Aspects of scattering amplitudes

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Lecture 2: leading singularities and two-loop integrands

- Last time we saw that the coefficients of special functions are the residues of integrals in Feynman parameter space

- Actually, they are residues also in the original momentum space

- These residues are computed by turning the integration region into a T^4L

- These leading singularities are the simplest information about a loop integral

- With several complex variables, a residue is defined as follows:

- If in local coordinates the measure is

$$\frac{da_1 da_2 \cdots da_n}{a_1 a_2 \cdots a_n}$$

- then $\mathrm{Res}(a_1, \ldots, a_n) = +1$.

- The residue is alternating in the a's:

$$\mathrm{Res}(a_2, a_1, \ldots, a_n) \frac{da_1 da_2 \cdots da_n}{a_1 a_2 \cdots a_n} = -1.$$

- Consider a general form

$$\frac{da_1 da_2 \cdots da_n}{f_1 f_2 \cdots f_n} g$$

- A residue is defined for every discrete solution a' for setting the f_i to 0:

$$\mathrm{Res}_{a'}(f_1, f_2, \ldots, f_n) \frac{da_1 da_2 \cdots da_n}{f_1 f_2 \cdots f_n} g \equiv \frac{1}{\mathrm{Det}(\frac{\partial f_i}{\partial a_i}(a'))} g(a')$$

- Note there is no absolute value

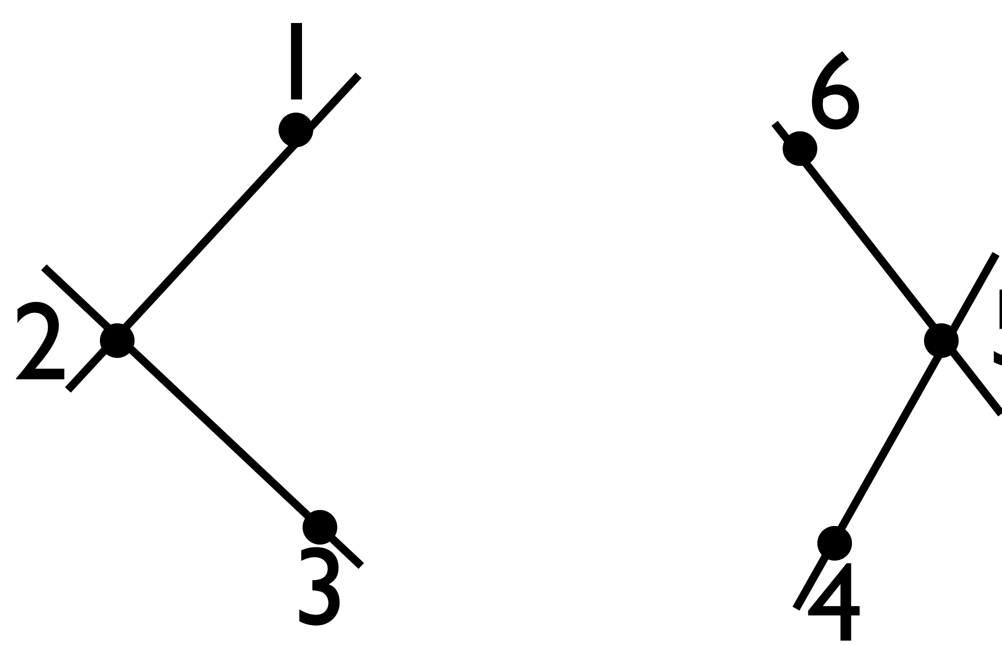
- This is called the "Poincaré residue"

(Griffiths & Harris)

- Computing a leading singularity takes two steps:
  1. Find the solution(s)
  2. Evaluate the Jacobian
- Momentum twistors are useful for both
- In momentum twistor space, the loop variable is a line AB
- *Schubert's problem*: given four lines, find a fifth one, AB, which intersects all four

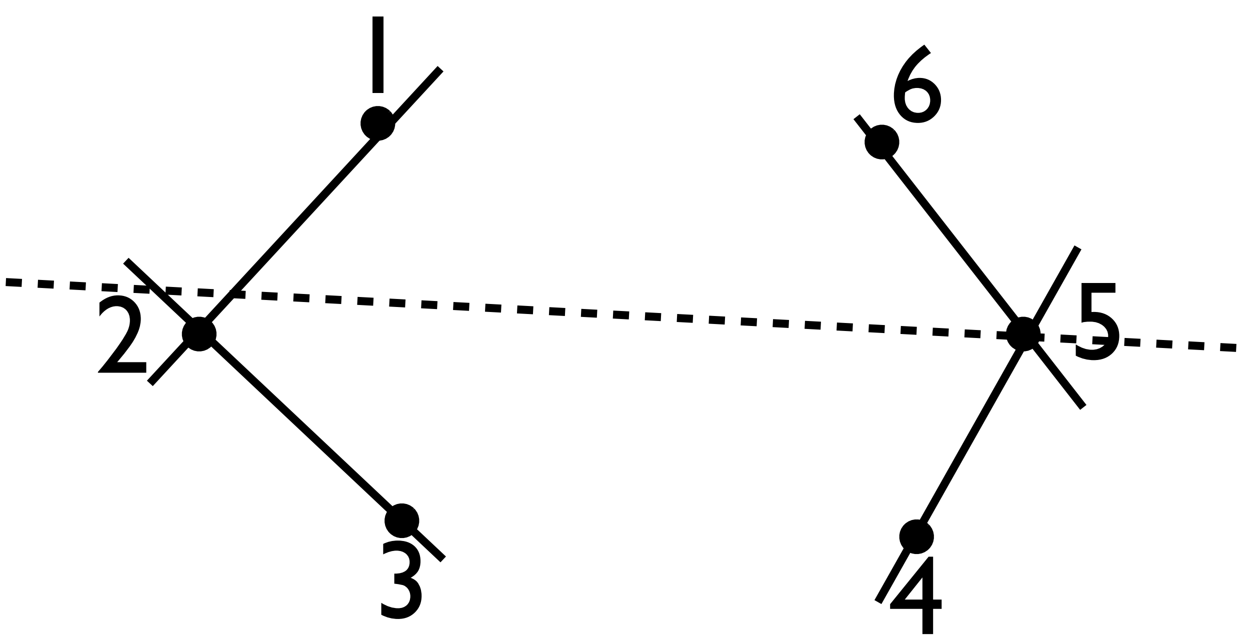
- Consider a "2-mass easy" box

$$\int \frac{d^4 Z_A d^4 Z_B}{\pi^2 \mathrm{vol}(\mathrm{GL}(2))} \frac{1}{\langle AB12\rangle\langle AB23\rangle\langle AB45\rangle\langle AB56\rangle}$$

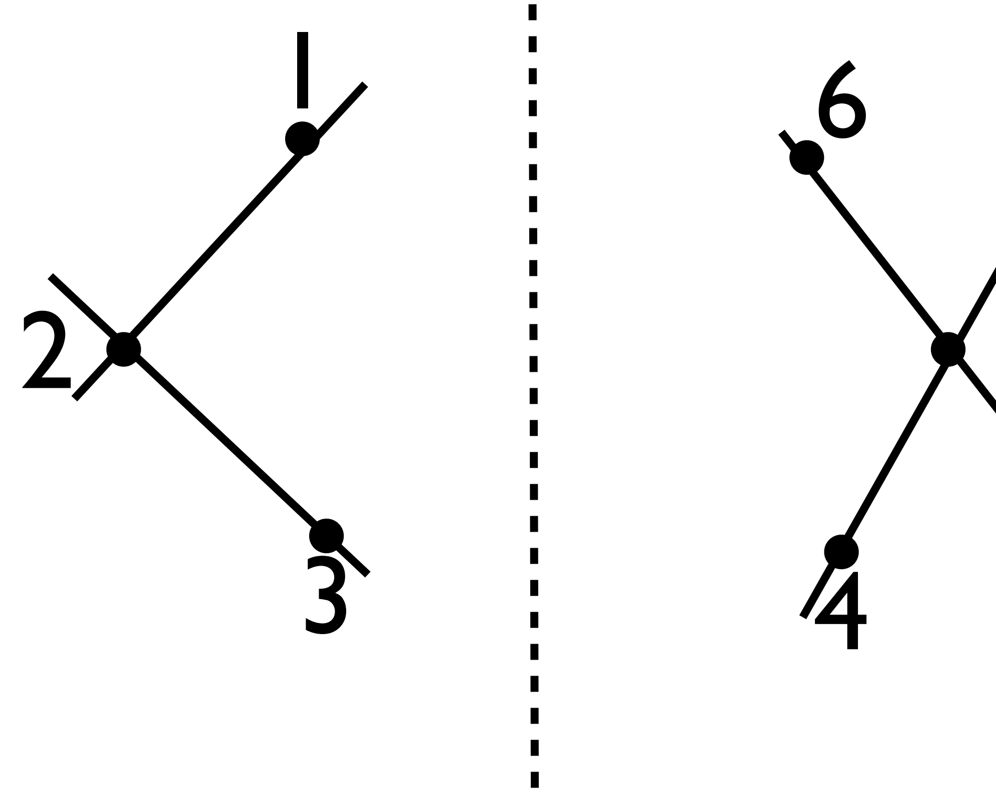


(the two 'wedges' define two different planes)

- The two solutions:



$$(AB)_L = 25$$



$$(AB)_R = (123) \cap (456)$$

- To computing the residue at AB=25, write:

$$
\begin{aligned}
Z_A &= Z_2 + \ {}_1 Z_1 + \ {}_2 Z_3 \\
Z_B &= Z_5 + \ {}_1 Z_4 + \ {}_2 Z_6
\end{aligned}
$$

where alpha and beta are small.

- The measure:

$$
\begin{aligned}
\int \frac{d^4 Z_A d^4 Z_B}{\mathrm{vol}(\mathrm{GL}(2))} &\equiv \int \langle AB d^2 Z_A \rangle \langle AB d^2 Z_B \rangle \\
&\rightarrow \langle 1235 \rangle \langle 4562 \rangle \int d^2 \alpha d^2 \beta
\end{aligned}
$$

- The denominators:

$$\begin{aligned}\langle AB12\rangle\langle AB23\rangle &\rightarrow \langle 1235\rangle^2\alpha_1\alpha_2,\\ \langle AB45\rangle\langle AB56\rangle &\rightarrow \langle 2456\rangle^2\beta_1\beta_2\end{aligned}$$

- Thus, near the leading singularity,

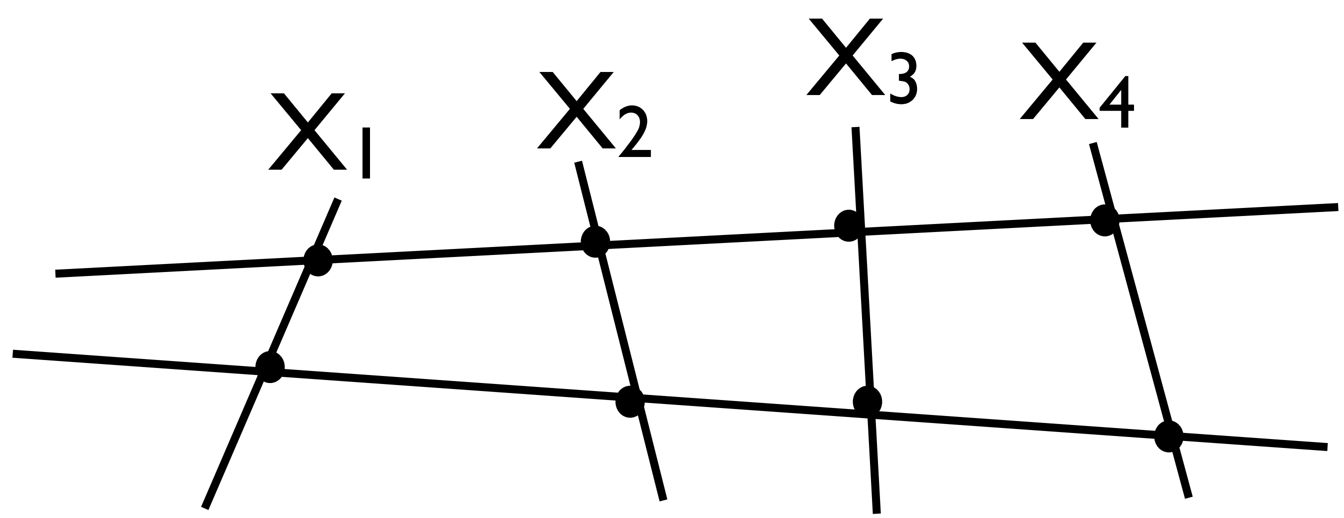
$$\int_{AB}\frac{1}{\langle AB12\rangle\langle AB23\rangle\langle AB45\rangle\langle AB56\rangle}\approx\frac{1}{\langle 1235\rangle\langle 2456\rangle}\int\frac{d\alpha_1 d\alpha_2 d\beta_1 d\beta_2}{\alpha_1\alpha_2\beta_1\beta_2}$$

- The other residue is equal and opposite

- The unit-leading singularity integral is thus:

$$\int_{AB}\frac{\langle 1235\rangle\langle 2456\rangle}{\langle AB12\rangle\langle AB23\rangle\langle AB45\rangle\langle AB56\rangle}$$

- The 4-mass box revisited



- There are always two solutions. Why?

- Finding these solutions requires solving a quadratic equation. The residue turns out to be:

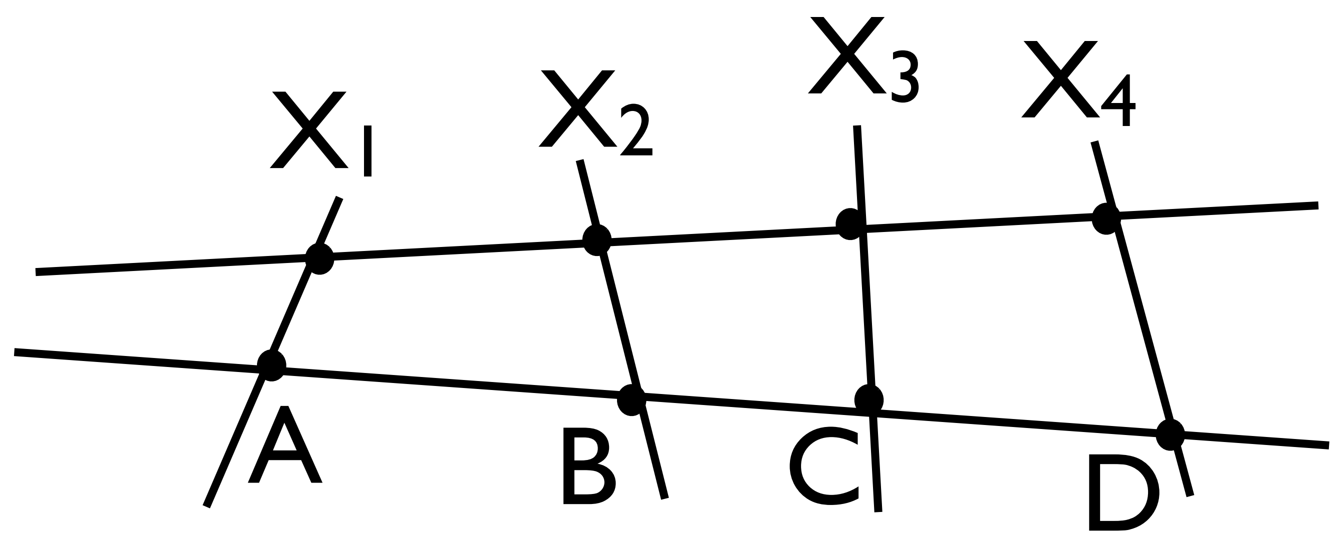
$$\frac{\pm 1}{x_{13}^2 x_{24}^2 \sqrt{(1-u-v)^2 - 4uv}}$$

(c.f. Britto, Cachazo &Feng, 2005)

- Note: the arguments of the dilogarithms,

$$I_{4m} = 2\mathrm{Li}_2(1-\alpha_+) - 2\mathrm{Li}_2(1-\alpha\quad) + \log v \log \frac{\alpha_+}{\alpha}$$

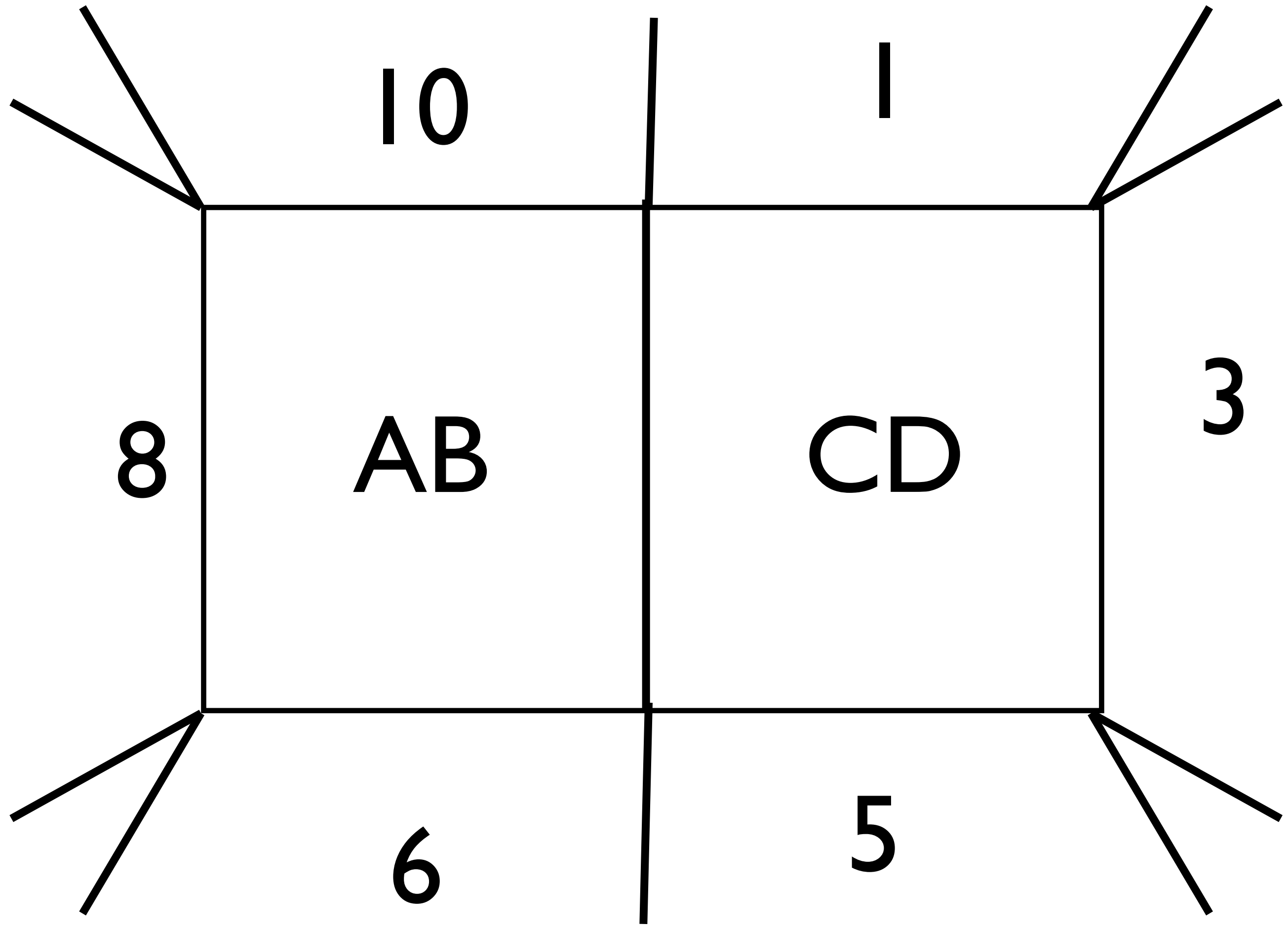
have a nice interpretation in twistor space:



- A,B,C,D are four point on the same line, we can take cross-ratios: $\alpha_\pm = \frac{\langle AB\rangle\langle CD\rangle}{\langle AC\rangle\langle BD\rangle}$

- This is the case for all 1-loop dilogarithms

- Note: some integrals have *no* leading singularities

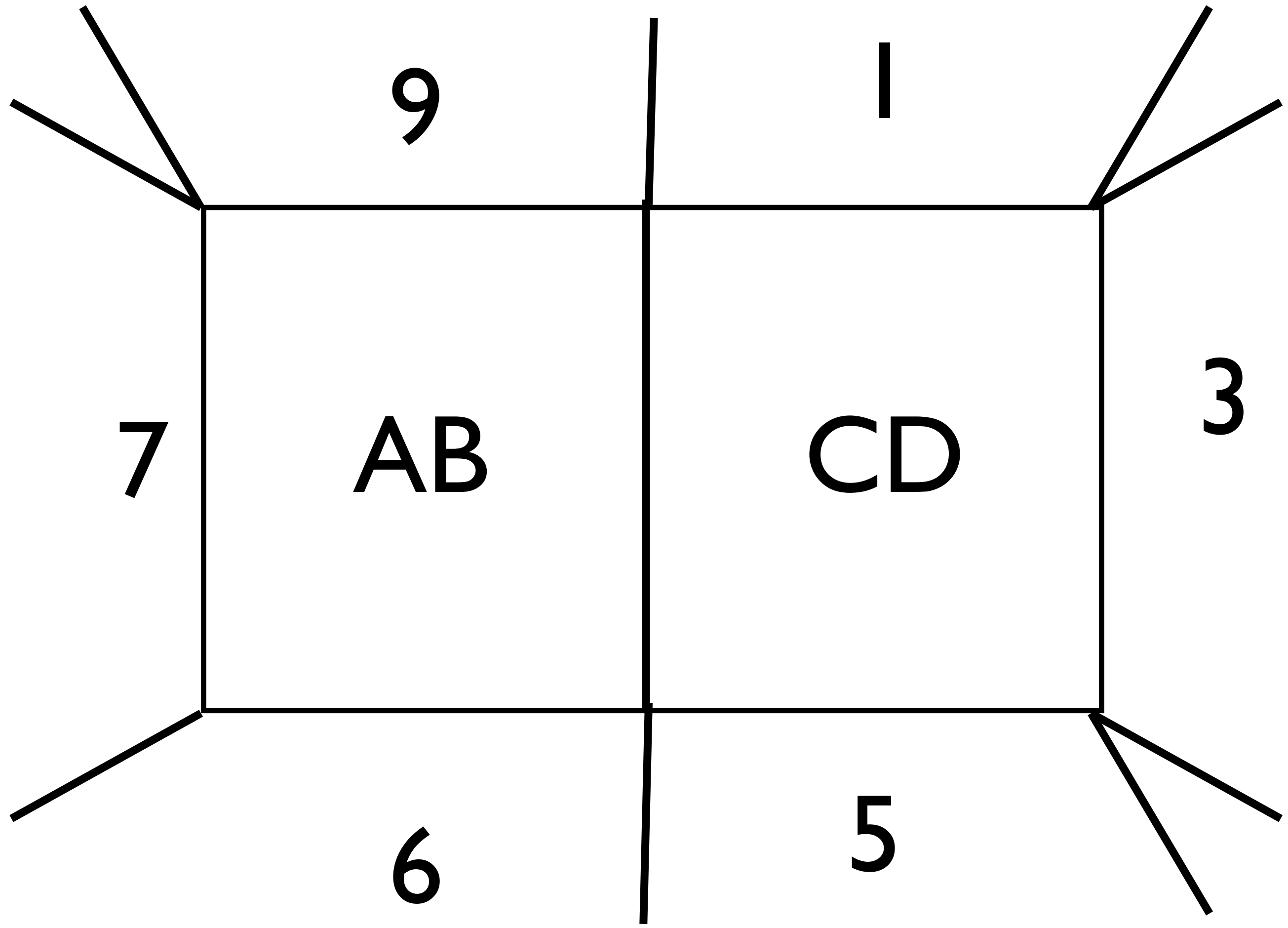


- Cutting all 7 propagators leaves

$$\int \frac{d\tau}{\sqrt{\text{quartic in } \tau}}$$

- Instead of poles, this integral has *periods*

- Note: some integrals have *no* leading singularities

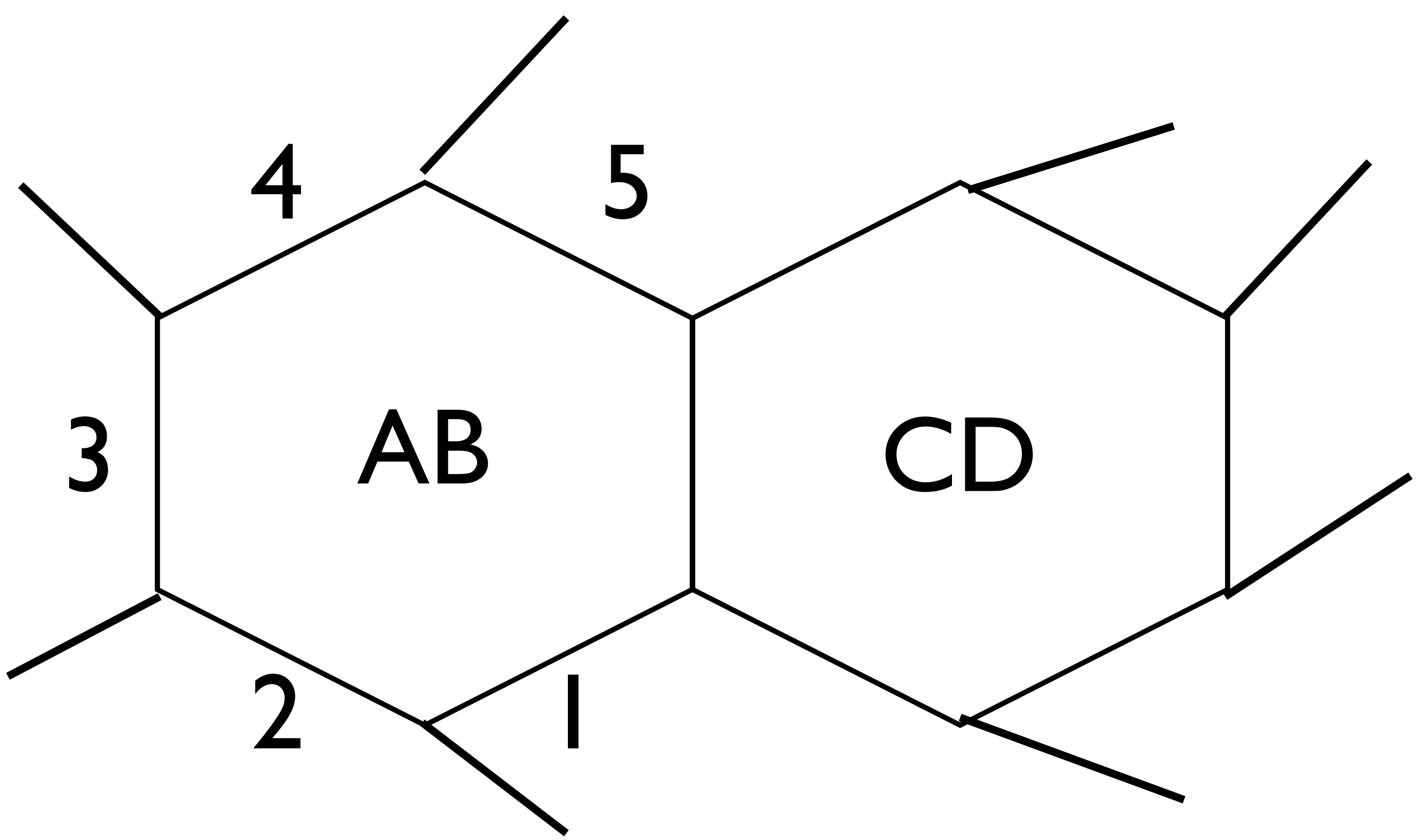


- Note: any degeneration will remove the elliptic integral
$$\int \frac{d\tau}{(\tau-\tau_0)\sqrt{\text{quadratic in } \tau}}$$

- 4-point case analyzed by (K.Larsen & D. Kosower)

Integral reduction at two-loops

- Consider some crazy, but conformal, double-hexagon integral (this looks academic, but in a second, we will learn something about 5- and 6-pt)



- By homogeneity, it will have a numerator quadratic in AB and quadratic in CD

Arkani-Hamed et al, "Local integrands for Planar Scattering Amplitudes"

- We can expand the AB numerators into a basis of $X_1, \ldots, X_5, \tilde{X},$ where

$$\tilde{X} = \epsilon^{i i_1 \cdots i_5} X_1^{i_1} \cdots X_5^{i_5},$$

similar to what we used at 1-loop.

- (Why can't we use CD?)

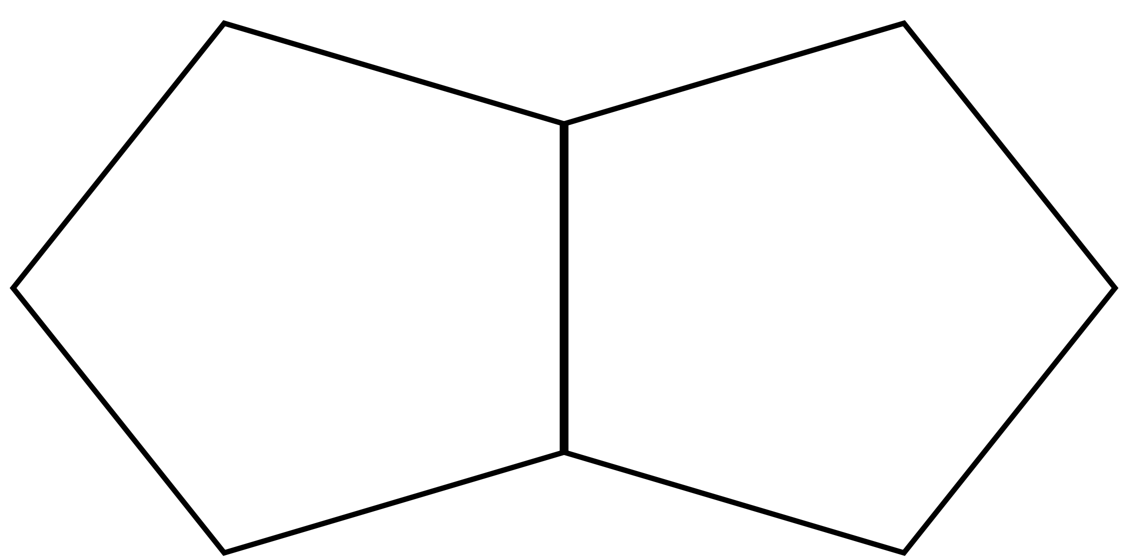
- Every AB numerator then removes some denominator, except $\langle AB\tilde{X}\rangle\langle AB\tilde{X}\rangle$

- To remove this one, there is a wonderful 21-term identity

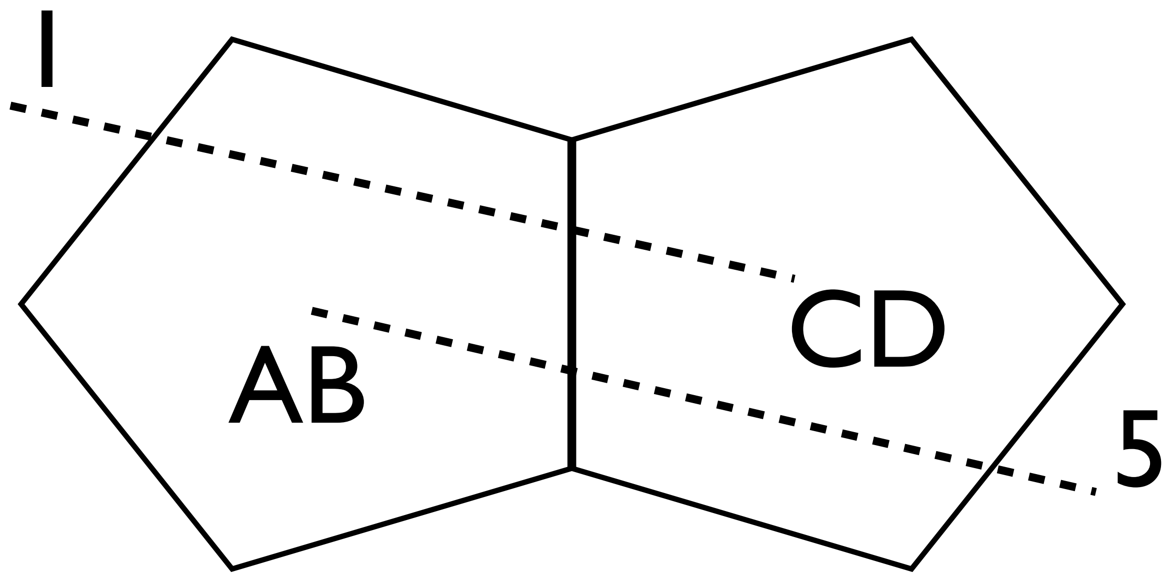
- 21-term identity:

$$0 = \langle ABAB \rangle = \sum_{i,j} c_{i,j} \langle ABX_i \rangle \langle ABX_j \rangle, \quad (i, j = 1, \ldots, 5, \tilde{\ })$$

- (Why does that exist?)
- Conclusion: hexagons are not needed at 2-loops in four dimensions



- Double pentagons require some choices for numerators. For instance:



- This picture is a shorthand for the numerator

$$\langle CDX_1\rangle\langle ABX_5\rangle$$

- Desirably, a basis of 'building blocks' should have *unit leading singularities*

- A successful strategy for constructing such a basis is the following:

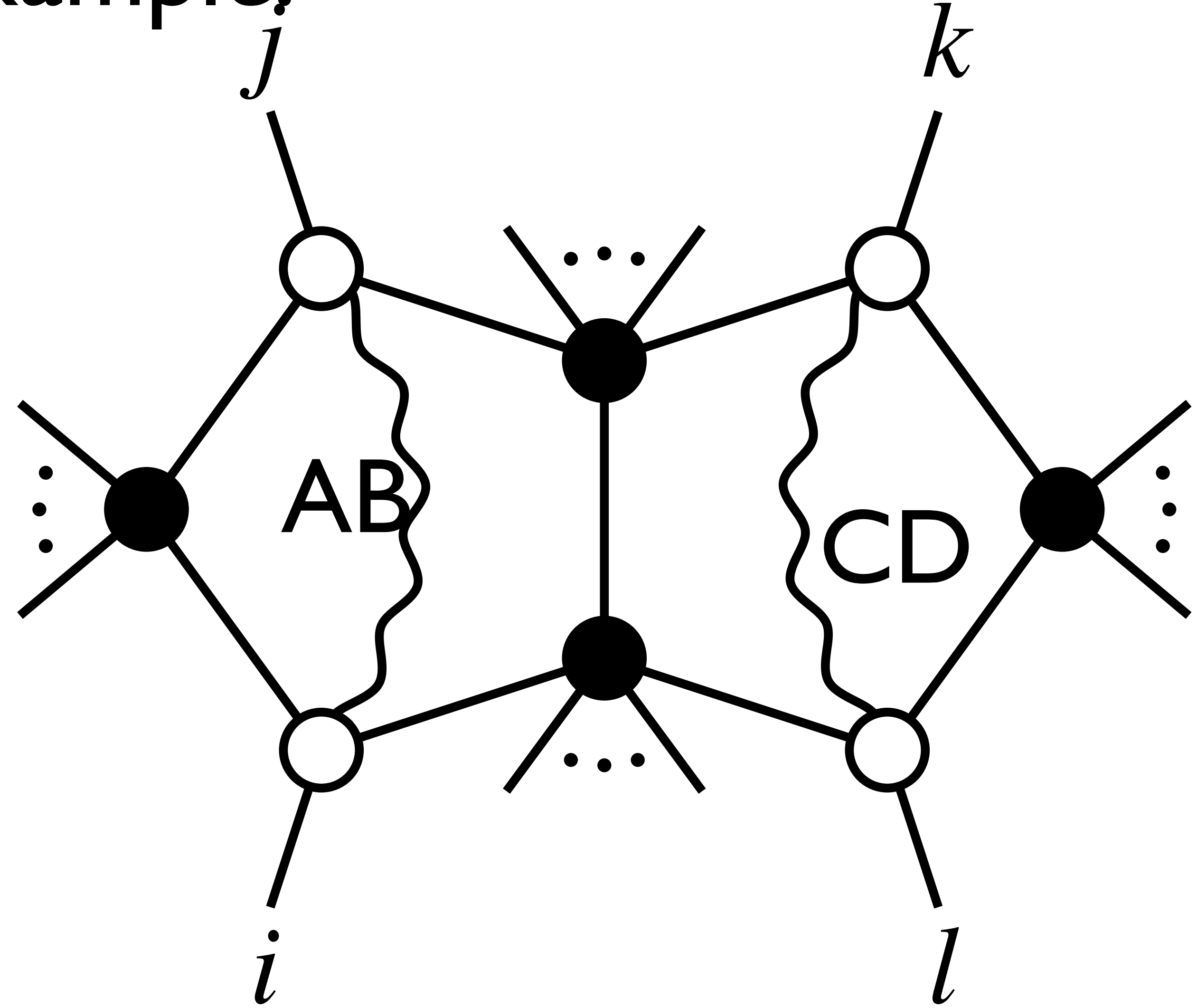
  try set as many leading singularities to zero as possible

- The idea is that nonzero leading singularities are related to each other by residue theorems

- Each pentagon has four external propagators; the 6 natural numerators for one pentagon are

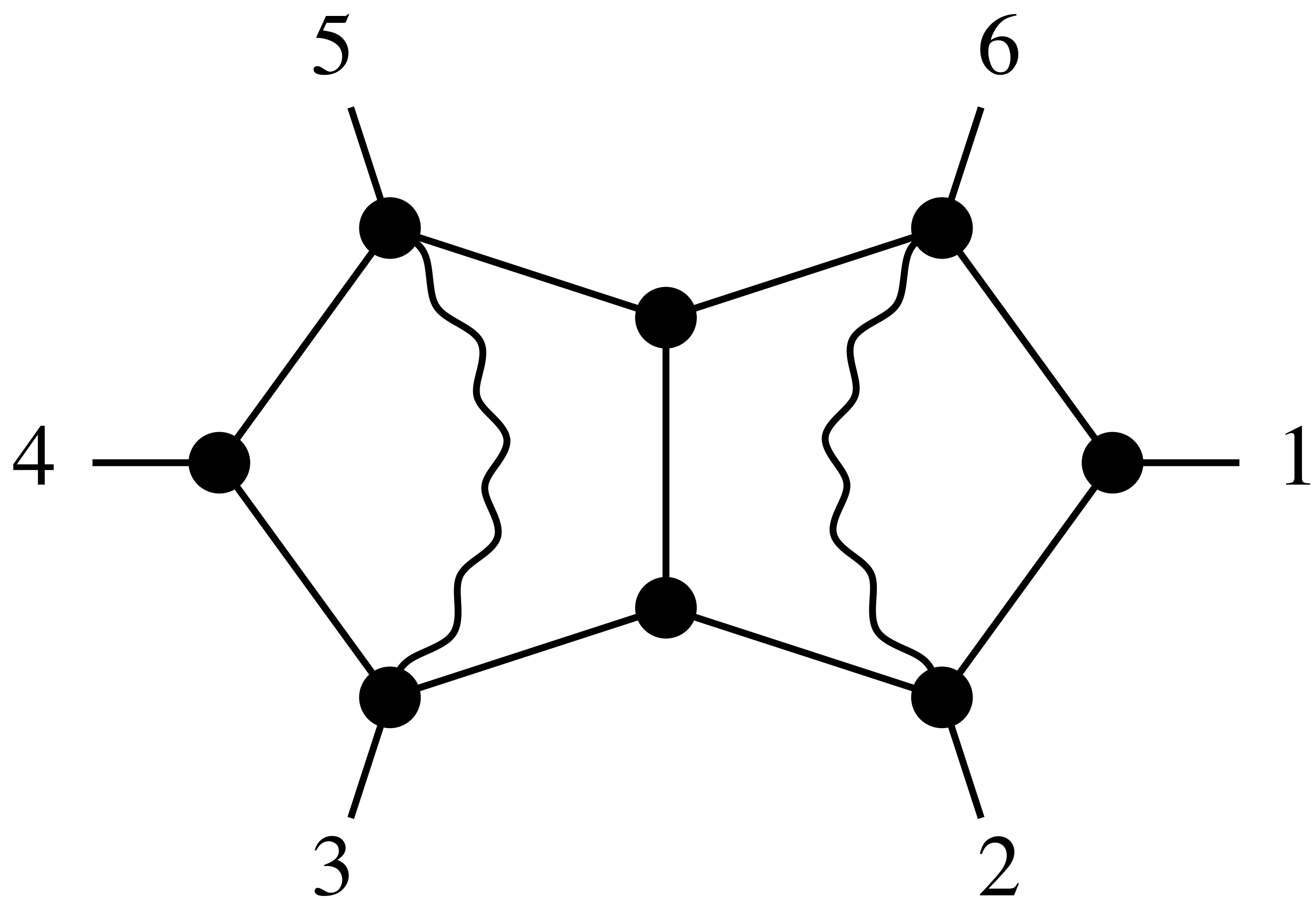
$$X_1, X_2, X_3, X_4, (LS)_1, (LS)_2$$

- Example;



$$
\left\{
\begin{array}{l}
\dfrac{\langle AB\,(i\text{–}1\,i\,i{+}1)\bigcap(j\text{–}1\,j\,j{+}1)\rangle\langle i\,j\,k\,l\rangle}{\langle AB\,i\text{–}1\,i\rangle\langle AB\,i\,i{+}1\rangle\langle AB\,j\text{–}1\,j\rangle\langle AB\,j\,j{+}1\rangle\langle AB\,CD\rangle} \\
\times\dfrac{\langle CD\,(k\text{–}1\,k\,k{+}1)\bigcap(l\text{–}1\,l\,l{+}1)\rangle}{\langle CD\,k\text{–}1k\rangle\langle CD\,k\,k{+}1\rangle\langle CD\,l\text{–}1\,l\rangle\langle CD\,l\,l{+}1\rangle}
\end{array}
\right\}
$$

- Even in the 6 particles case, this integral is *finite*

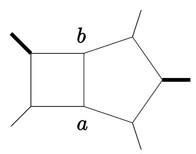


- Why?

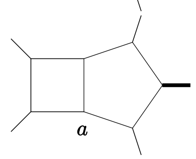
- Example:

The 2-loop MHV amplitude, n-particles, in N=4 SYM

1. No legs attached

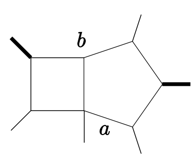


$$\frac{1}{4}x^2_{ab}x^2_{a+1,q}\left(x^2_{a,b+1}x^2_{a-1,b} - x^2_{ab}x^2_{a-1,b+1}\right) \tag{60}$$

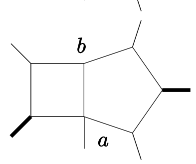


$$\frac{1}{2}x^2_{a,a+2}x^2_{a+1,q}\left(x^2_{a-1,a+2}x^2_{a,a+3} - x^2_{a-1,a+3}x^2_{a,a+2}\right) \tag{61}$$

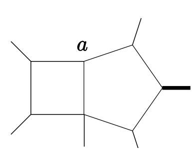
2. One massless leg attached



$$\frac{1}{4}\left(x^2_{a-1,b+1}x^2_{ab} - x^2_{a-1,b}x^2_{a,b+1}\right)\left(x^2_{a+1,q}x^2_{a+2,b} - x^2_{a+1,b}x^2_{a+2,q}\right) \tag{62}$$



$$\frac{1}{4}x^2_{a-1,b}\left(x^2_{ab}x^2_{a+1,q}x^2_{b-1,b+1} + x^2_{a,b+1}x^2_{a+1,b}x^2_{b-1,q} - x^2_{ab}x^2_{a+1,b+1}x^2_{b-1,q}\right) \tag{63}$$

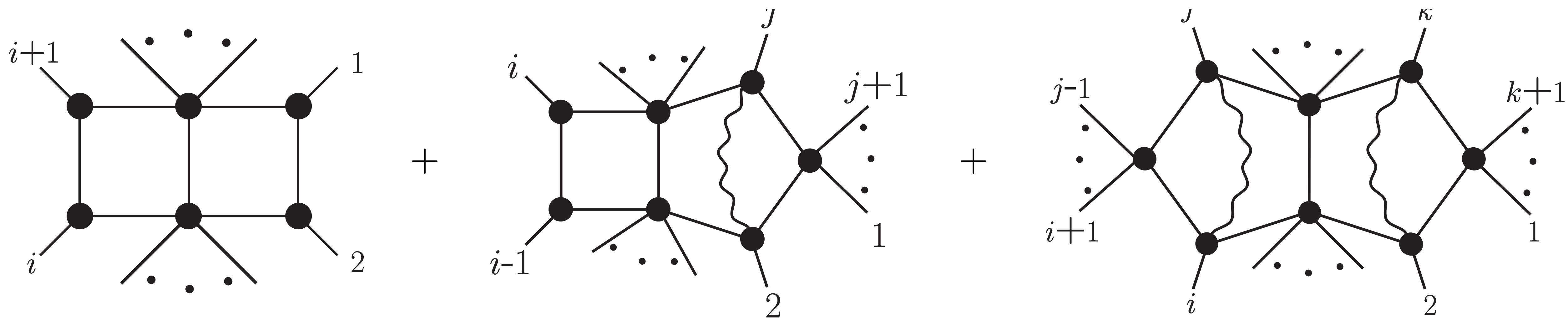


$$\frac{1}{4}\Big(x^2_{a-4,a}x^2_{a-3,a}x^2_{a-2,q}x^2_{a-1,a+1} - x^2_{a-4,a+1}x^2_{a-3,a}x^2_{a-2,a}x^2_{a-1,q} + $$
$$+ 2x^2_{a-4,a}x^2_{a-3,a+1}x^2_{a-2,a}x^2_{a-1,q} - x^2_{a-4,a}x^2_{a-3,a}x^2_{a-2,a+1}x^2_{a-1,q}\Big)$$

...

(C. Vergu 0908.2394)

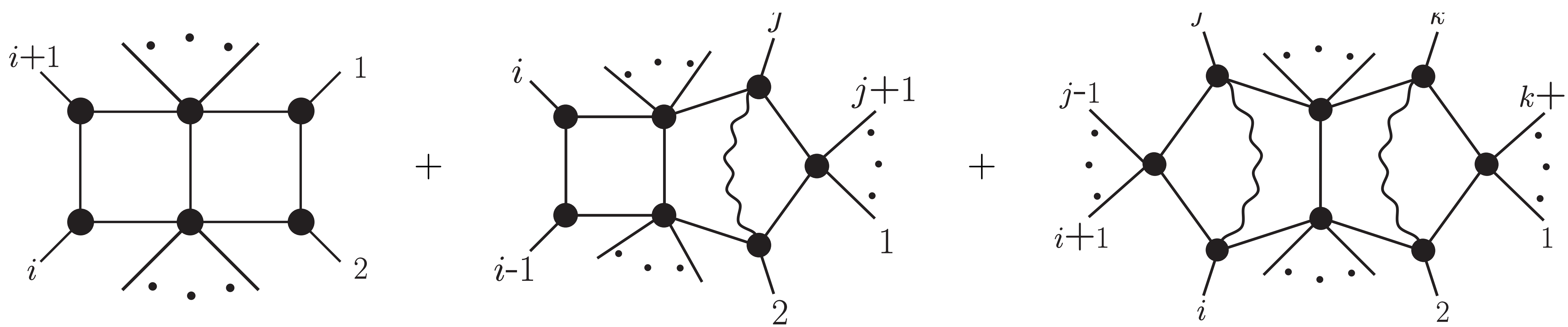
- Using the unit leading singularity basis, the *same* expression (actually, including also the parity-odd part) becomes:



Arkani-Hamed et al, 1008.2958

- The reason this looks simpler, is all due to the choice of basis

- Using the unit leading singularity basis, the *same* expression (actually, including also the parity-odd part) becomes:



Note: $\sim \frac{1}{\epsilon^4}$ $\sim \frac{1}{\epsilon}$ Finite

- These integrals beg to be integrated....

- So far, we haven't used symmetrical integration

- In the non-conformal cases, we can have high powers of the infinity point downstairs

$$\int_{AB,CD} \frac{1}{\langle ABI \rangle^2 \cdots}$$

- To remove these, one has to remove terms which integrate to zero

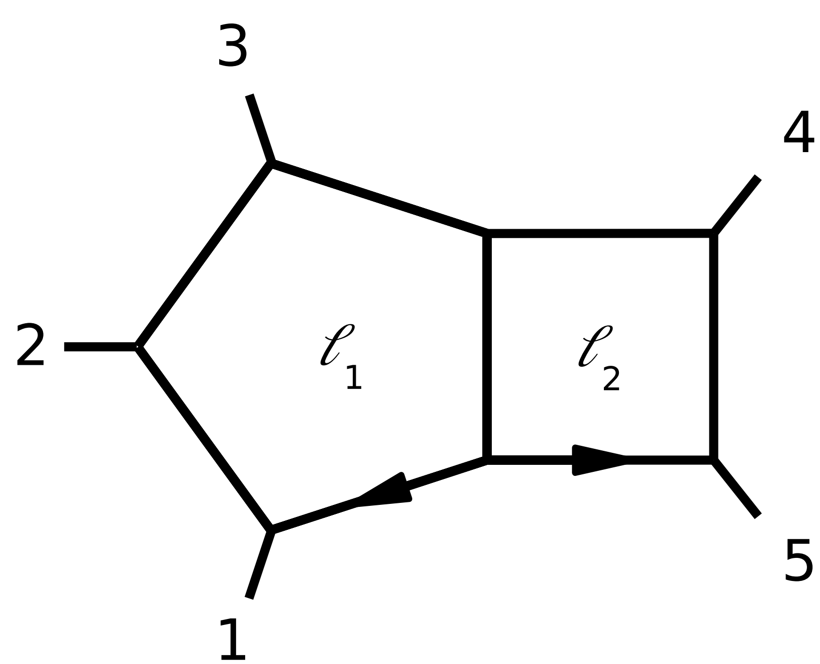
- *Integration by parts* (IBP) identities:

$$\int_{\ell_1,\ell_2} \left(v_1^\mu \frac{\partial}{\partial \ell_1^\mu} + v_2^\mu \frac{\partial}{\partial \ell_2^\mu} \right) \cdots = 0$$

- Important results on IBP identities at n=4,5,6 points were obtained by

Gluza, Kajda& Kosower (2010)

- For instance, all pentaboxes can be reduced to 3 *masters* $P^{**}_{3,2}[1]\,,P^{**}_{3,2}[k_1\cdot\ell_2]\,,P^{**}_{3,2}[k_5\cdot\ell_1]\,.$



- The theory we have just described applies to these masters

- Ii is important and nontrivial, to correctly estimate the "complexity" of an integral (discuss the double-pentagon case)

- Discuss the pentabox with numerator example (divergent integral -> finite, unit LS integral)