Aspects of scattering amplitudes

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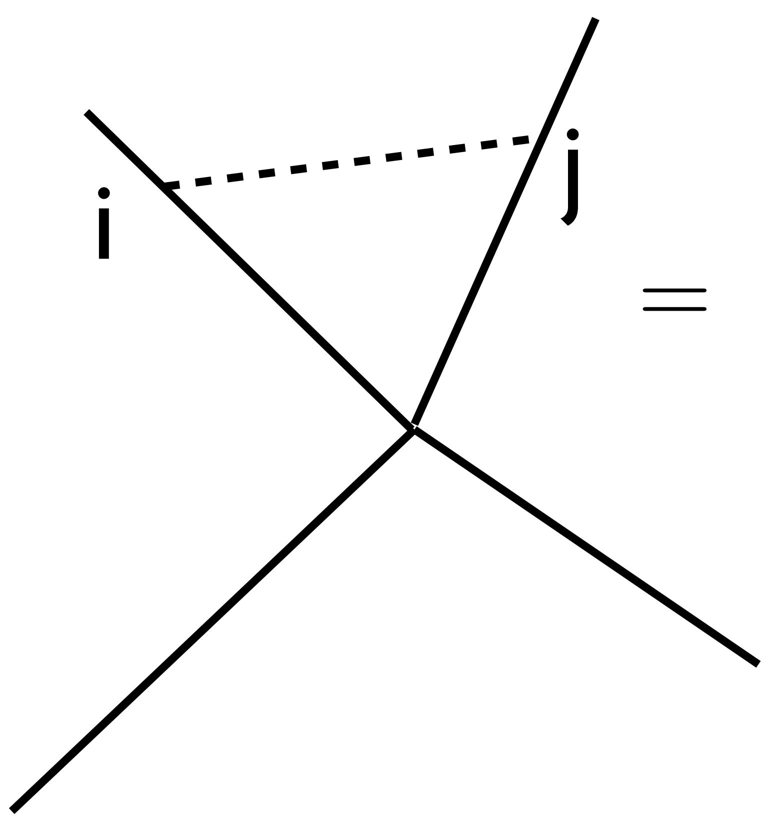
Lecture 3: More on infrared singularities; Differential equations, I

- Let us discuss infrared divergences of scattering amplitudes a bit longer

- Soft gluons effectively see Wilson lines



$$= \sum_{i\neq j} g^2_{\mathrm{YM}} T^a_i \otimes T^a_j \times$$

$$\frac{1}{2}\int^{\infty} dt_1 \int^{\infty} dt_2 \frac{i^2 p_i \cdot p_j}{4\pi^2 (tp_i - t_2 p_j)^{2(1-\epsilon)}}$$

- The integral is simple

$$\frac{1}{2}\int^{\infty} dt_1 \int^{\infty} dt_2 \frac{i^2 p_i \cdot p_j}{4\pi^2 (tp_i - t_2 p_j)^{2(1-\epsilon)}} \sim \frac{1}{16\pi^2\epsilon^2} + \text{finite}$$

- By overall color conservation,

$$\begin{aligned} \sum_i T_i^a &= 0, \\ \sum_{i\neq j} T_i^a \otimes T_j^a &= -\sum_i T_i^a \otimes T_i^a \equiv -\sum_i C_i \end{aligned}$$

- Thus $A^{\text{div}} \sim -\frac{1}{\epsilon^2} A^{\text{tree}} \times \sum_i \frac{g_{\text{YM}}^2 C_i}{16\pi^2} + \mathcal{O}(\frac{1}{\epsilon})$

- Simple physical interpretation:
  an amplitude for *n* particles is proportional to the probability of not emitting additional ones

- In a gauge theory this probability is very small

- The reason for exponentiation is that quanta emitted at different scales do not interfere

- In a full computation, divergences will always cancel against real emission:

$$\frac{1}{\epsilon^2} \to \frac{1}{2} \log \frac{Q^2}{\mu_{\rm IR}^2}$$

- Historical note: factorization theorems were developed for QCD starting ~30 years ago

Collins, Soper & Sterman
Korchemsky& Marchesini
Dixon, Sterman & Magnea
...

- Spectacularly verified in planar N=4 SYM:

$$A_n = e^{\text{“}\Gamma_{\rm cusp}(\lambda) M_n^{\rm MHV, 1\text{-}loop}\text{”}} \times [\text{Finite}]$$

↓ BDS Ansatz

↓ $= A_n^{\rm tree}, n = 4, 5$

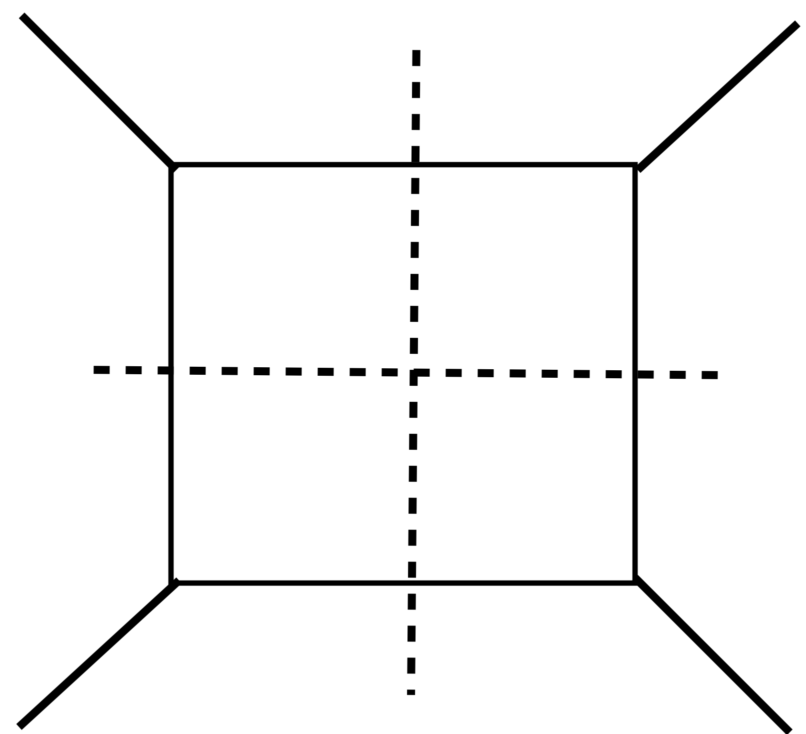
(Bern, Dixon & Smirnov)

- Confirmed at strong coupling

(Alday & Maldacena)

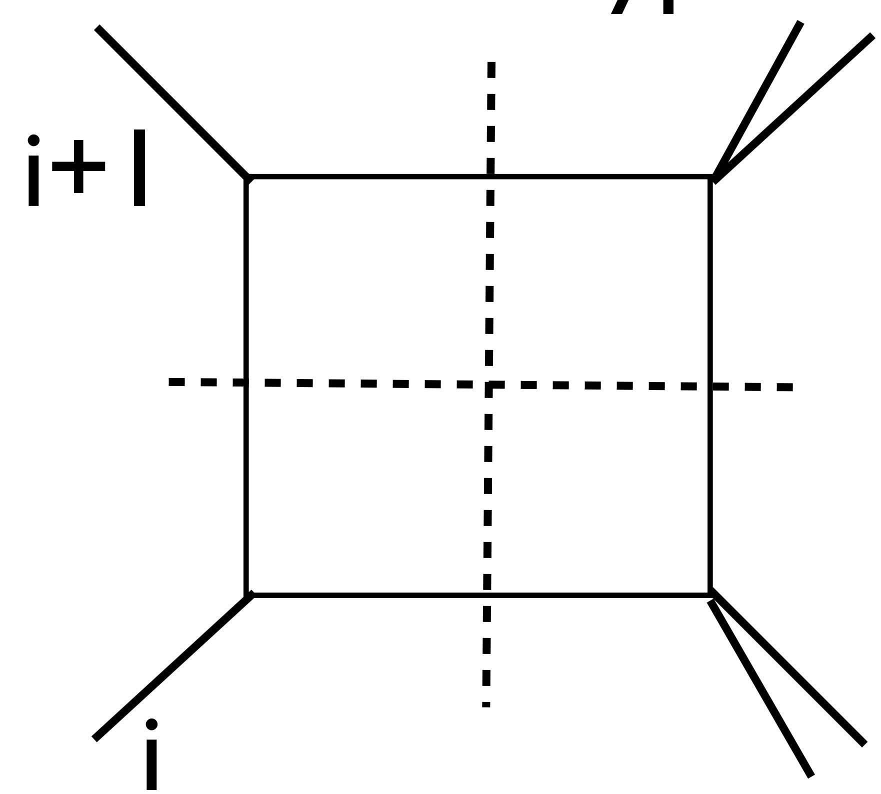
- At 1-loop, infrared exponentiation together with generalized unitarity had unexpected implications

Generalized unitarity:

Box coefficients = 

(See Johansson's lectures)

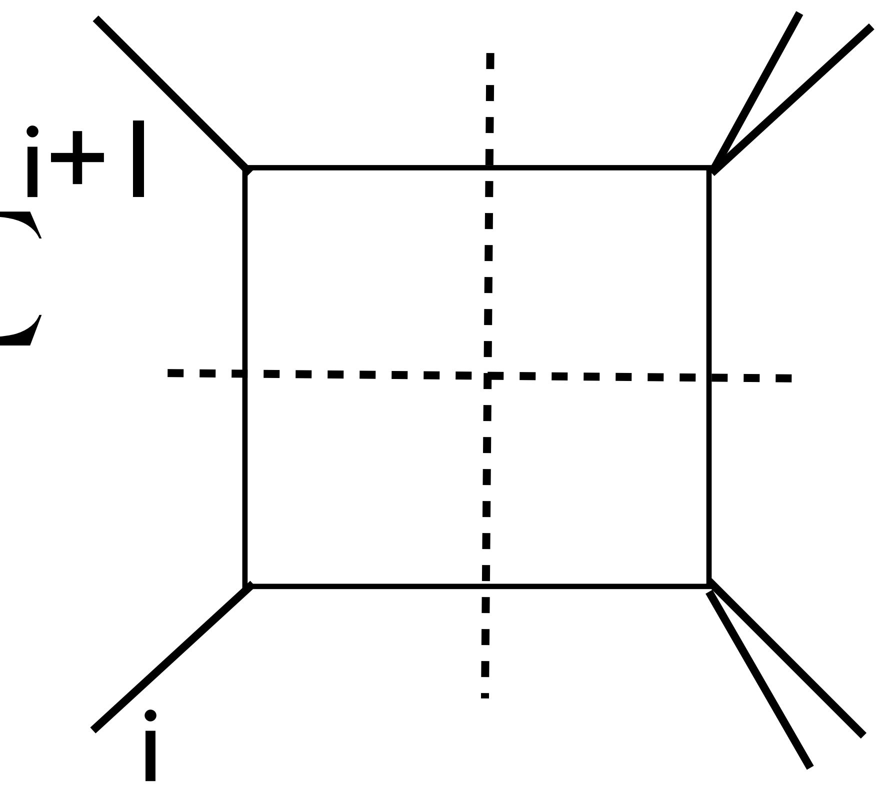
- In N=4 SYM, only boxes
- But we have seen that soft-collinear region between particles i and i+1 comes only from the 2mh or 1m-type boxes:



+ degenerations

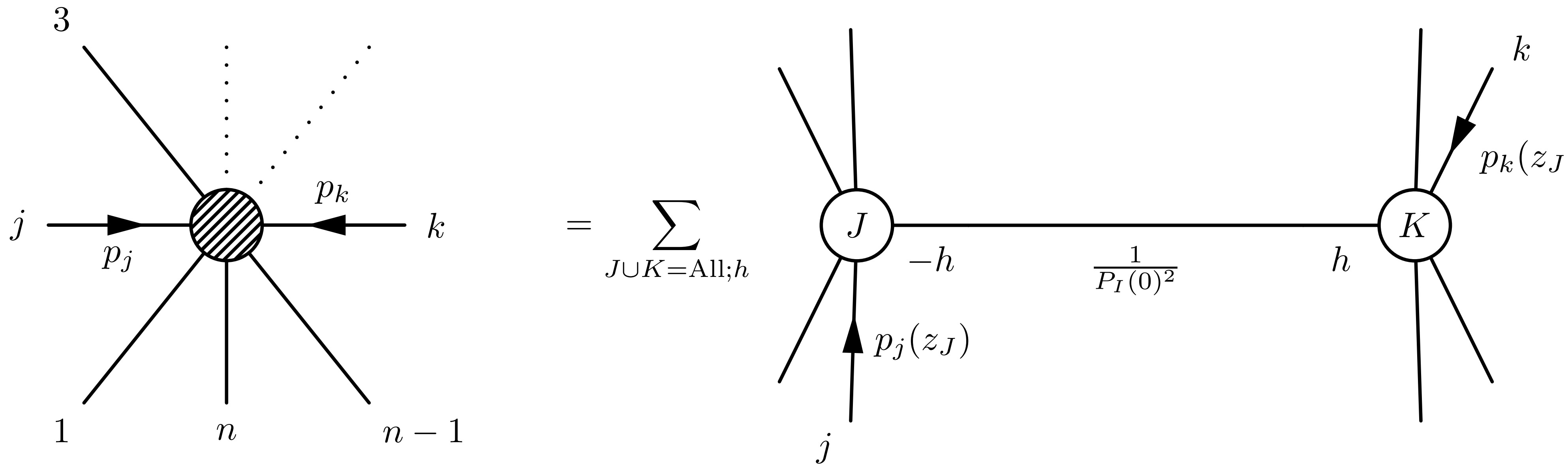
- There is a clash unless

$$A_n^{\text{tree}} = \sum$$



- Britto, Cachazo and Feng evaluated these quad cuts, and found that these were products of tree amplitudes

- BCF(W) recursion relation



- Schematically, in many different allowed representations,

$$A_n^{\text{tree}} = \sum_{m=2}^{n-2} A_{m+1}^{\text{tree}} A_{n-m+1}^{\text{tree}}$$

- Everything is on-shell!

Britto, Cachazo, Feng,
Britto,Cachazo,Feng&Witten,
Arkani-Hamed & Kaplan

- Proof that the `unphysical dilogarithms' always cancel:

Differential equations, I

- Throughout these lectures, I emphasized leading singularities and the importance of properly normalized integrals
- Let us call these, *pure integrals*
- Pure integrals are expected to produce pure transcendental functions
- They obey nice differential equations

(many authors; see Henn's lecture)

- Why pure integrals? Consider the simplest one,

$$\frac{dx(a-b)}{(x-a)(x-b)}$$

- This can be verified to have unit residues

- If we differentiate with respect to a, we get a total derivative of something rational:

$$\begin{aligned}\frac{d}{da}\frac{dx(a-b)}{(x-a)(x-b)} &= \frac{d}{da}\left(\frac{dx}{x-a}-\frac{dx}{x-b}\right)\\ &= -dx\frac{d}{dx}\left(\frac{1}{x-a}\right)\end{aligned}$$

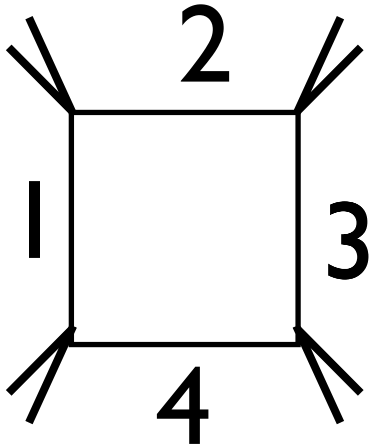
- *Principle:* the differential of a pure integral is a total derivative (of a rational function)

- We find that this property is true generally, in particular for all Feynman parameter integrals which appear in two-loop computations

(Arkani-Hamed& SCH, to appear)

- This is a powerful statement, which can become an engine for computations

- Let us check this on our favorite integral:



$$\equiv I_4^{4m} = \int_X \frac{\sqrt{\text{Det }G}}{X\cdot X_1 \cdots X\cdot X_4}, G = \text{Det}[X_i \cdot X_j].$$

- Turns out,

$$X_1\cdot\frac{d}{dX_4}I_4^{4m} \quad = \quad \cancelto{0}{\int_X V\cdot\frac{d}{dX}\frac{\sqrt{\text{Det }G}}{X\cdot X_2 X\cdot X_3 X\cdot X_4}}$$

$$+\int_X K\cdot\frac{d}{dX}\frac{X\cdot V\sqrt{\text{Det }G}}{X\cdot X_2 X\cdot X_3 X\cdot X_4 X\cdot K}$$

where $V = \dfrac{1}{2\text{Det }G}\dfrac{d}{dX_4}\text{Det }G,$ and K is one of the LS's.

- Any derivative produces a total derivative!
- Q: Why is

$$X_1\cdot\frac{d}{dX_4}I_4^{4m} \quad = \quad \int_X K\cdot\frac{d}{dX}\frac{X\cdot V\sqrt{\text{Det }G}}{X\cdot X_2 X\cdot X_3 X\cdot X_4 X\cdot K}$$

  nonzero?

- In next lecture, we analyze similar phenomena in a toy model for Feynman parameter space, where the boundaries are more obvious than here