UCI University of California, Irvine

The 10th Workshop on
Flavor Symmetries and Consequences
in Accelerators and Cosmology
FLASY2024

Modular Invariant Holomorphic Observables

Xueqi Li

arXiv: 2401.04738

Mu-Chun Chen, Xiang-Gan Liu, Xueqi Li, Omar Medina, Michael Ratz

Flavor Puzzle

The origin of the parameters in the flavor sector

For example, in SUSY, the lepton masses can be generated via superpotential

$$\mathscr{W} = Y_e^{ij} L_i H_d \bar{E}_j + \frac{1}{2} \kappa^{ij} L_i H_u L_j H_u$$

Flavor Puzzle

The origin of the parameters in the flavor sector

For example, in SUSY, the lepton masses can be generated via superpotential

$$\mathscr{W} = Y_e^{ij} L_i H_d \bar{E}_j + \frac{1}{2}\kappa^{ij} L_i H_u L_j H_u$$

As Higgs obtained a vacuum expectation value (vev), in the basis in which Y_e^{ij} is diagonal, this then gives a mass matrix for neutrino

$$\Rightarrow m_\nu^{ij} = \kappa^{ij} v_u^2$$

What is the origin to the structure of κ^{ij}?

Flavor + Modular

One Possibile Solution

One solution is to apply modular symmetry in flavor physics.

[arXiv:1706.08749 Ferruccio Feruglio]

A theory in modular symmetry content

- a modulus τ takes values in the upper half of the complex plane
- regular matter fields,
- and a modular group in which the theory is invariant under.

Will be explained later…

Flavor + Modular

One Possibile Solution

Impose following 3 requirement to the coupling, **then these conditions are so strong and the couplings are almost unique.**

1. **Modular invariance / covariance** (◎), required by the modular symmetry
   - Invariant under modular transformations
2. **Meromorphic** ($\not{\bar{\tau}}$), required by SUSY
   - The coupling depends only on modulus τ but not on its conjugate $\bar{\tau}$
3. **Finite** ($\not\infty$)
   - The coupling are finite for all values of τ in the upper half of the complex plane, including at $i\infty$.

1. **Modular invariance / covariance** (◎), required by the modular symmetry
   - Invariant under modular transformations
2. **Meromorphic** ($\not{\bar{\tau}}$), required by SUSY
   - The coupling depends only on modulus τ but not on its conjugate $\bar{\tau}$
3. **Finite** ($\not\infty$)
   - The coupling are finite for all values of τ in the upper half of the complex plane, including at $i\infty$.

Question answered in our work:

Can observables have these features?

Introduction to Modular Flavor Theory Framework

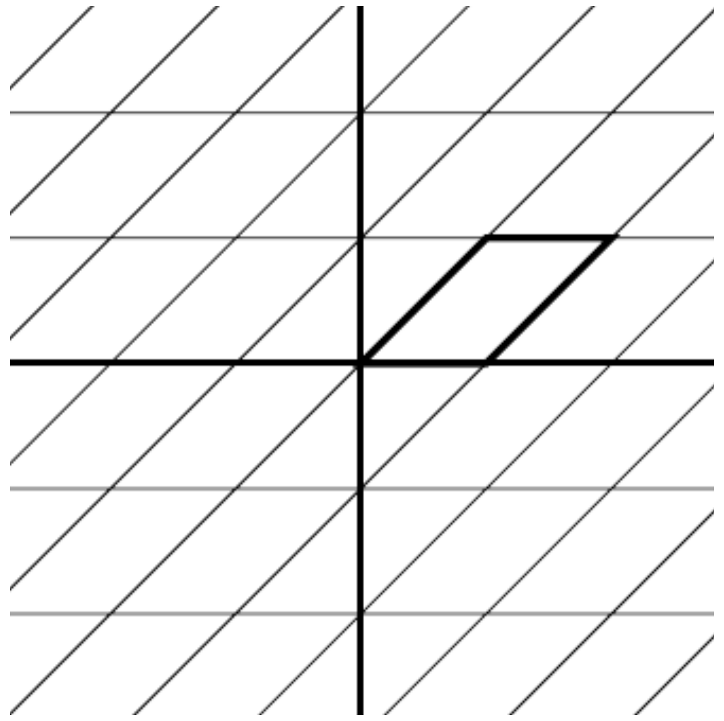
What is Modular symmetry

Modular symmetry presenting a geometry structure of extra dimension

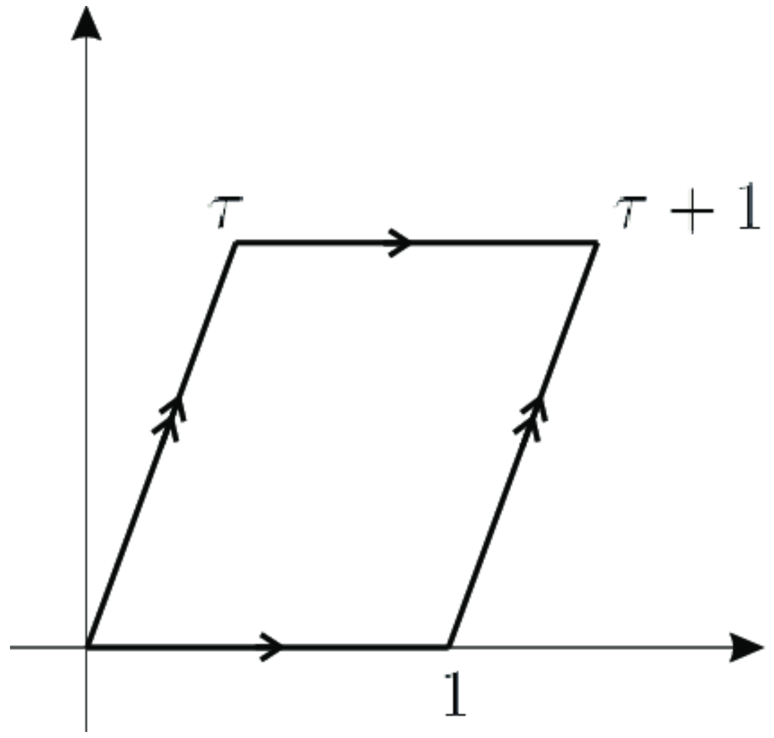
$$\text{Torus: } T^2 = S^1 \times S^1$$

Characterized by τ

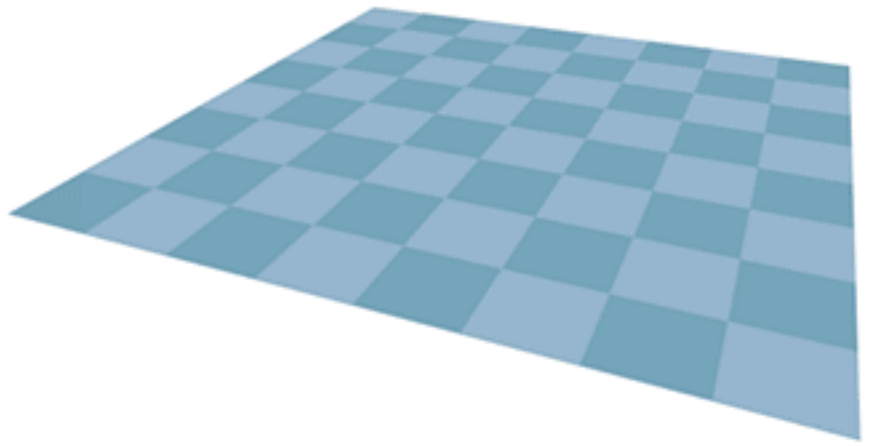
Symmetry: SL$(2,\mathbb{Z})$



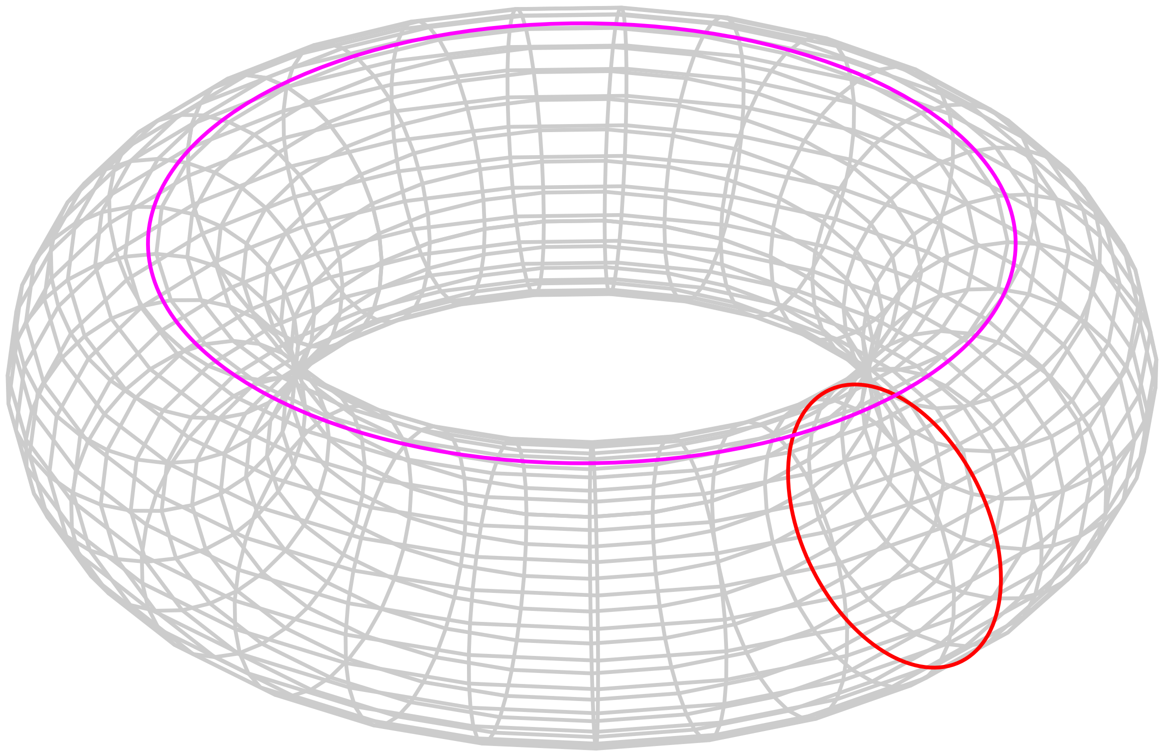
Lattice



Characterized by τ



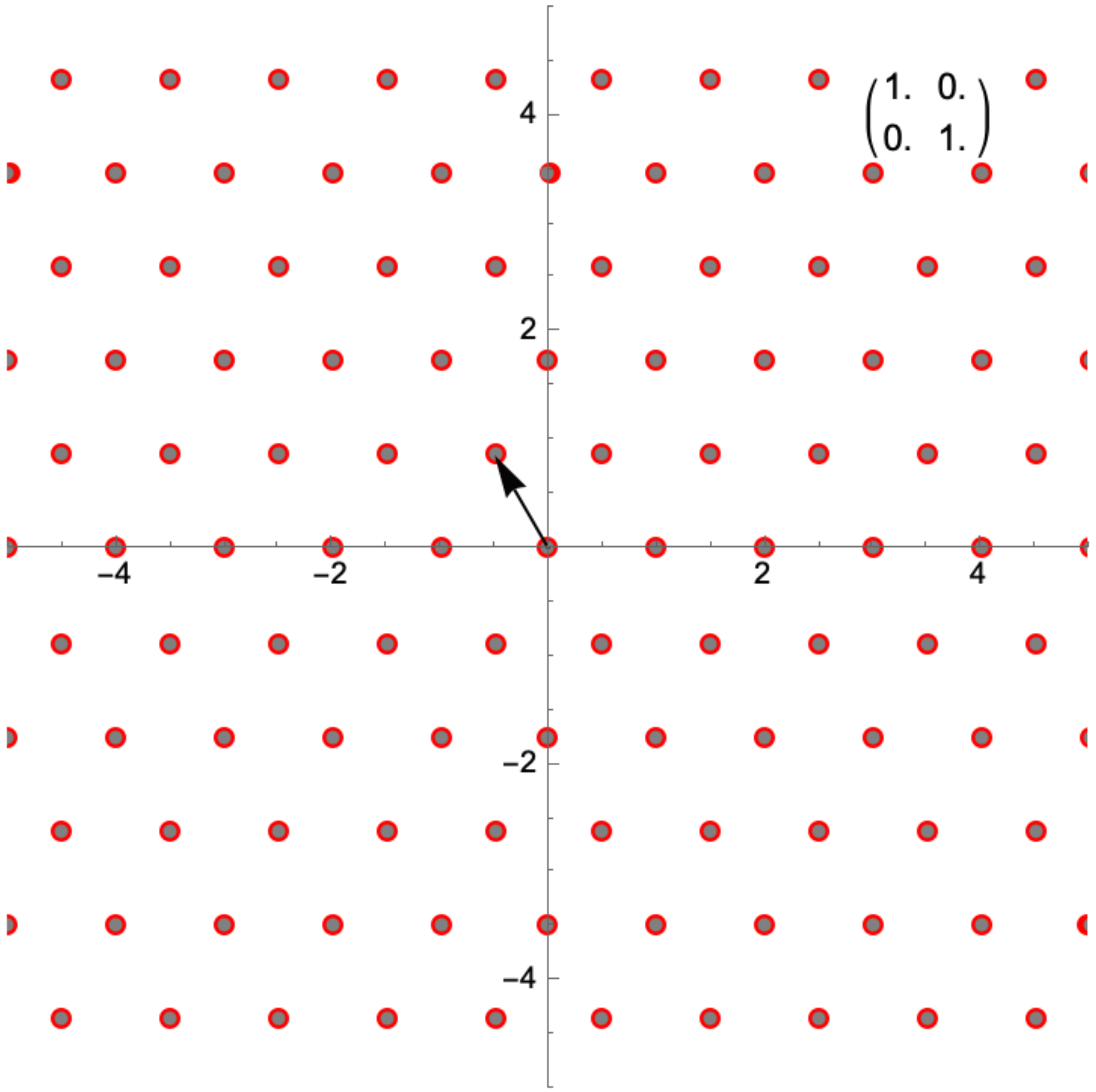
Torus



Wikipedia

What is Modular symmetry

Modular transformation changes the basis on lattice



$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \tau \mapsto -\frac{1}{\tau}$$

A modular symmetry SL$(2,\mathbb{Z})$ are generated via two generator of SL$(2,\mathbb{Z})$:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

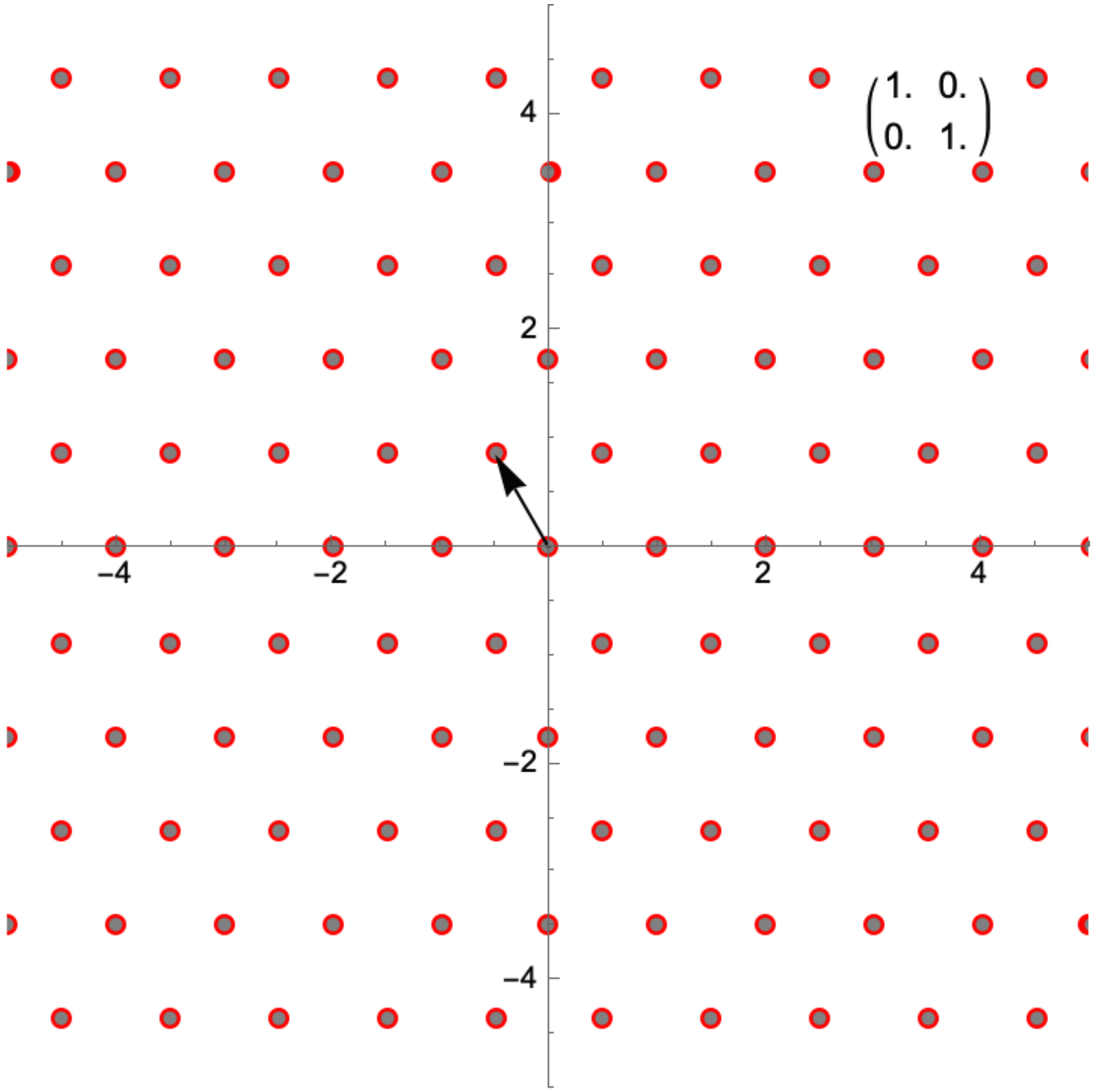
A chiral supermultiplet τ, known as modulus, transform under an element

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ SL$(2,\mathbb{Z})$ as

$$\tau \xrightarrow{\gamma} \frac{a\tau + b}{c\tau + d}$$

What is Modular symmetry

Modular transformation changes the basis on lattice



$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \tau \mapsto \tau + 1$$

A modular symmetry SL$(2,\mathbb{Z})$ are generated via two generator of SL$(2,\mathbb{Z})$:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

A chiral supermultiplet τ, known as modulus, transform under an element

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ SL$(2,\mathbb{Z})$ as

$$\tau \xrightarrow{\gamma} \frac{a\tau + b}{c\tau + d}$$

What is Modular symmetry

Modular symmetry presenting a geometry structure of extra dimension

Regular matter fields Φ transform under $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ as

$$\Phi \xrightarrow{\gamma} (c\tau + d)^{-k_\Phi}\rho_\Phi(\gamma)\Phi$$

where ρ_Φ is the representation of SL$(2,\mathbb{Z})$ and k_Φ is known as the modular weight of Φ.

A Theory with Modular Symmetry

Kahler potential

A Lagrangian in SUSY

$$\mathscr{L} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\ \mathscr{K}(\Phi_i, \Phi_i^\dagger) + \left(\int \mathrm{d}^2\theta \mathscr{W}(\Phi_i) + \text{h.c.} \right)$$

where $\mathscr{K}$ is the Kahler potential and $\mathscr{W}$ is the superpotential.

In general, a modular invariance Kahler potential can are of the form

$$\mathscr{K} = \sum_i \frac{\Phi_i^\dagger \Phi_i}{(-i\tau + i\bar{\tau})^{k_i}} + \cdots$$

where the term presented are the minimal modular invariance Kahler potential.

A Theory with Modular Symmetry

Superpotential

Now we move on the the superpotential, in general, the superpotential have Yukawa terms like

$$\mathscr{W} \supset g Y^{ijk}(\tau)\Phi_i\Phi_j\Phi_k$$

by requiring the theory to be modular invariant, recall $\Phi \xrightarrow{\gamma} (c\tau + d)^{-k_\Phi}\rho_\Phi(\gamma)\Phi$, we then require Yukawa coupling transform as

$$Y^{ijk}(\tau) \xrightarrow{\gamma} Y^{ijk}\left(\gamma(\tau)\right) = (c\tau + d)^{k_Y}\rho_Y(\gamma)Y^{ijk}(\tau)$$

$$k_Y = k_{\Phi_i} + k_{\Phi_j} + k_{\Phi_k}$$

Flavor + Modular

Now we apply <u>modular symmetry</u> in <u>flavor physics</u>. To do so, we impose 3 requirements to the coupling

1. **<u>Modular invariance / covariance</u>** ($\odot$), required by the <u>modular symmetry</u>
    - Invariant under modular transformations
2. **<u>Meromorphic</u>** $(\not{\bar{\tau}})$, required by SUSY
    - The coupling depends only on modulus τ but not on its conjugate $\bar{\tau}$
3. **<u>Finite</u>** $(\not{\infty})$
    - The coupling are finite for all values of τ in the upper half of the complex plane, including at $i\infty$.

Holomorphic $\begin{cases} \text{Meromorphic}(\not{\bar{\tau}}) \\ \text{Finiteness}(\not{\infty}) \end{cases}$

3 Requirements

1. **Modular invariance / covariance** (⊚), required by the modular symmetry
    - Invariant under modular transformations
2. **Meromorphic** ($\not\bar{\tau}$), required by SUSY
    - The coupling depends only on modulus τ but not on its conjugate $\bar{\tau}$
3. **Finite** ($\not\infty$)
    - The coupling are finite for all values of τ in the upper half of the complex plane, including at $i\infty$.

Recall the coupling in superpotential

$$\mathscr{W} \supset gY^{ijk}(\tau)\Phi_i\Phi_j\Phi_k$$

where under a modular transformation,

$$Y^{ijk}(\tau) \xrightarrow{\gamma} Y^{ijk}\left(\gamma(\tau)\right) = (c\tau + d)^{k_Y}\rho_Y(\gamma)Y^{ijk}(\tau)$$

These 3 requirements and above transformation then uniquely determine $Y^{ijk}(\tau)$ to be **vector-valued modular forms**.

3 Requirements

1. **Modular invariance / covariance** (◎), required by the modular symmetry
   - Invariant under modular transformations
2. **Meromorphic** ($\not\bar{\tau}$), required by SUSY
   - The coupling depends only on modulus τ but not on its conjugate $\bar{\tau}$
3. **Finite** ($\not\infty$)
   - The coupling are finite for all values of τ in the upper half of the complex plane, including at $i\infty$.

Can we impose the same restriction on the observable?

Naive Attempt

Usual physical observables?

The usual observables, say m, the mass, are finite $(\not\infty)$ and modular invariant $(\circledcirc)$ in a theory with modular symmetry.

Therefore we only need to check meromorphy $(\not\hbar)$.

Naive Attempt

Usual physical observables?

Consider a toy model

$$\mathscr{W} = \frac{1}{2}\mathscr{M}(\tau)\Phi^2,\ \mathscr{K} = \frac{\Phi^\dagger\Phi}{(-i\tau + i\bar{\tau})^{k_\Phi}}$$

To find the physical mass, we look into the scalar potential

$$V(\phi) \overset{?}{=} \left|\frac{\partial\mathscr{W}}{\partial\phi}\right|^2 = |\mathscr{M}(\tau)|^2|\phi|^2$$

Naive Attempt

Usual physical observables?

$$\mathscr{W} = \frac{1}{2}\mathscr{M}(\tau)\Phi^2,\ \mathscr{K} = \frac{\Phi^\dagger\Phi}{(-i\tau + i\bar{\tau})^{k_\Phi}}$$

However, notice we need to canonically normalize Kahler potential, therefore we must introduce the Kahler metric:

$$V(\phi) = \frac{\partial \mathscr{W}^\dagger}{\partial \phi^\dagger} K^{\Phi^\dagger\Phi} \frac{\partial \mathscr{W}}{\partial \phi},\ \text{where } K^{\Phi^\dagger\Phi} = (-i\tau + i\bar{\tau})^{k_\Phi}$$

which gives rise an additional terms in the mass

$$m^2 = |\mathscr{M}(\tau)|^2(-i\tau + i\bar{\tau})^{k_\Phi}$$

which depends on $\bar{\tau}$.

Naive Attempt

Non-holomorphic observables

Physical mass is not meromorphic $(\hbar)$ (thus also not holomorphic).

Actually, one can show in general **all observables** discussed before our paper were non-holomorphic.

Problem:

Kahler metric enter into the mass, which is in general not **meromorphic** $(\hbar)$.

Therefore we cannot use the nice uniqueness argument here.

Modular Invariant Holomorphic Observables

Modular Invariant Holomorphic Observables

Idea: Remove the non-holomorphic terms coming from the Kahler metric

Recall the supoerpotential of lepton sector

$$\mathscr{W} = Y_e^{ij} L_i H_d \bar{E}_j + \frac{1}{2}\kappa^{ij}(\tau) L_i H_u L_j H_u$$

where now we add the modular (τ) dependence.

In the simplest setting, let's say we are in the basis in which charge lepton Yukawa is diagonal.

And we make use of the minimal Kahler potential.

Modular Invariant Holomorphic Observables

Idea: Remove the non-holomorphic terms coming from the Kahler metric

Consider

$$I_{ij}(\tau) := \frac{\kappa_{ii}(\tau)\kappa_{jj}(\tau)}{(\kappa_{ij}(\tau))^2} = \frac{m_{ii}(\tau, \bar{\tau})m_{jj}(\tau, \bar{\tau})}{(m_{ij}(\tau, \bar{\tau}))^2}$$

where $m_{ij}(\tau, \bar{\tau}) := (-i\tau + i\bar{\tau})^{(k_{L_j}+k_{L_j})/2}\kappa_{ij}(\tau)v_u^2$ is the neutrino mass matrix we obtained after Kahler potential are canonically normalized.

We see now that by doing ratio of the mass matrix entries, we cancel the non-holomorphic terms from the Kahler metric, we therefore obtained $I_{ij}(\tau)$ is now meromorphic ($\not{\bar{\tau}}$).

After a modular transformation, the automorphic factor $(c\tau + d)^{k_{L_i}}$ is also canceled. This object is therefore modular invariant ($\odot$).

Modular Invariant Holomorphic Observables

Invariant in terms of physical observables

We can write the mass matrix m_ν in terms of observables:

- Neutrino masses $\{m_1, m_2, m_3\}$, and
- PMNS matrix U, depends on mixing and phase $\{\theta_{12}, \theta_{13}, \theta_{23}, \delta_{\text{CP}}, \varphi_1, \varphi_2\}$, where δ_{CP} is the CP violation phase, and φ_1 and φ_2 are two Majorana phases.

$$m_\nu = U^* \text{diag}(m_1, m_2, m_3) U^\dagger$$

Modular Invariant Holomorphic Observables

Invariant in terms of physical observables

Plug in, we obtained the invariants are

$$I_{12} = \frac{a_0\left[\widetilde{m}_1\left(e^{i\delta_{\rm CP}}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)^2+\widetilde{m}_2\left(e^{i\delta_{\rm CP}}c_{12}c_{23}-s_{12}s_{13}s_{23}\right)^2+e^{2i\delta_{\rm CP}}m_3c_{13}^2s_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(e^{i\delta_{\rm CP}}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)+\widetilde{m}_2s_{12}\left(s_{12}s_{13}s_{23}-e^{i\delta_{\rm CP}}c_{12}c_{23}\right)-e^{2i\delta_{\rm CP}}m_3s_{13}s_{23}\right]^2}\,,$$

$$I_{13} = \frac{a_0\left[\widetilde{m}_1\left(c_{12}c_{23}s_{13}-e^{i\delta_{\rm CP}}s_{12}s_{23}\right)^2+\widetilde{m}_2\left(c_{23}s_{12}s_{13}+e^{i\delta_{\rm CP}}c_{12}s_{23}\right)^2+e^{2i\delta_{\rm CP}}m_3c_{13}^2c_{23}^2\right]}{c_{13}^2\left[\widetilde{m}_1c_{12}\left(c_{12}c_{23}s_{13}-e^{i\delta_{\rm CP}}s_{12}s_{23}\right)+\widetilde{m}_2s_{12}\left(c_{23}s_{12}s_{13}+e^{i\delta_{\rm CP}}c_{12}s_{23}\right)-e^{2i\delta_{\rm CP}}m_3c_{23}s_{13}\right]^2}\,,$$

$$I_{23} = \left[e^{2i\delta_{\rm CP}}m_3c_{13}^2s_{23}^2+\widetilde{m}_1\left(e^{i\delta_{\rm CP}}c_{23}s_{12}+c_{12}s_{13}s_{23}\right)^2+\widetilde{m}_2\left(e^{i\delta_{\rm CP}}c_{12}c_{23}-s_{12}s_{13}s_{23}\right)^2\right]$$
$$\times\frac{4\left[e^{2i\delta_{\rm CP}}m_3c_{13}^2c_{23}^2+\widetilde{m}_2\left(c_{23}s_{12}s_{13}+e^{i\delta_{\rm CP}}c_{12}s_{23}\right)^2+\widetilde{m}_1\left(c_{12}c_{23}s_{13}-e^{i\delta_{\rm CP}}s_{12}s_{23}\right)^2\right]}{\left[\widetilde{m}_1a_1+\widetilde{m}_2a_2-e^{2i\delta_{\rm CP}}m_3\sin(2\theta_{23})c_{13}^2\right]^2}\,,$$

$$a_0 = (\widetilde{m}_1c_{12}^2+\widetilde{m}_2s_{12}^2)c_{13}^2+e^{2i\delta_{\rm CP}}m_3s_{13}^2\,,$$
$$a_1 = \left[\left(e^{2i\delta_{\rm CP}}s_{12}^2-c_{12}^2s_{13}^2\right)\sin(2\theta_{23})-e^{i\delta_{\rm CP}}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}\right]\,,$$
$$a_2 = \left[e^{i\delta_{\rm CP}}\cos(2\theta_{23})\sin(2\theta_{12})s_{13}+\left(e^{2i\delta_{\rm CP}}c_{12}^2-s_{12}^2s_{13}^2\right)\sin(2\theta_{23})\right]\,.$$

$$s_{ij}=\sin\theta_{ij},\ c_{ij}=\cos\theta_{ij},\ \widetilde{m}_i=m_ie^{i\varphi_2}$$

Modular Invariant Holomorphic Observables

$$I_{ij}(\tau) := \frac{\kappa_{ii}(\tau)\kappa_{jj}(\tau)}{(\kappa_{ij}(\tau))^2} = \frac{m_{ii}(\tau,\bar{\tau})m_{jj}(\tau,\bar{\tau})}{(m_{ij}(\tau,\bar{\tau}))^2}$$

Remark

1. They are <u>modular invariant</u> ($\odot$)
2. They are <u>meromorphic</u> ($\not{\bar{\tau}}$)
3. **Not** necessary <u>finite</u> everywhere ($\not\infty$)

Additionally

4. These invariant only depend on the observables
5. They are actually renormalization group invariant [arXiv:hep-ph/0205147 Sanghyeon Chang, T. K. Kuo]

Apply it in an Example Model

A Model in modular group A_4

[arXiv:1706.08749 Ferruccio Feruglio]

field/coupling	(E_1^c, E_2^c, E_3^c)	L	$H_{u/d}$	φ_T	$Y_{\mathbf{3}}^{(2)}(\tau)$
$SU(2)_L \times U(1)_Y$	$(\mathbf{1}, 1)$	$(\mathbf{2}, -1/2)$	$(\mathbf{2}, \pm 1/2)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$
$\Gamma_3 \cong A_4$	$(\mathbf{1}, \mathbf{1}'', \mathbf{1}')$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{3}$
k_I	$(2, 2, 2)$	1	0	−3	−2

$$\mathcal{W}_e = \alpha E_1^c H_d (L\varphi_T)_{\mathbf{1}} + \beta E_2^c H_d (L\varphi_T)_{\mathbf{1}'} + \gamma E_3^c H_d (L\varphi_T)_{\mathbf{1}''} \,,$$

$$\mathcal{W}_\nu = \frac{1}{\Lambda}\left(H_u \cdot L\, H_u \cdot L\, Y_{\mathbf{3}}^{(2)}\right)_{\mathbf{1}} \,.$$

$$\langle \varphi_T \rangle = (u, 0, 0)$$

$$M_e = u\, v_d\, \mathrm{diag}(\alpha, \beta, \gamma) \,,$$

$$m_\nu(\tau, \bar\tau) = (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau)\frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1(\tau) & -Y_2(\tau) & -Y_3(\tau) \\ -Y_2(\tau) & 2Y_3(\tau) & -Y_1(\tau) \\ -Y_3(\tau) & -Y_1(\tau) & 2Y_2(\tau) \end{pmatrix}$$

$$=: (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau) v_u^2 \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & \kappa_{22} & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & \kappa_{33} \end{pmatrix} \,.$$

Modular weight k and their representation uniquely fixed Yukawa as modular forms

$$Y_{\mathbf{3}}^{(2)} := \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

A Model in modular group A_4

Invariant

The invariants are then given as

$$I_{12}(\tau) = 4\frac{Y_1(\tau)\,Y_3(\tau)}{\big(Y_2(\tau)\big)^2},\; I_{13}(\tau) = 4\frac{Y_1(\tau)\,Y_2(\tau)}{\big(Y_3(\tau)\big)^2},\; I_{23}(\tau) = 4\frac{Y_2(\tau)\,Y_3(\tau)}{\big(Y_1(\tau)\big)^2}$$

A modular invariant meromorphic ($\odot$, $\not{t}$) function are either

- τ independent constant (which are finite ($\not\infty$), thus holomorphic)
- or it has pole

(There is no modular invariant holomorphic function except constant functions)

A Model in modular group A_4
Invariant

$$I_{12}(\tau) = 4\frac{Y_1(\tau)\,Y_3(\tau)}{\big(Y_2(\tau)\big)^2},\ I_{13}(\tau) = 4\frac{Y_1(\tau)\,Y_2(\tau)}{\big(Y_3(\tau)\big)^2},\ I_{23}(\tau) = 4\frac{Y_2(\tau)\,Y_3(\tau)}{\big(Y_1(\tau)\big)^2}$$

1. I_{13} has a singularity at $\tau = i\infty$; I_{23} has a singularity at $\tau = \dfrac{-3+i\sqrt{3}}{6}$, and vanishes at $\tau = i\infty$.
2. Y_i satisfy $Y_2^2 + 2Y_1Y_3 = 0$, therefore $I_{12}(\tau) = -2$, **a constraint that is independent of τ**
3. One can also showed $I_{13}I_{23} = -32$, **a constraint that is independent of τ**

In addition

- Mass matrix has a sum rule: $m_3 = \begin{cases} m_2 + m_1 & \text{for normal ordering (NO)}\ , \\ m_2 - m_1 & \text{for inverted ordering (IO)}\ . \end{cases}$

Use relation $I_{12}(\tau) = -2$

I_{12} **is a <u>Modular Invariant Holomorphic</u> Observables** $(\circledcirc, \not{t}, \not\infty)$

Using the sum rule and the known mass square difference Δm^2_{sol} and Δm^2_{atm}, we can fixed all the mass. We also know the mixing angles from oscillation experiments. We therefore look into the phases

$$\{\delta_{\text{CP}}, \varphi_1, \varphi_2\}$$

and therefore determine the neutrinoless double beta decay matrix element

$$\langle m_{ee} \rangle = \sum_i U_{ei}^2 m_i$$

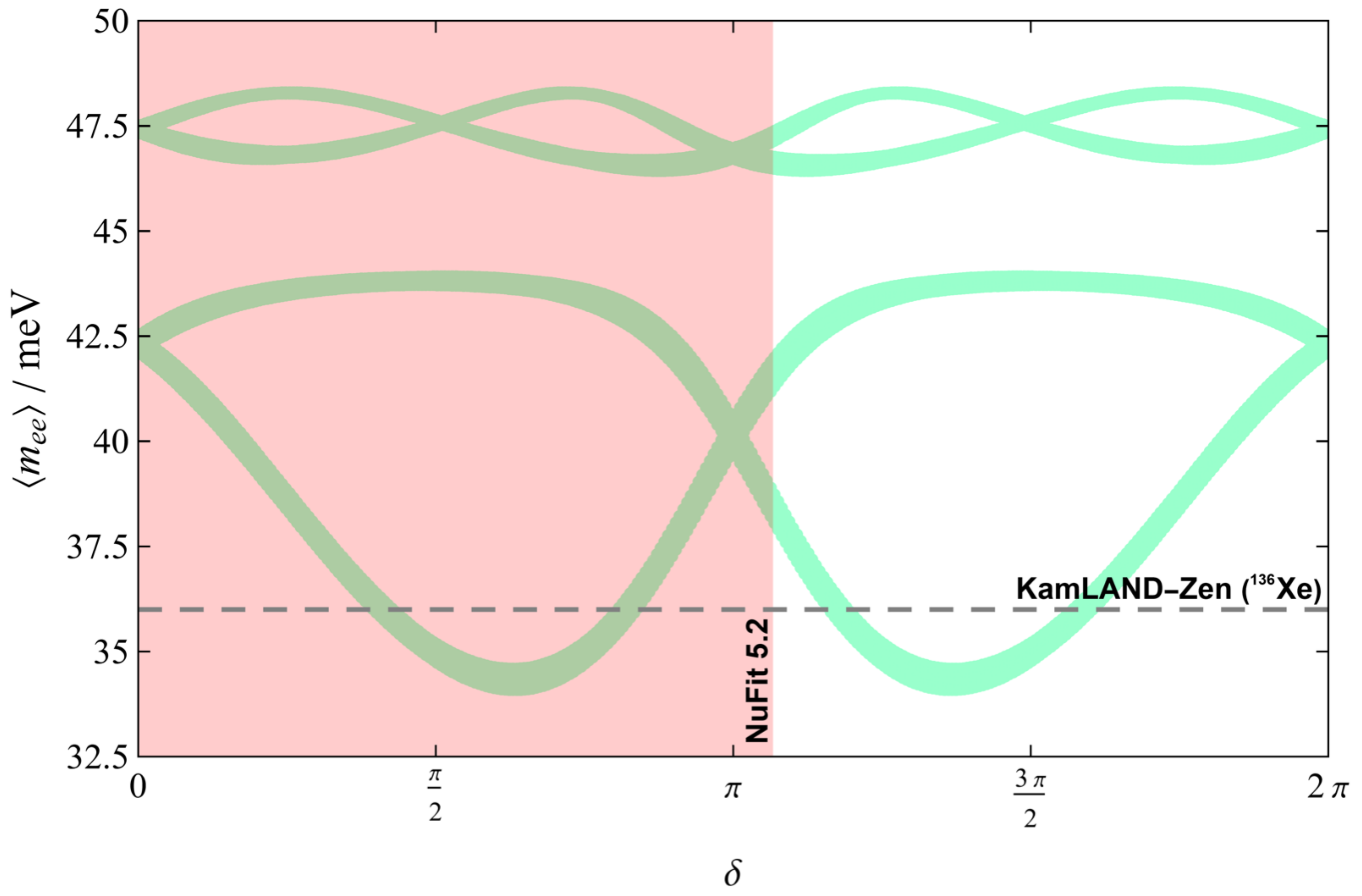
Use relation $I_{12}(\tau) = -2$

I_{12} **is a <u>Modular Invariant Holomorphic</u> Observables (◎ , ŧ, ꝺ)**

Once we impose $I_{12}(\tau) = -2$, the allowed $\langle m_{ee} \rangle$ are shown in the plot

Notice this result

- Independent of the value of τ
- We only impose 1 out of 3 relations (or 2 out of 6 real relations)



Use relation $I_{13}I_{23} = -32$

$I_{13}I_{23}$ **is a <u>Modular Invariant Holomorphic</u> Observables** (◎ , $\not\tau$, $\not\infty$)

We can now use the relation, $I_{13}I_{23} = -32$.

This gives 2 more real constraint and the systems is **over-constrained**.

We have verified that **cannot** satisfy relations while still being consistent with data.

Therefore this model is **ruled out**. Agree with analyses done by previous work.

We arrive at this conclusion without doing any fit nor a scan over τ.

Conclusion

Conclusion

Modular Invariant Holomorphic Observables

$$I_{ij}(\tau) := \frac{\kappa_{ii}(\tau)\kappa_{jj}(\tau)}{(\kappa_{ij}(\tau))^2} = \frac{m_{ii}(\tau,\bar{\tau})m_{jj}(\tau,\bar{\tau})}{(m_{ij}(\tau,\bar{\tau}))^2}$$

There **exist** observables that are

1. <u>Modular invariant</u> ($\odot$)
2. <u>Meromorphic</u> ($\not\bar{\tau}$)
3. Some are <u>finite</u> everywhere ($\not\infty$)

Moreover...

4. Usually, we can use I_{ij} to construct observables that are also <u>finite</u> ($\not\infty$), which lead to a <u>modular invariant holomorphic</u> observables ($\odot$, $\not\bar{\tau}$, $\not\infty$).

and...

5. They are also independent of renormalization scale.

Conclusion

Modular Invariant Holomorphic Observables

- They are highly constrained by their symmetries and properties
- Composed solely of quantities that can be measured experimentally
- Gives rise robust, important, immediate useful information and phenomenological constraints without need to perform a scan of the parameter space

Open question

Modular Invariant Holomorphic Observables

- Apply the same idea to quark sector?
- In the case in which the charge lepton mass matrix is not diagonal?
- Do these invariants have more physical meaning or even can be directly measured?
- In the case where Kahler potential is not minimal?
- …

Thank you!

Modular Invariant Holomorphic Observables
RG Invariant

RG Equation for κ: $16\pi^2 \dfrac{\mathrm{d}}{\mathrm{d}t}\kappa = P^{\mathsf{T}}\,\kappa + \kappa\,P + \alpha\,\kappa$

at one-loop $P = C_e\, Y_e^{\dagger} Y_e$ $\quad C_e = 1$ in the MSSM
$C_e = -3/2$ in the SM

If P is diagonal, then $\Delta\kappa_{ij} = \dfrac{\Delta t}{16\pi^2}\,\kappa_{ij}\left(P_{ii} + P_{jj} + \alpha\right)$

Therefore I_{ij} is RG Invariant $\quad I_{ij}(\tau) := \dfrac{\kappa_{ii}(\tau)\kappa_{jj}(\tau)}{(\kappa_{ij}(\tau))^2} = \dfrac{m_{ii}(\tau,\bar{\tau})m_{jj}(\tau,\bar{\tau})}{(m_{ij}(\tau,\bar{\tau}))^2}$

Modular Invariant Holomorphic Observables

RG Invariant

$$\frac{\mathrm{d}}{\mathrm{d}t}\kappa = \widetilde{P}\,\kappa\,\widetilde{Q}^\mathsf{T} + \widetilde{Q}\,\kappa\,\widetilde{P}^\mathsf{T} + \widetilde{\alpha}\,\kappa\ , \tag{61}$$

where $\widetilde{P}$, $\widetilde{Q}$ and $\widetilde{\alpha}$ are composed of the renormalizable couplings of the theory and diagonal,

$$\widetilde{P} = \mathrm{diag}(\widetilde{P}_1, \widetilde{P}_2, \widetilde{P}_3)\ , \tag{62a}$$

$$\widetilde{Q} = \mathrm{diag}(\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{Q}_3)\ . \tag{62b}$$

At 1-loop, $\widetilde{P} = \frac{1}{16\pi^2}P$, $\widetilde{Q} = \mathbb{1}$, and $\widetilde{\alpha} = \frac{1}{16\pi^2}\alpha$. Equation (61) implies that

$$\dot{\kappa}_{ij} = \kappa_{ij}\left(\widetilde{P}_i\,\widetilde{Q}_j + \widetilde{P}_j\,\widetilde{Q}_i + \widetilde{\alpha}\right)\ , \tag{63}$$

where no summation over i or j is implied. This means that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_{ij} &= \frac{\dot{\kappa}_{ii}\,\kappa_{jj}}{\kappa_{ij}^2} + \frac{\kappa_{ii}\,\dot{\kappa}_{jj}}{\kappa_{ij}^2} - 2\frac{\kappa_{ii}\,\kappa_{jj}}{\kappa_{ij}^3}\,\dot{\kappa}_{ij} \\ &= 2\left(\widetilde{P}_i - \widetilde{P}_j\right)\left(\widetilde{Q}_i - \widetilde{Q}_j\right) I_{ij}\ .\end{aligned} \tag{64}$$

This has two immediate consequences:

1. At 1-loop, where $\widetilde{Q}_i = 1$ for all i, I_{ij} are RG invariant.
2. Zeros and poles of I_{ij} remain zeros and poles at *all orders*.

A Model in modular group A_4

Invariant

$$I_{12}(\tau) = 4\frac{Y_1(\tau)\,Y_3(\tau)}{\left(Y_2(\tau)\right)^2},\ I_{13}(\tau) = 4\frac{Y_1(\tau)\,Y_2(\tau)}{\left(Y_3(\tau)\right)^2},\ I_{23}(\tau) = 4\frac{Y_2(\tau)\,Y_3(\tau)}{\left(Y_1(\tau)\right)^2}$$

1. I_{13} has a singularity at $\tau = i\infty$; I_{23} has a singularity at $\tau = \dfrac{-3+i\sqrt{3}}{6}$, and vanishes at $\tau = i\infty$.
2. Y_i satisfy $Y_2^2 + 2Y_1Y_3 = 0$, therefore $I_{12}(\tau) = -2$, **a constraint that is independent of τ**
3. One can also showed $I_{13}I_{23} = -32$, **a constraint that is independent of τ**

$$I_{12}(\tau) = -2\ ,$$
$$I_{13}(\tau) = -2\left(1 + \frac{1}{3}j_3(\tau)\right)^3\ ,$$
$$I_{23}(\tau) = -\frac{32}{I_{13}} = \frac{16}{\left(1+\frac{1}{3}j_3(\tau)\right)^3}\ . \qquad j_3(\tau) := \eta(\tau/3)^3/\eta(3\tau)^3$$

$$I_{13} = -\frac{2}{27}q^{-1} - \frac{10}{9} - 4q + \frac{152}{27}q^2 + 18q^3 - 88q^4 + \frac{2768}{27}q^5 + 216q^6 + \ldots\ ,$$
$$I_{23} = 432q - 6480q^2 + 73872q^3 - 725328q^4 + 6503328q^5 - 54855792q^6 + \ldots\ .$$
$$q = e^{2\pi i \tau}$$

A Model in modular group A_5

[arXiv:1903.12588 Gui-Jun Ding, Stephen F.King, Xiang-Gan Liu]

field/coupling	E^c	L	$H_{u/d}$	χ	φ	$Y_{\mathbf{5}}^{(2)}(\tau)$
$SU(2)_L \times U(1)_Y$	$(\mathbf{1}, 1)$	$(\mathbf{2}, -1/2)$	$(\mathbf{2}, \pm 1/2)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$
$\Gamma_5 \cong A_5$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{5}$
k_I	2	1	0	−3/2	−3/2	−2

$$\mathcal{W}_e = \left[\alpha(E^c L)_{\mathbf{1}}\chi^2 + \beta(E^c L)_{\mathbf{1}}(\varphi^2)_{\mathbf{1}} + \gamma(E^c L)_{\mathbf{5}}(\varphi^2)_{\mathbf{5}} + \zeta(E^c L)_{\mathbf{3}}(\chi\varphi)_{\mathbf{3}}\right]_{\mathbf{1}} H_d \ ,$$

$$\mathcal{W}_\nu = \frac{1}{\Lambda}\left(H_u \cdot L\, H_u \cdot L\, Y_{\mathbf{5}}^{(2)}\right)_{\mathbf{1}} \ .$$

$$\langle\chi\rangle = v_\chi \ ,$$

$$\langle\varphi\rangle = v_\varphi\,(1,0,0) \ .$$

$$M_e = v_d \begin{pmatrix} \mu_e + 4\gamma v_\varphi^2 & 0 & 0 \\ 0 & 0 & \mu_e - 2\gamma v_\varphi^2 + \zeta v_\chi v_\varphi \\ 0 & \mu_e - 2\gamma v_\varphi^2 - \zeta v_\chi v_\varphi & 0 \end{pmatrix} ,$$

$$m_\nu(\tau,\bar\tau) = (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau)\frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1(\tau) & -\sqrt{3}Y_5(\tau) & -\sqrt{3}Y_2(\tau) \\ -\sqrt{3}Y_5(\tau) & \sqrt{6}Y_4(\tau) & -Y_1(\tau) \\ -\sqrt{3}Y_2(\tau) & -Y_1(\tau) & \sqrt{6}Y_3(\tau) \end{pmatrix}$$

$$=: (-\mathrm{i}\,\tau + \mathrm{i}\,\bar\tau) v_u^2 \begin{pmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} \\ \kappa_{12} & \kappa_{22} & \kappa_{23} \\ \kappa_{13} & \kappa_{23} & \kappa_{33} \end{pmatrix} ,$$

Modular weight k and their representation uniquely fixed Yukawa as modular forms

$$Y_{\mathbf{5}}^{(2)}(\tau) := \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}$$

A Model in modular group A_5
Invariant

$$I_{12} = \frac{2\sqrt{6}}{3}\frac{Y_1(\tau)Y_4(\tau)}{Y_5^2(\tau)},\ I_{13} = \frac{2\sqrt{6}}{3}\frac{Y_1(\tau)Y_3(\tau)}{Y_2^2(\tau)},\ I_{23} = 6\frac{Y_3(\tau)Y_4(\tau)}{Y_1^2(\tau)}$$

1. All of them have poles
2. We can still make combination of them which is Modular Invariant Holomorphic($\odot$, $\not{t}$, $\not\infty$)

$$-4 = 18I_{12} + 18I_{13} + 9I_{12}I_{13} + I_{12}I_{13}I_{23}\ ,$$

$$\begin{aligned}-8 = {} & 12I_{12} - 108I_{12}^2 + 12I_{13} + 414I_{12}I_{13} + 108I_{12}^2I_{13} - 108I_{13}^2 + 108I_{12}I_{13}^2 + 81I_{12}^2I_{13}^2 \\ & -I_{12}^2I_{23} - I_{13}^2I_{23}\ .\end{aligned}$$

A Model in modular group A_5

Invariant

$$
\begin{aligned}
-4 &= 18I_{12} + 18I_{13} + 9I_{12}I_{13} + I_{12}I_{13}I_{23} \ , \\
-8 &= 12I_{12} - 108I_{12}^2 + 12I_{13} + 414I_{12}I_{13} + 108I_{12}^2I_{13} - 108I_{13}^2 + 108I_{12}I_{13}^2 + 81I_{12}^2I_{13}^2 \\
&\quad -I_{12}^2I_{23} - I_{13}^2I_{23} \ .
\end{aligned}
$$

Invariant under exchange $I_{12} \leftrightarrow I_{13}$

At the level of observables, this is $\theta_{23} \mapsto \theta_{23} + \dfrac{\pi}{2}$

Know as $\mu \leftrightarrow \tau$ symmetry