Momentum twistors, special functions and symbols

Lecture 3

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Atrani, 06/10 - 11/10 2011

- 1st lecture: Kinematics
  - ➡ What are the arguments of the special functions?
- 2nd lecture:
  - ➡ What are the kind of functions that can appear in loop computations?
- How to deal with these functions?
  - ➡ The functions satisfy complicated identities, and we need/want to know ALL of them, e.g., to simplify expressions.

The six-point remainder function

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \tag{H.1}$$

$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+$$
$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+$$
$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$
$$\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+$$
$$\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+$$
$$G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-$$
$$\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+$$
$$G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-$$

+ 17 more pages...

The six-point remainder function

$$R(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\,\mathrm{Li}_4(1-1/u_i)\right)$$
$$-\frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \chi\frac{\pi^2}{12}\left(J^2+\zeta(2)\right)$$

$$x_i^\pm = u_i x^\pm, \;\; x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3}, \;\; \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

- The square roots turn out to 'cross ratios' in twistor space!

$$u_1 = \frac{\langle 1234\rangle\langle 4561\rangle}{\langle 1245\rangle\langle 3461\rangle}, \qquad x_1^+ = -\frac{\langle 1456\rangle\langle 2356\rangle}{\langle 1256\rangle\langle 3456\rangle}$$

[Goncharov, Spradlin, Volovich, Vergu]

The one-mass hexagon in D=6

$$-\frac{(\log(-\mathrm{y2p}\,u(4))-\log(-\mathrm{y2m}\,u(4)))\log^2(u(4))}{(\mathrm{y2m}\,u(4)-\mathrm{y2p}\,u(4))\,(u(2,5)-1)}+$$

$$\frac{\left(-\frac{\log(u(4))\log(-\mathrm{y2m}\,u(4))}{\mathrm{y2m}\,u(4)-\mathrm{y2p}\,u(4)}+\frac{\log(u(4))\log(-\mathrm{y2p}\,u(4))}{\mathrm{y2m}\,u(4)-\mathrm{y2p}\,u(4)}+\frac{\mathrm{Li}_2(\mathrm{y2m}+1)}{\mathrm{y2m}\,u(4)-\mathrm{y2p}\,u(4)}-\frac{\mathrm{Li}_2(\mathrm{y2p}+1)}{\mathrm{y2m}\,u(4)-\mathrm{y2p}\,u(4)}\right)\log(u(4))}{u(2,5)-1}+$$

$$\frac{\left(\log^2(-\mathrm{y2p})-\log^2(-\mathrm{y2m})\right)\log(u(4))}{(2\,\mathrm{y2m}-2\,\mathrm{y2p})\,u(4)\,(u(2,5)-1)}-\frac{(\mathrm{Li}_2(\mathrm{y2m}+1)-\mathrm{Li}_2(\mathrm{y2p}+1))\log(u(4))}{(\mathrm{y2m}-\mathrm{y2p})\,u(4)\,(u(2,5)-1)}+$$

$$\frac{(\mathrm{Li}_2(\mathrm{y2m}\,u(4)+1)-\mathrm{Li}_2(\mathrm{y2p}\,u(4)+1))\log(u(4))}{(\mathrm{y2m}\,u(4)-\mathrm{y2p}\,u(4))\,(u(2,5)-1)}-\frac{(\log(-\mathrm{y1p})-\log(-\mathrm{y1m}))\log(u(2,5))\log(u(4))}{(\mathrm{y1m}-\mathrm{y1p})\,(u(4)+u(2,5)\,u(6,2)-1)}+$$

$$\frac{(\mathrm{Li}_2(\mathrm{y1m}+1)-\mathrm{Li}_2(\mathrm{y1p}+1))\log(u(4))}{(\mathrm{y1m}-\mathrm{y1p})\,(u(4)+u(2,5)\,u(6,2)-1)}-\frac{(\mathrm{Li}_2(\mathrm{y1m}\,u(6,2)+1)-\mathrm{Li}_2(\mathrm{y1p}\,u(6,2)+1))\,u(6,2)\log(u(4))}{(\mathrm{y1m}\,u(6,2)-\mathrm{y1p}\,u(6,2))\,(u(4)+u(2,5)\,u(6,2)-1)}-$$

$$\frac{1}{u(4)}\log(1-u(2,5))\left(-\frac{\log(1-u(2,5))\log(-\mathrm{y2m}\,(1-u(2,5)))}{\mathrm{y2m}\,(1-u(2,5))-\mathrm{y2p}\,(1-u(2,5))}+\frac{\log(1-u(2,5))\log(-\mathrm{y2p}\,(1-u(2,5)))}{\mathrm{y2m}\,(1-u(2,5))-\mathrm{y2p}\,(1-u(2,5))}+\right.$$

$$\left.\frac{\mathrm{Li}_2(\mathrm{y2m}+1)}{\mathrm{y2m}\,(1-u(2,5))-\mathrm{y2p}\,(1-u(2,5))}-\frac{\mathrm{Li}_2(\mathrm{y2p}+1)}{\mathrm{y2m}\,(1-u(2,5))-\mathrm{y2p}\,(1-u(2,5))}\right)+$$

$$\frac{1}{u(4)+u(2,5)\,u(6,2)-1}\log(u(2,5))\,(1-u(2,5)\,u(6,2))$$

$$\left(-\frac{\log(1-u(2,5))\log(-\mathrm{y1m}\,(1-u(2,5)\,u(6,2)))}{\mathrm{y1m}\,(1-u(2,5)\,u(6,2))-\mathrm{y1p}\,(1-u(2,5)\,u(6,2))}+\frac{\log(1-u(2,5))\log(-\mathrm{y1p}\,(1-u(2,5)\,u(6,2)))}{\mathrm{y1m}\,(1-u(2,5)\,u(6,2))-\mathrm{y1p}\,(1-u(2,5)\,u(6,2))}+\right.$$

$$\left.\frac{\mathrm{Li}_2\left(\frac{\mathrm{y1m}\,(1-u(2,5)\,u(6,2))}{1-u(2,5)}+1\right)}{\mathrm{y1m}\,(1-u(2,5)\,u(6,2))-\mathrm{y1p}\,(1-u(2,5)\,u(6,2))}-\frac{\mathrm{Li}_2\left(\frac{\mathrm{y1p}\,(1-u(2,5)\,u(6,2))}{1-u(2,5)}+1\right)}{\mathrm{y1m}\,(1-u(2,5)\,u(6,2))-\mathrm{y1p}\,(1-u(2,5)\,u(6,2))}\right)+$$

$$\frac{1}{u(4)\,(u(2,5)-1)}\log(1-u(2,5)\,u(6,2))\,u(2,5)\,u(6,2)\left(-\frac{\log(1-u(2,5)\,u(6,2))\log(-\mathrm{y2m}\,(1-u(2,5)\,u(6,2)))}{\mathrm{y2m}\,(1-u(2,5)\,u(6,2))-\mathrm{y2p}\,(1-u(2,5)\,u(6,2))}+\right.$$

$$\frac{\log(1-u(2,5)\,u(6,2))\log(-\mathrm{y2p}\,(1-u(2,5)\,u(6,2)))}{\mathrm{y2m}\,(1-u(2,5)\,u(6,2))-\mathrm{y2p}\,(1-u(2,5)\,u(6,2))}+\frac{\mathrm{Li}_2(\mathrm{y2m}+1)}{\mathrm{y2m}\,(1-u(2,5)\,u(6,2))-\mathrm{y2p}\,(1-u(2,5)\,u(6,2))}-$$

$$\left.\frac{\mathrm{Li}_2(\mathrm{y2p}+1)}{\mathrm{y2m}\,(1-u(2,5)\,u(6,2))-\mathrm{y2p}\,(1-u(2,5)\,u(6,2))}\right)-\frac{1}{u(4)\,(u(2,5)-1)}\log(1-u(2,5)\,u(6,2))$$

$$\left(-\frac{\log(1-u(2,5)\,u(6,2))\log(-\mathrm{y2m}\,(1-u(2,5)\,u(6,2)))}{\mathrm{y2m}\,(1-u(2,5)\,u(6,2))-\mathrm{y2p}\,(1-u(2,5)\,u(6,2))}+\frac{\log(1-u(2,5)\,u(6,2))\log(-\mathrm{y2p}\,(1-u(2,5)\,u(6,2)))}{\mathrm{y2m}\,(1-u(2,5)\,u(6,2))-\mathrm{y2p}\,(1-u(2,5)\,u(6,2))}+\right.$$

$$\left.\frac{\mathrm{Li}_2(\mathrm{y2m}+1)}{\mathrm{y2m}\,(1-u(2,5)\,u(6,2))-\mathrm{y2p}\,(1-u(2,5)\,u(6,2))}-\frac{\mathrm{Li}_2(\mathrm{y2p}+1)}{\mathrm{y2m}\,(1-u(2,5)\,u(6,2))-\mathrm{y2p}\,(1-u(2,5)\,u(6,2))}\right)+$$

$$\frac{\log^2(1-u(2,5))\,(\log(-\mathrm{y2p}\,(1-u(2,5)))-\log(-\mathrm{y2m}\,(1-u(2,5))))}{u(4)\,(\mathrm{y2m}\,(1-u(2,5))-\mathrm{y2p}\,(1-u(2,5)))}-$$

$$\frac{\log(1-u(2,5))\,(\mathrm{Li}_2(\mathrm{y2m}\,(1-u(2,5))+1)-\mathrm{Li}_2(\mathrm{y2p}\,(1-u(2,5))+1))}{u(4)\,(\mathrm{y2m}\,(1-u(2,5))-\mathrm{y2p}\,(1-u(2,5)))}+$$

$$\left(-\frac{1}{2}\log(\mathrm{y2m}\,(1-u(2,5))+1)\log^2(-\mathrm{y2m}\,(1-u(2,5)))-\log^2(u(2,5))\log(-\mathrm{y2m}\,(1-u(2,5)))-\right.$$

$$\log\left(1+\frac{1}{\mathrm{y2m}}\right)\log(\mathrm{y2m}\,(1-u(2,5))+1)\log(-\mathrm{y2m}\,(1-u(2,5)))+\log\left(1+\frac{1}{\mathrm{y2m}}\right)\log(u(2,5))$$

$$\log(-\mathrm{y2m}\,(1-u(2,5)))+\log(\mathrm{y2m}\,(1-u(2,5))+1)\log(u(2,5))\log(-\mathrm{y2m}\,(1-u(2,5)))+$$

$$\log(1-u(2,5))\log(u(2,5))\log(-\mathrm{y2m}\,(1-u(2,5)))+\frac{1}{2}\log(\mathrm{y2p}\,(1-u(2,5))+1)\log^2(-\mathrm{y2p}\,(1-u(2,5)))+$$

$$\log(-\mathrm{y2p}\,(1-u(2,5)))\log^2(u(2,5))+\log\left(1+\frac{1}{\mathrm{y2p}}\right)\log(\mathrm{y2p}\,(1-u(2,5))+1)\log(-\mathrm{y2p}\,(1-u(2,5)))-$$

$$\log\left(1+\frac{1}{\mathrm{y2p}}\right)\log(-\mathrm{y2p}\,(1-u(2,5)))\log(u(2,5))-\log(\mathrm{y2p}\,(1-u(2,5))+1)\log(-\mathrm{y2p}\,(1-u(2,5)))$$

$$\log(u(2,5))-\log(1-u(2,5))\log(-\mathrm{y2p}\,(1-u(2,5)))\log(u(2,5))+\log(1-u(2,5))\,\mathrm{Li}_2(\mathrm{y2m}\,(1-u(2,5))+$$

$$\log(1-u(2,5))\,\mathrm{Li}_2(\mathrm{y2p}\,(1-u(2,5))+1)-\log(1-u(2,5))\,\mathrm{Li}_2\left(\frac{\mathrm{y2m}\,(1-u(2,5))+1}{u(2,5)}\right)+$$

$$\log(1-u(2,5))\,\mathrm{Li}_2\left(\frac{\mathrm{y2p}\,(1-u(2,5))+1}{u(2,5)}\right)-\mathrm{Li}_3\left(1+\frac{1}{\mathrm{y2m}}\right)-\mathrm{Li}_3\left(-\frac{1}{\mathrm{y2m}}\right)+\mathrm{Li}_3\left(1+\frac{1}{\mathrm{y2p}}\right)+$$

$$\mathrm{Li}_3\left(-\frac{1}{\mathrm{y2p}}\right)+\mathrm{Li}_3(\mathrm{y2m}\,(1-u(2,5))+1)-\mathrm{Li}_3(\mathrm{y2p}\,(1-u(2,5))+1)-\mathrm{Li}_3\left(\frac{\mathrm{y2m}\,(1-u(2,5))+1}{u(2,5)}\right)+$$

$$\mathrm{Li}_3\left(-\frac{\mathrm{y2m}\,(1-u(2,5))+1}{\mathrm{y2m}\,u(2,5)}\right)+\mathrm{Li}_3\left(\frac{\mathrm{y2p}\,(1-u(2,5))+1}{u(2,5)}\right)-\mathrm{Li}_3\left(-\frac{\mathrm{y2p}\,(1-u(2,5))+1}{\mathrm{y2p}\,u(2,5)}\right)-$$

$$\mathrm{Li}_3\left(\frac{-\mathrm{y2m}\,(1-u(2,5))+u(2,5)-1}{u(2,5)}\right)+\mathrm{Li}_3\left(-\frac{-\mathrm{y2m}\,(1-u(2,5))+u(2,5)-1}{\mathrm{y2m}\,(1-u(2,5))\,u(2,5)}\right)+$$

$$\left.\mathrm{Li}_3\left(\frac{-\mathrm{y2p}\,(1-u(2,5))+u(2,5)-1}{u(2,5)}\right)-\mathrm{Li}_3\left(-\frac{-\mathrm{y2p}\,(1-u(2,5))+u(2,5)-1}{\mathrm{y2p}\,(1-u(2,5))\,u(2,5)}\right)\right)\Big/$$

$$(u(4)\,(\mathrm{y2m}\,(1-u(2,5))-\mathrm{y2p}\,(1-u(2,5))))+\frac{\left(\log^2(-\mathrm{y2p})-\log^2(-\mathrm{y2m})\right)\log(1-u(2,5))}{(2\,\mathrm{y2m}-2\,\mathrm{y2p})\,u(4)\,(u(2,5)-1)}-$$

$$\frac{\left(\log^2(-\mathrm{y2p})-\log^2(-\mathrm{y2m})\right)\log(u(3,6))}{(2\,\mathrm{y2m}-2\,\mathrm{y2p})\,u(4)\,(u(2,5)-1)}-$$

$$\frac{\left(\log^2(-\mathrm{y2p})-\log^2(-\mathrm{y2m})\right)\log(1-u(2,5)\,u(6,2))}{(2\,\mathrm{y2m}-2\,\mathrm{y2p})\,u(4)\,(u(2,5)-1)}-$$

$$\frac{\log(1-u(2,5))\,(\mathrm{Li}_2(\mathrm{y2m}+1)-\mathrm{Li}_2(\mathrm{y2p}+1))}{(\mathrm{y2m}-\mathrm{y2p})\,u(4)\,(u(2,5)-1)}+$$

$$\frac{\log(u(3,6))\,(\mathrm{Li}_2(\mathrm{y2m}+1)-\mathrm{Li}_2(\mathrm{y2p}+1))}{(\mathrm{y2m}-\mathrm{y2p})\,u(4)\,(u(2,5)-1)}+$$

$$\frac{\log(1-u(2,5)\,u(6,2))\,(\mathrm{Li}_2(\mathrm{y2m}+1)-\mathrm{Li}_2(\mathrm{y2p}+1))}{(\mathrm{y2m}-\mathrm{y2p})\,u(4)\,(u(2,5)-1)}+$$

$$\frac{1}{u(4)\,(u(2,5)-1)}$$

$$\left(-\frac{\log(-\mathrm{y2m})\log^2\left(\frac{1}{u(4)}\right)}{\mathrm{y2m}-\mathrm{y2p}}+\frac{\log(-\mathrm{y2p})\log^2\left(\frac{1}{u(4)}\right)}{\mathrm{y2m}-\mathrm{y2p}}-\frac{\log^2(-\mathrm{y2m}\,u(4))\log\left(\frac{1}{u(4)}\right)}{2\,(\mathrm{y2m}-\mathrm{y2p})}+\frac{\log^2(-\mathrm{y2p}\,u(4))\log\left(\frac{1}{u(4)}\right)}{2\,(\mathrm{y2m}-\mathrm{y2p})}+\right.$$

$$\frac{\mathrm{Li}_2(\mathrm{y2m}\,u(4)+1)\log\left(\frac{1}{u(4)}\right)}{\mathrm{y2m}-\mathrm{y2p}}-\frac{\mathrm{Li}_2(\mathrm{y2p}\,u(4)+1)\log\left(\frac{1}{u(4)}\right)}{\mathrm{y2m}-\mathrm{y2p}}-\frac{\log^2(-\mathrm{y2m}\,u(4))\log(\mathrm{y2m}\,u(4)+1)}{2\,(\mathrm{y2m}-\mathrm{y2p})}+$$

$$\left.\frac{\log^2(-\mathrm{y2p}\,u(4))\log(\mathrm{y2p}\,u(4)+1)}{2\,(\mathrm{y2m}-\mathrm{y2p})}-\frac{\mathrm{Li}_3(-\mathrm{y2m}\,u(4))}{\mathrm{y2m}-\mathrm{y2p}}+\frac{\mathrm{Li}_3(-\mathrm{y2p}\,u(4))}{\mathrm{y2m}-\mathrm{y2p}}\right)+\frac{1}{u(4)\,(u(2,5)-1)}$$

$$\left(-\frac{\log(-\mathrm{y2m})\log^2\left(\frac{1}{1-u(2,5)}\right)}{\mathrm{y2m}-\mathrm{y2p}}+\frac{\log(-\mathrm{y2p})\log^2\left(\frac{1}{1-u(2,5)}\right)}{\mathrm{y2m}-\mathrm{y2p}}-\frac{\log^2(-\mathrm{y2m}\,(1-u(2,5)))\log\left(\frac{1}{1-u(2,5)}\right)}{2\,(\mathrm{y2m}-\mathrm{y2p})}+\right.$$

$$\frac{\log^2(-\mathrm{y2p}\,(1-u(2,5)))\log\left(\frac{1}{1-u(2,5)}\right)}{2\,(\mathrm{y2m}-\mathrm{y2p})}+\frac{\mathrm{Li}_2(\mathrm{y2m}\,(1-u(2,5))+1)\log\left(\frac{1}{1-u(2,5)}\right)}{\mathrm{y2m}-\mathrm{y2p}}-$$

$$\frac{\mathrm{Li}_2(\mathrm{y2p}\,(1-u(2,5))+1)\log\left(\frac{1}{1-u(2,5)}\right)}{\mathrm{y2m}-\mathrm{y2p}}-\frac{\log(\mathrm{y2m}\,(1-u(2,5))+1)\log^2(-\mathrm{y2m}\,(1-u(2,5)))}{2\,(\mathrm{y2m}-\mathrm{y2p})}+$$

$$\left.\frac{\log(\mathrm{y2p}\,(1-u(2,5))+1)\log^2(-\mathrm{y2p}\,(1-u(2,5)))}{2\,(\mathrm{y2m}-\mathrm{y2p})}-\frac{\mathrm{Li}_3(-\mathrm{y2m}\,(1-u(2,5)))}{\mathrm{y2m}-\mathrm{y2p}}+\frac{\mathrm{Li}_3(-\mathrm{y2p}\,(1-u(2,5)))}{\mathrm{y2m}-\mathrm{y2p}}\right)-$$

$$\frac{1}{u(4)\,(u(2,5)-1)}\left(-\frac{\log(-\mathrm{y2m})\log^2\left(\frac{1}{1-u(2,5)\,u(6,2)}\right)}{\mathrm{y2m}-\mathrm{y2p}}+\frac{\log(-\mathrm{y2p})\log^2\left(\frac{1}{1-u(2,5)\,u(6,2)}\right)}{\mathrm{y2m}-\mathrm{y2p}}-\right.$$

$$\frac{\log^2(-\mathrm{y2m}\,(1-u(2,5)\,u(6,2)))\log\left(\frac{1}{1-u(2,5)\,u(6,2)}\right)}{2\,(\mathrm{y2m}-\mathrm{y2p})}+\frac{\log^2(-\mathrm{y2p}\,(1-u(2,5)\,u(6,2)))\log\left(\frac{1}{1-u(2,5)\,u(6,2)}\right)}{2\,(\mathrm{y2m}-\mathrm{y2p})}+$$

$$\frac{\mathrm{Li}_2(\mathrm{y2m}\,(1-u(2,5)\,u(6,2))+1)\log\left(\frac{1}{1-u(2,5)\,u(6,2)}\right)}{\mathrm{y2m}-\mathrm{y2p}}-\frac{\mathrm{Li}_2(\mathrm{y2p}\,(1-u(2,5)\,u(6,2))+1)\log\left(\frac{1}{1-u(2,5)\,u(6,2)}\right)}{\mathrm{y2m}-\mathrm{y2p}}-$$

$$\frac{\log^2(-\mathrm{y2m}\,(1-u(2,5)\,u(6,2)))\log(\mathrm{y2m}\,(1-u(2,5)\,u(6,2))+1)}{2\,(\mathrm{y2m}-\mathrm{y2p})}+$$

$$\frac{\log^2(-\mathrm{y2p}\,(1-u(2,5)\,u(6,2)))\log(\mathrm{y2p}\,(1-u(2,5)\,u(6,2))+1)}{2\,(\mathrm{y2m}-\mathrm{y2p})}-$$

$$\left.\frac{\mathrm{Li}_3(-\mathrm{y2m}\,(1-u(2,5)\,u(6,2)))}{\mathrm{y2m}-\mathrm{y2p}}+\frac{\mathrm{Li}_3(-\mathrm{y2p}\,(1-u(2,5)\,u(6,2)))}{\mathrm{y2m}-\mathrm{y2p}}\right)-$$

$$\left(-\log(-\mathrm{y2m}\,u(4))\log^2(1-u(4))+\log(-\mathrm{y2p}\,u(4))\log^2(1-u(4))+\log\left(1+\frac{1}{\mathrm{y2m}}\right)\log(-\mathrm{y2m}\,u(4))\log(1-u(4))+\right.$$

$$\log(u(4))\log(-\mathrm{y2m}\,u(4))\log(1-u(4))-\log\left(1+\frac{1}{\mathrm{y2p}}\right)\log(-\mathrm{y2p}\,u(4))\log(1-u(4))-$$

$$\log(u(4))\log(-\mathrm{y2p}\,u(4))\log(1-u(4))+\log(-\mathrm{y2m}\,u(4))\log(\mathrm{y2m}\,u(4)+1)\log(1-u(4))-$$

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The one-mass hexagon in D=6

$$
\begin{aligned}
&\mathcal{I}_{6,m}(u_1,u_2,u_3,u_4)\\
&\quad= \frac{1}{\sqrt{\Delta}}\Bigg[-\sum_{i=1}^{8}\sum_{j=1}^{2}\Big(L_3(x^+_{i,j},x^-_{i,j}) - \frac{1}{6}\,\bar{\ell}_1(x^+_{i,j},x^-_{i,j})^3 - \frac{\pi^2}{6}\,\bar{\ell}_1(x^+_{i,j},x^-_{i,j})\Big)\\
&\qquad +\tfrac{1}{2}\Big(\bar{\ell}_1(x^+_{2,1},x^-_{2,1}) + \bar{\ell}_1(x^+_{2,2},x^-_{2,2})\Big)\Big(2\bar{\ell}_1(x^+_{1,1},x^-_{1,1})\,\bar{\ell}_1(x^+_{1,2},x^-_{1,2})\\
&\qquad +\bar{\ell}_1(x^+_{1,1},x^-_{1,1})\,\bar{\ell}_1(x^+_{3,1},x^-_{3,1}) + \bar{\ell}_1(x^+_{1,1},x^-_{1,1})\,\bar{\ell}_1(x^+_{3,2},x^-_{3,2}) + \bar{\ell}_1(x^+_{1,2},x^-_{1,2})\,\bar{\ell}_1(x^+_{3,1},x^-_{3,1})\\
&\qquad +\bar{\ell}_1(x^+_{1,2},x^-_{1,2})\,\bar{\ell}_1(x^+_{3,2},x^-_{3,2}) + 2\bar{\ell}_1(x^+_{3,1},x^-_{3,1})\,\bar{\ell}_1(x^+_{3,2},x^-_{3,2})\Big)\Bigg], \qquad (4.1)
\end{aligned}
$$

$$
L_3(x^+,x^-) = \sum_{k=0}^{2}\frac{(-1)^k}{(2k)!!}\,\ln^k(x^+x^-)\,\big(\ell_{3-k}(x^+) - \ell_{3-k}(x^-)\big)\,,
$$

$$
\ell_n(x) = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\mathrm{Li}_n(1/x)\right)\,,
$$

and all the arguments are cross ratios of twistors.

Symbols

Definition and properties

Aim

- Main idea: To each expression in terms of polylogarithms we associate a *tensor*.

- Conjecture: ALL the complicated *functional* equations among polylogarithms correspond to *algebraic* relations in the tensor algebra

- In other words, if we have a complicated expression, and if we can find another expression that has the same symbol, then there are functional equations that bring you from one to the other (they are not necessarily equal!)

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- Provided that $\mathcal{S}(\pi^2) = 0\,.$

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- Similarly:

$$\mathcal{S}(\ln z\ln(1-z)) = z\otimes(1-z) + (1-z)\otimes z$$

$$d[\ln(1-z)\ln z] = \ln z\,d\ln(1-z) + \ln(1-z)d\ln z$$

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- Zeta values vanish under the symbol map.

$$\mathcal{S}(\zeta_m) = 0$$

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- In this case however they are absent. [Why?]

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$$\mathrm{Li}_2(z^2) = 2\mathrm{Li}_2(z) + 2\mathrm{Li}_2(-z)$$

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- What about branch cuts and zeta values?

Exercise

- Proof Abel's identity (aka the five-term relation)

$$\ln(1-x)\ln(1-y) = \mathrm{Li}_2\left(\frac{x}{1-y}\right)+\mathrm{Li}_2\left(\frac{y}{1-x}\right)-\mathrm{Li}_2\left(x\right)-\mathrm{Li}_2\left(y\right)-\mathrm{Li}_2\left(\frac{xy}{(1-x)(1-y)}\right)$$

Symbols

- Assume you have a function $F(x_1, \ldots, x_n)$ that satisfies

$$dF(x_1, \ldots, x_n) = \sum_i F_i(x_1, \ldots, x_n)\, d\ln R_i$$

Then the symbol of F is defined by

$$\mathcal{S}(F) = \sum_i \mathcal{S}(F_i) \otimes R_i$$

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- Multiple polylogarithms satisfy the differential equation (for generic arguments):

$$dG(a_{n-1}, \ldots, a_1; a_n) = \sum_{i=1}^{n-1} G(a_{n-1}, \ldots, \hat{a}_i, \ldots, a_1; a_n)\, d\log\left(\frac{a_i - a_{i+1}}{a_i - a_{i-1}}\right)$$

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- Some care is needed though for non generic arguments!

Symbols

- Symbols of classical polylogarithms:

$$\mathcal{S}\big(Li_m(x)\big) = -\ \big((1-x)\otimes \underbrace{x\otimes\cdots\otimes x}_{m-1\ factors}\big)$$

- Symbol of a generic multiple polylogarithm of weight 2:

$$\mathcal{S}\left(G(a,b;z)\right) = \left(1-\frac{z}{a}\right)\otimes\left(1-\frac{a}{b}\right) + \left(1-\frac{z}{b}\right)\otimes\left(1-\frac{z}{a}\right) - \left(1-\frac{z}{b}\right)\otimes\left(1-\frac{b}{a}\right)$$

- N.B.: Weight of the polylogarithm = rank of the tensor!

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- Shuffles

$$\mathcal{S}\left(G(\vec{a};x)\,G(\vec{b};y)\right) = \mathcal{S}\left(G(\vec{a};x)\right) ⧢ \mathcal{S}\left(G(\vec{b};y)\right)$$

where on the rhs we use the shuffle product in the space of symbols, e.g.,

$$a ⧢ b = a \otimes b + b \otimes a$$

$$(a \otimes b) ⧢ c = a \otimes b \otimes c + a \otimes c \otimes b + c \otimes a \otimes b$$

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- In other words, so far we have only dealt with the question of how to obtain the symbol of a function.
- Now, we also need to know how to *integrate* a symbol back to a function (which is in general much more complicated!).

Symbols

Integration

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- Given a tensor (symbol)

$$S = \sum_{I=(i_1,\ldots,i_m)} c_I \; \omega_{i_1} \otimes \cdots \otimes \; \omega_{i_m}$$

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can we always find a function whose symbol is equal to S?

- Answer: NO! This is only possible if S satisfies a certain integrability condition

$$\sum_{I=(i_1,\ldots,i_m)} c_I\ \left[d\log\omega_{i_j} \wedge d\log\omega_{i_{j+1}}\right] \omega_{i_1} \otimes \cdots \otimes \widehat{\omega}_{i_j} \otimes \widehat{\omega}_{i_{j+1}} \otimes \cdots \otimes\ \omega_{i_m}\ = \ 0$$

- Exercise: Prove the integrability of the multiple polylogarithm of weight 2, $G(a,b;z)$.

Integration

- Assume an integrable symbol *S*.
- Decide on the functions that should appear (‘basis functions’).
- Construct suitable arguments (most difficult step!)
- Write a linear combination of the basis functions in these arguments.
- Solve a linear system for the coefficients.
- Fix the remaining ambiguity.

Example

$$-(1-2x)\otimes(1-x)-(1-x)\otimes(1-2x)-(1+x)\otimes x$$

- We have to decide on some weight two functions that could appear in the answer:

$$\sum_i c_i\,\mathrm{Li}_2(f_i(x))+\sum_{j,k} c_{jk}\,\log(g_j(x))\,\log(h_k(x))$$

- Next we should find the potential arguments of the polylogarithms. This is the most difficult step.
- If we have this, we can equate the symbol of our ansatz to the original symbol, and solve for the coefficients.
- It turns out that here this is not even needed...

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- But this is nothing but the imaginary part of $\mathcal{S}(\mathrm{Li}_2(-x))$!

Example

- Next, subtract off the contribution you have found:

$$S - \mathcal{S}(\mathrm{Li}_2(-x)) = -(1-2x)\otimes(1-x) - (1-x)\otimes(1-2x)$$

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- Note however that the symbol does not uniquely fix the function, because we could add terms with vanishing symbol (zeta values, branch cuts, etc.)

Indecomposable functions

- For the integration process, it is good to know which functions of a given weight are (a priori) independent.
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➡ Weight 2: $\mathrm{Li}_2, \mathrm{Li}_{1,1}$

➡ Weight 3: $\mathrm{Li}_3, \mathrm{Li}_{2,1}, \mathrm{Li}_{1,2}, \mathrm{Li}_{1,1,1}$

➡ Weight 4: $\mathrm{Li}_4, \mathrm{Li}_{3,1}, \mathrm{Li}_{1,3}, \mathrm{Li}_{2,2}, \mathrm{Li}_{2,1,1}, \mathrm{Li}_{1,2,1}, \mathrm{Li}_{1,1,2}, \mathrm{Li}_{1,1,1,1}$

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- For the integration process, it is good to know which functions of a given weight are (a priori) independent.
- ‘Mathematical Folklore’: Only polylogarithms where no index is equal to 1 are independent.

➡ Weight 1: $\ln, \text{Li}_1$

➡ Weight 2: $\text{Li}_2, \text{Li}_{1,1}$

➡ Weight 3: $\text{Li}_3, \text{Li}_{2,1}, \text{Li}_{1,2}, \text{Li}_{1,1,1}$

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- Consequence: Multiple polylogarithms become important for the first time at weight 4.

Summary of lecture 3

- We have defined symbols of multiple polylogarithms, and discussed their properties.

- We also briefly discussed the problem of integrating symbols, i.e., finding a function whose symbol matches a given tensor.

- Question: Can we obtain the symbols of Feynman integrals without having to do the integrals? [See Caron-Huot's lecture]