Flavor Symmetry Flashes

in memory of Eileen (1949-2022)

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ANTEATERS
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Strong Flavor Symmetry

1932: Neutron was discovered with the same mass as the proton.

1932: Heisenberg (and others) postulated isotopic spin [strong flavor SU(2)].

1953: Gell-Mann (and others) added strangeness.

1612: Francis Bacon proclaimed "There is no excellent beauty that hath not some strangeness in the proportion".

1961: Gell-Mann and Ne'eman postulated [strong flavor] SU(3).

1964: Gell-Mann and Zweig postulated quarks.

1972: Fritzsch/Gell-Mann/Leutwyler postulated QCD [color SU(3)] leading eventually to a paradigm shift in the understanding of strong flavor symmetry.

Instead of $m_p \simeq m_n$ for the origin of SU(2), and $m_p \simeq m_n \simeq m_\Lambda$ for that of SU(3), the correct understanding is $m_u, m_d << m_s << \Lambda_{QCD}$.

In other words, the strong flavor symmetries are not sensitive to the masses of the particles involved, a lesson also to be learned later for weak flavor symmetry. [Flash 1]

S_3 Reps and Apps

The non-Abelian discrete symmetry S_3 may be chosen to act on the three vertices of an equilateral triangle.

In the (x, y) plane, let

$1 \sim (1,0),\ 2 \sim (-1/2, \sqrt{3}/2),\ 3 \sim (-1/2, -\sqrt{3}/2)$,

then the 6 group elements are 2×2 matrices permuting these 3 vertices. All applications of S_3 use this representation before 1991.

Ma, PRD 43, 2761 (1991): Use the basis $(x + iy, x - iy)$ instead, then $1 \sim (1,1),\ 2 \sim (\omega, \omega^2),\ 3 \sim (\omega^2, \omega)$, with $\omega^3 = 1$.

class	element	real rep	complex rep
C_1	(123)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
C_2	(231)	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$
C_2	(312)	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$
C_3	(132)	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
C_3	(321)	$\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}$
C_3	(213)	$\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$	$\begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$

S_3 multiplication rule: $\mathbf{2} \times \mathbf{2} = 1 + 1' + 2$.

rep	1	$1'$	2
real	$11 + 22$	$12 - 21$	$\begin{pmatrix} 12 + 21 \\ 11 - 22 \end{pmatrix}$
complex	$12 + 21$	$12 - 21$	$\begin{pmatrix} 22 \\ 11 \end{pmatrix}$

If $(\psi_1, \psi_2) \sim 2$ then $(\psi_1^*, \psi_2^*) \sim 2$ in the real rep, but in the complex rep, $(\psi_2^*, \psi_1^*) \sim 2$ instead.

The invariant product of three 2's is
$121 + 211 + 112 - 222$ in the real rep.
It is simply $111 + 222$ in the complex rep.

1964: Yamaguchi proposed S_3 for strong flavor symmetry.

1978: Pakvasa/Sugawara used it for weak flavor symmetry.

In the SM, let $(u,c)_{L,R} \sim \mathbf{2}$, $(d,s)_{L,R} \sim \mathbf{2}$, $\Phi_0 \sim 1$, $\Phi_{1,2} \sim \mathbf{2}$, then

$$\mathcal{M}_{uc}, \mathcal{M}_{ds} \sim \begin{pmatrix} b & a \\ a & c \end{pmatrix},$$

from $\langle \Phi_0 \rangle \propto a$, $\langle \Phi_1 \rangle = b$, $\langle \Phi_2 \rangle \propto c$, in the complex rep.

If $b = 0$, then $\theta_C \simeq \sqrt{m_d/m_s} - \sqrt{m_u/m_c}$.

In the real rep, $\mathcal{M}_{uc}, \mathcal{M}_{ds} \sim \begin{pmatrix} a' + c' & b' \\ b' & a' - c' \end{pmatrix}$.

To obtain the desirable structure, $c' = -a'$ has to be assumed. So why does the choice of representation matter? The answer is that if the S_3 symmetry remains exact and all multiplets are available, representations do not make any difference, but for any physical application, we first choose the fermion multiplets, then the Higgs fields which break the symmetry. How this is done depends on the representation. [Flash 2]

[Babu/Xu/Yu(arXiv:2312.15828) uses the real rep.]

$$\text{Suppose we want} \quad \mathcal{M}_q = \begin{pmatrix} 0 & a & 0 \\ a & b & c \\ 0 & d & e \end{pmatrix},$$

then with $S_3 \times Z_2$ in the complex representation, let
$(\bar{d}_L, \bar{s}_L) \sim (2, +)$, $\bar{b}_L \sim (1, -)$,
$(d_R, s_R) \sim (2, +)$, $b_R \sim (1, -)$;
$(\bar{u}_L, \bar{c}_L) \sim (2, +)$, $\bar{t}_L \sim (1, -)$,
$(u_R, c_R) \sim (2, -)$, $t_R \sim (1, +)$.
$\Phi = (\phi^+, \phi^0)$ with $\Phi_0 \sim (1, +)$, $\Phi'_0 \sim (1, -)$,
$(\Phi_1, \Phi_2) \sim (2, +)$, $(\Phi'_1, \Phi'_2) \sim (2, -)$, $\langle \phi^0_0 \rangle = v_0$,
$\langle {\phi'}^0_0 \rangle = v'_0$, $\langle (\phi^0_1, \phi^0_2) \rangle = (0, v_2)$, $\langle ({\phi'}^0_1, {\phi'}^0_2) \rangle = (v_1, 0)$.

To enforce the $(0, v_2)$ and $(v_1, 0)$ alignment, a Higgs triplet $\xi = (\xi^{++}, \xi^{+}, \xi^{0}) \sim (1, -)$ is added, so that $\xi^\dagger(\Phi_1\Phi'_2 + \Phi_2\Phi'_1)$ is allowed.

Whereas the Higgs potential is invariant under $S_3 \times Z_2$, the assumed breaking pattern retains a different Z_2 under which Φ_1 and Φ'_2 are odd, and all other fields even.

Note that the term $\Phi_1{}^\dagger\Phi'_1 + \Phi_2^\dagger\Phi'_2$ is forbidden by the input Z_2, whereas $(\Phi_1{}^\dagger\Phi'_1 + \Phi_2^\dagger\Phi'_2)^2$ is allowed by both the input Z_2 and the residual Z_2.

The Higgs triplet ξ will be used for neutrino masses. Details to be worked out [Ezzat/Khalil/Ma(2024)].

The desired 3×3 quark mass matrix patterns

$$\mathcal{M}_q = \begin{pmatrix} 0 & a & 0 \\ a & b & c \\ 0 & d & e \end{pmatrix}$$

are obtained with the zeros guaranteed by the Z_2 residual symmetry of the Higgs potential.

For $\mathcal{M}_d$, $a \sim (\bar{d}_L s_R + \bar{s}_L d_R) v_0$, $b \sim \bar{s}_L s_R v_2$, $c \sim \bar{s}_L b_R v_1$, $d \sim \bar{b}_L s_R v_1$, $e \sim \bar{b}_L b_R v_0$.

For $\mathcal{M}_u$, $a \sim (\bar{u}_L c_R + \bar{c}_L u_R) {v'_0}^*$, $b \sim \bar{c}_L c_R v_1^*$, $c \sim \bar{c}_L t_R v_2^*$, $d \sim \bar{t}_L c_R v_2^*$, $e \sim \bar{t}_L t_R {v'_0}^*$.

A_4 Facts and Lore

How the Ancients viewed the contents of the Universe:

Element	Greek	Hindu	Buddhist	Chinese
Aether	–	√	√	–
Air	√	√	√	–
Earth	√	√	√	√
Fire	√	√	√	√
Water	√	√	√	√
Wood	–	–	–	√
Metal	–	–	–	√

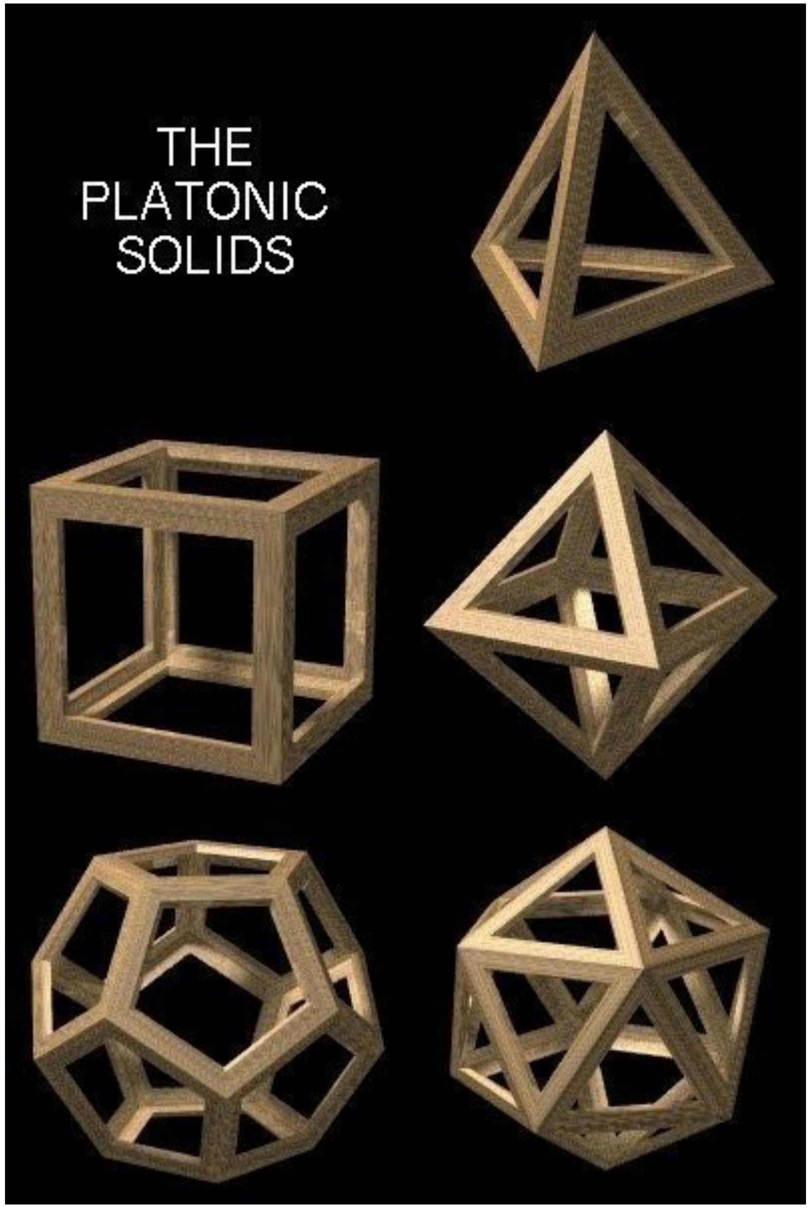
Theaetetus (c. 417 B.C. to c. 369 B.C.) proved that there are five and only five perfect geometric solids.

Plato (c. 427 B.C. to c. 347 B.C.) then proposed that there should also be a fifth element (quintessence), which is invisible and pervades the cosmos.
He also assigned each element to a solid.

Hollywood (1997): Fifth Element = Milla Jovovich.

In 4 dimensions, there are 6 perfect geometric solids.
In 5 (or more) dimensions, there are only 3.

Golden Ratio: $\phi = [2\cos(2\pi/5)]^{-1} = 1.618$.


THE
PLATONIC
SOLIDS

Perfect Three-Dimensional Geometric Solids:

solid	faces	vertices	Plato	group
tetrahedron	4	4	fire	A_4
octahedron	8	6	air	S_4
cube	6	8	earth	S_4
icosahedron	20	12	water	A_5
dodecahedron	12	20	quintessence	A_5

Amusingly, there are also 5 string theories in 10 dimensions: Type I is dual to Heterotic SO(32), Type IIA is dual to Heterotic $E_8 \times E_8$, and Type IIB is self-dual.

In the (x, y, z) space, let the 4 vertices of the tetrahedron be $1 \sim (1, 1, 1)$, $2 \sim (1, -1, -1)$, $3 \sim (-1, 1, -1)$, $4 \sim (-1, -1, 1)$, then the 12 group elements are 3×3 matrices permuting these 4 vertices.

class	n	h	χ_1	χ_2	χ_3	χ_4
C_1	1	1	1	1	1	3
C_2	4	3	1	ω	ω^2	0
C_3	4	3	1	ω^2	ω	0
C_4	3	2	1	1	1	-1

As a rough visual guide, imagine the 4 vertices at Los Angeles, Kabul, Sydney, and Buenos Aires.

1978: Cabibbo and Wolfenstein speculated

$$U_{l\nu} = U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$.

In the PDG convention, this implies $s_{23} = c_{23} = 1/\sqrt{2}$, $s_{12} = c_{12} = 1/\sqrt{2}$, $s_{13} = 1/\sqrt{3}$, $c_{13} = \sqrt{2/3}$, and $\delta = \pi/2$. If $\omega \leftrightarrow \omega^2$, then $\delta = -\pi/2$.

Note that θ_{23} and δ_{CP} are very close to present neutrino data.

2001: Ma/Rajasekaran obtained U_ω using A_4.

This non-Abelian discrete symmetry has 12 elements and 4 irreducible representations: $\underline{1}, \underline{1}', \underline{1}'', \underline{3}$. Using

$$\underline{3} \times \underline{3} = \underline{1} + \underline{1}' + \underline{1}'' + \underline{3} + \underline{3}.$$

the following decompositions are obtained:

$$\begin{aligned} \underline{1} &= 11 + 22 + 33, \\ \underline{1}' &= 11 + \omega 22 + \omega^2 33, \\ \underline{1}'' &= 11 + \omega^2 22 + \omega 33. \end{aligned}$$

Let $(\nu, l)_i \sim \underline{3}$, $l_i^c \sim \underline{1}, \underline{1}', \underline{1}''$, and $\Phi_i \sim \underline{3}$, then

$$\mathcal{M}_l = \begin{pmatrix} f_e v_1^* & f_\mu v_1^* & f_\tau v_1^* \\ f_e v_2^* & f_\mu \omega v_2^* & f_\tau \omega^2 v_2^* \\ f_e v_3^* & f_\mu \omega^2 v_3^* & f_\tau \omega v_3^* \end{pmatrix}$$

$$= \begin{pmatrix} v_1^* & 0 & 0 \\ 0 & v_2^* & 0 \\ 0 & 0 & v_3^* \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} f_e & 0 & 0 \\ 0 & f_\mu & 0 \\ 0 & 0 & f_\tau \end{pmatrix} .$$

For $v_1 = v_2 = v_3$, a residual Z_3 symmetry exists with $U_\omega^\dagger$ as the link between $\mathcal{M}_l$ and $\mathcal{M}_\nu$, where the 3 charged lepton masses are arbitrary.

If neutrino masses come from Higgs triplets transforming as $\underline{1}, \underline{1}', \underline{1}''$, then $\mathcal{M}_\nu$ would be diagonal with 3 arbitrary masses, and the Cabibbo/Wolfenstein conjecture would be correct. In other words, a nontrivial mixing matrix is derived independent of the masses of the particles involved. [Flash 3]

2002: Harrison/Perkins/Scott proposed tribimaximal mixing, i.e.

$$U_{TBM} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$$

This means that

$$\mathcal{M}_\nu = \begin{pmatrix} m_2 & 0 & 0 \\ 0 & (m_1 - m_3)/2 & (m_1 + m_3)/2 \\ 0 & (m_1 + m_3)/2 & (m_1 - m_3)/2 \end{pmatrix},$$

which is diagonized by $U_\omega U_{TBM}$. Note again that the neutrino masses are arbitrary. The residual symmetry in the neutrino sector is now Z_2, as opposed to Z_3 in the charged lepton sector.

2012-3-8 Daya Bay announced that θ_{13} had been measured at 8.8°, thus ending tribimaximal mixing.

Cobimaximal Mixing

Special Form of $\mathcal{M}_\nu$: Ma(2002), Babu/Ma/Valle(2003):

In the basis where the charged-lepton mass matrix is diagonal, it was proposed that

$$\mathcal{M}_\nu = \begin{pmatrix} A & C & C^* \\ C & D^* & B \\ C^* & B & D \end{pmatrix},$$

where A, B are real. This was shown to lead to cobimaximal mixing: $\theta_{13} \neq 0$, $\theta_{23} = \pi/4$, $\delta_{CP} = \pm\pi/2$.

2004: Grimus/Lavoura first recognized that this pattern of $\mathcal{M}_\nu$ is in fact protected by a symmetry, i.e. $e \rightarrow e$ and $\mu \leftrightarrow \tau$ exchange with CP conjugation. [Flash 4]

Cobimaximal mixing is close to present data which indicate a preference for $\delta_{CP} = -\pi/2$. Note that this special form predicts that $|U_{\mu i}| = |U_{\tau i}|$. This harkens back to the original U_ω of 1978, where indeed this is satisfied. It is strongly suggestive that U_ω itself must have something to do with the realization of this special form of $\mathcal{M}_\nu$.

2000: Fukuura/Miura/Takasugi/Yoshimura noted that if $U_{l\nu} = U_\omega{}^\dagger \mathcal{O}$, where $\mathcal{O}$ is orthogonal, then $U^*_{2i} = U_{3i}$ for $i = 1, 2, 3$. Compared this to the PDG form of $U_{l\nu}$, i.e.

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

it is obvious that after rotating the phases of the third column and the second and third rows, the two matrices are identical if and only if $s_{23} = c_{23}$ and $\cos\delta = 0$, i.e. $\theta_{23} = \pi/4$ and $\delta_{CP} = \pm\pi/2$.

Origin of Fermion Masses

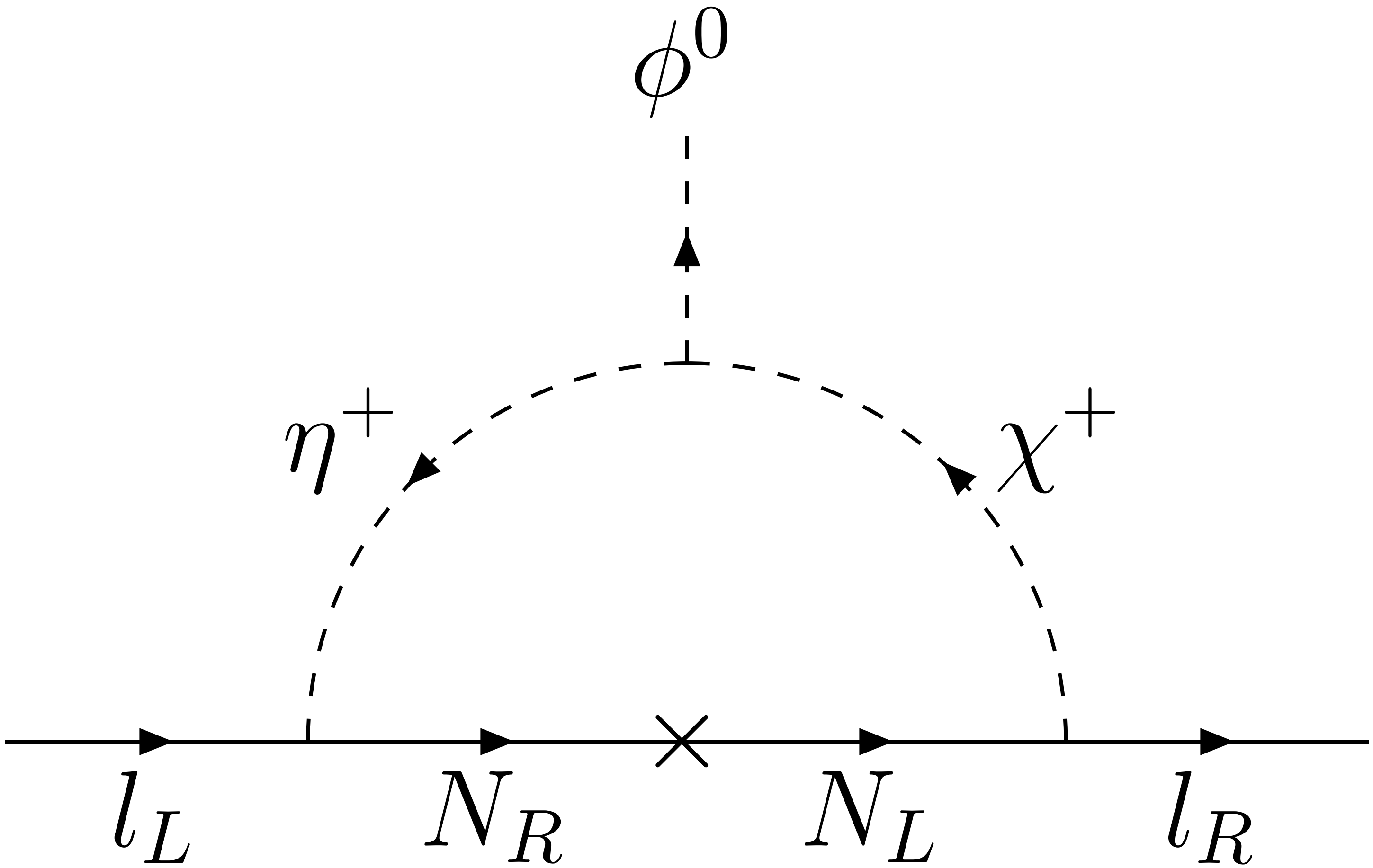
Ma, PRL 112, 091801(2014): There is only one Higgs, but fermion masses do not couple to it because of a flavor symmetry and a dark symmetry. This is natural using the scotogenic model [Ma(2006),Tao(1996)]. All SM fermion masses come from the soft breaking terms in the dark sector, which may have whatever residual symmetries that are desired. [Flash 5]. As an example using $A_5 \to A_4$, see my FLASY2022 talk [Ma(2022)]. To derive cobimaximal mixing, a set of real scalars in the dark sector may be used. [Ma and He (2015)].

With apology to J. R. R. Tokien:

Three families of quarks and leptons,

one Higgs to rule them all,

and in the darkness bind them.


ϕ^0
η^+
χ^+
l_L
N_R
N_L
l_R

The predicted new scalars which connect the quarks and leptons to their common dark-matter antecedents, i.e. $N_{1,2,3}$, are possibly observable at the LHC.
They may also change significantly the SM couplings of the one Higgs [Fraser/Ma(2014)] which are being measured. The doublet (η^+, η^0) and singlet χ^+ mix through the term $\mu(\eta^+\phi^0 - \eta^0\phi^+)\chi^-$, where $\langle\phi^0\rangle = v/\sqrt{2}$. Thus

$$\mathcal{M}^2_{\eta\chi} = \begin{pmatrix} m^2_\eta & \mu v/\sqrt{2} \\ \mu v/\sqrt{2} & m^2_\chi \end{pmatrix},$$

Let the mass eigenstates be $\zeta_1 = \eta\cos\theta + \chi\sin\theta$, and $\zeta_2 = \chi\cos\theta - \eta\sin\theta$ with masses m_1 and m_2, then $\mu v/\sqrt{2} = \sin\theta\cos\theta(m_1^2 - m_2^2)$. The one-loop mass is

$$m_l = \frac{f_\eta f_\chi \sin\theta\cos\theta m_N}{16\pi^2}\left(\frac{x_1\ln x_1}{x_1 - 1} - \frac{x_2\ln x_2}{x_2 - 1}\right),$$

where $x_{1,2} = m_{1,2}^2/m_N^2$.

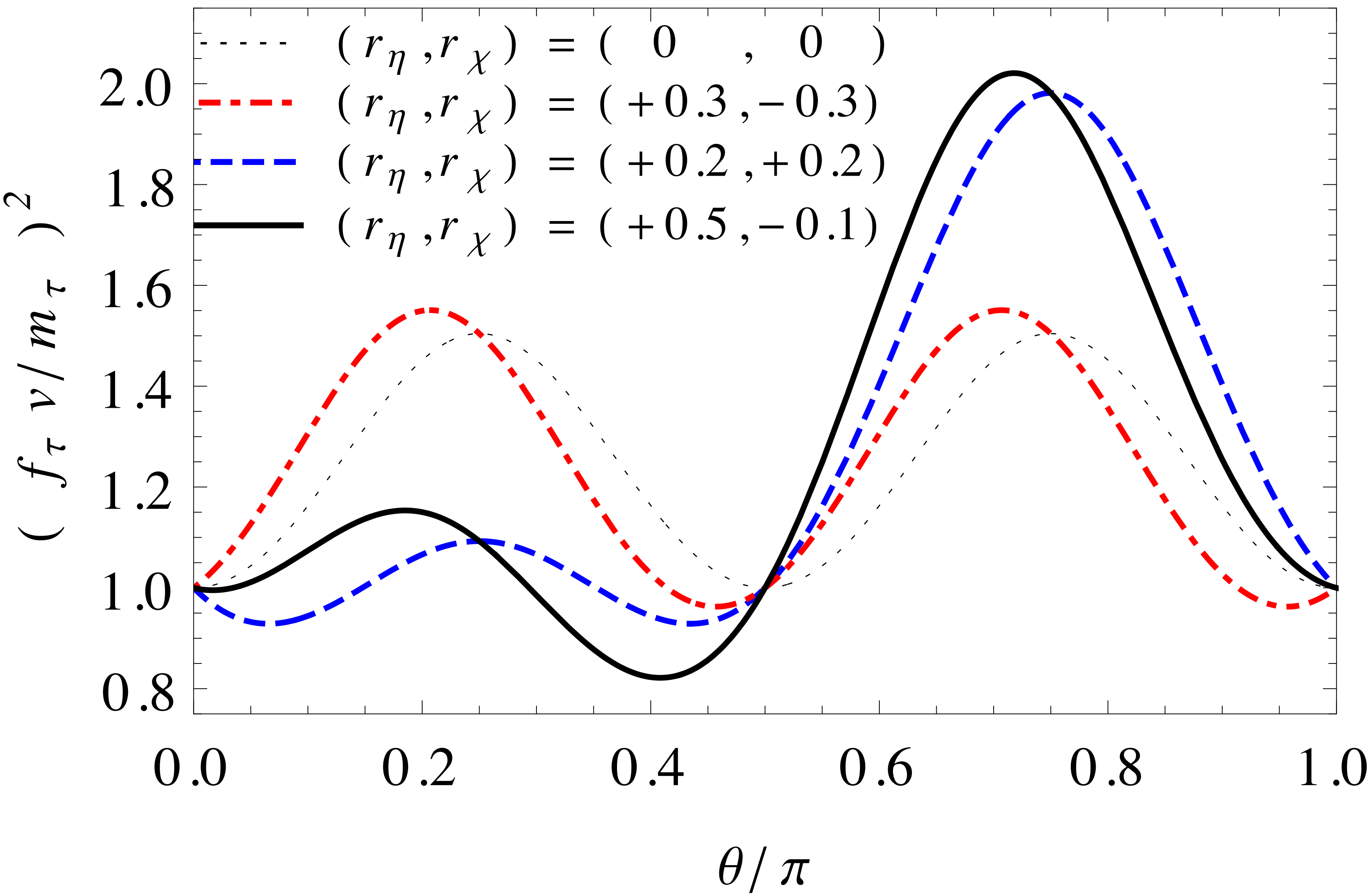
The Yukawa coupling of h to $\bar{l}l$ is now not exactly equal to m_l/v. It has three contributions, through $\eta^+\eta^-$, $\chi^+\chi^-$, and $\eta^\pm\chi^\mp$. Let $r_{\eta,\chi} = \lambda_{\eta,\chi}v\sqrt{2}/\mu$, then

$$f_l v/m_l = 1 + a_+F_+ + a_-F_-,$$

where $a_+ = (\sin 2\theta)^2/2 + \sin 4\theta(r_\eta - r_\chi)/4$,
$a_- = \sin 2\theta(r_\eta + r_\chi)/2$, and
$F_+ = [F(x_1, x_1) + F(x_2, x_2)]/2F(x_1, x_2) - 1$,
$F_- = [F(x_1, x_1) - F(x_2, x_2)]/2F(x_1, x_2)$, with

$$F(x_1, x_2) = \frac{1}{x_1 - x_2}\left(\frac{x_1 \ln x_1}{x_1 - 1} - \frac{x_2 \ln x_2}{x_2 - 1}\right).$$

$$F(x, x) = \frac{1}{x - 1} - \frac{\ln x}{(x-1)^2}.$$


$(r_\eta, r_\chi) = (0, 0)$
$(r_\eta, r_\chi) = (+0.3, -0.3)$
$(r_\eta, r_\chi) = (+0.2, +0.2)$
$(r_\eta, r_\chi) = (+0.5, -0.1)$
$(f_\tau v/m_\tau)^2$
θ/π
0.0
0.2
0.4
0.6
0.8
1.0
0.8
1.0
1.2
1.4
1.6
1.8
2.0

Personal Remarks

Our understanding of flavor symmetries has evolved over the years. Some of the insights [flashes] have been discussed in this talk. I hope it has been at least a refresher for some of you, and an informative review for the younger participants. For a recent comprehensive update, see [Ding/Valle(arXiv:2402.16963)]