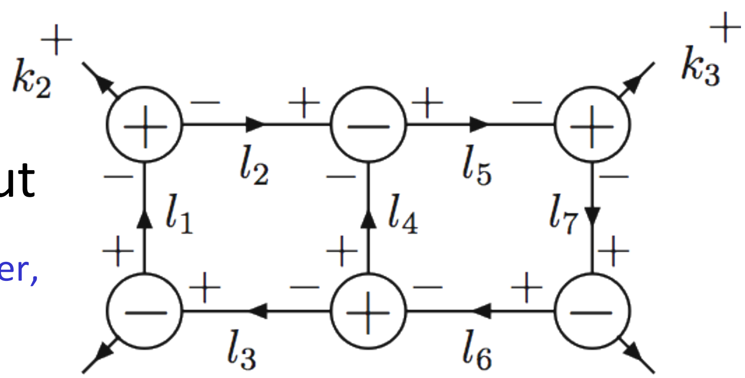
$$\ominus = A(-++)$$

complex loop momentum
$\longrightarrow$ phase space splits into
different branches

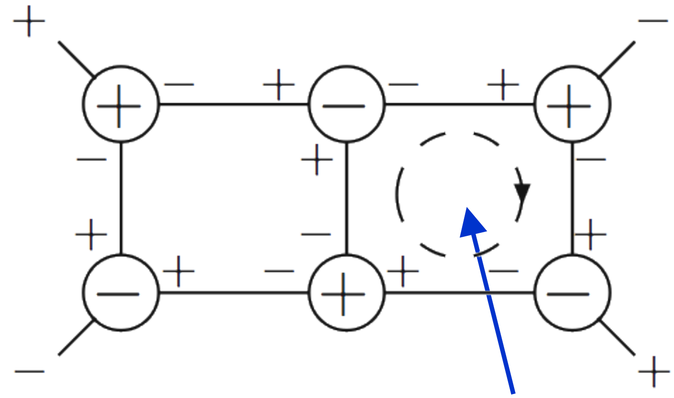
for 'singlets' only gluons
are allowed
$\longrightarrow$N=4 cuts same as QCD

hepta-cut

[Buchbinder,
Cachazo]



purely gluonic

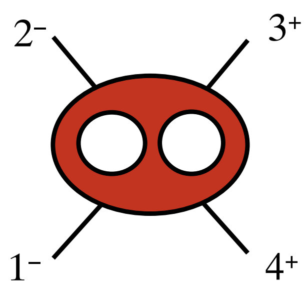


Full N =4 multiplet

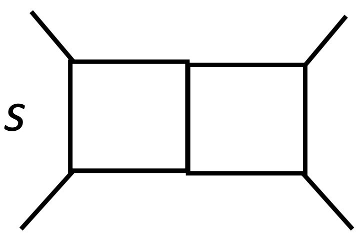
Two-loop amplitude

- Homework: Work out the two-loop amplitude at 4pts.

- Amplitude:



$$= stA^{tree} \times s$$



$$+ \; t$$