Near-Critical Behavior in Modular Flavor Models

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27 June 2024

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1 Flavor puzzle and flavor symmetry

2 Modular flavor symmetry

3 Near-Critical Behavior in $\mathrm{SL}(2,\mathbb{Z})$ modular models

4 Near-Critical behavior in $\mathrm{Sp}(4,\mathbb{Z})$ modular models

5 Summary and Outlook

Outline

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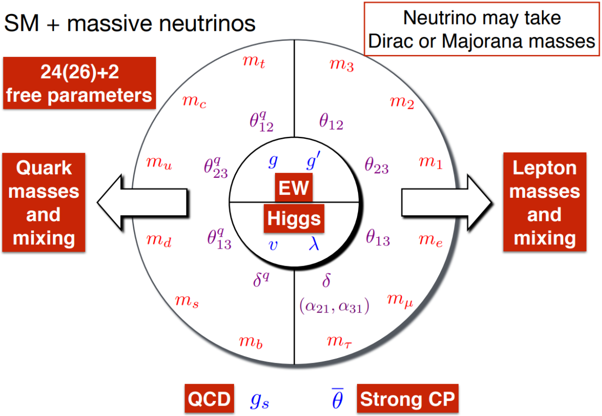
Parameters of the SM

"Pizza pie" of SM



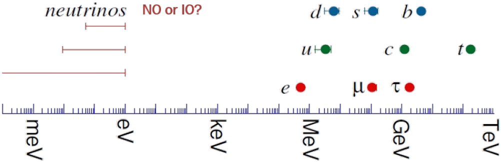
- 3 gauge couplings
- 2 Higgs parameters
- 1 QCD angle $\bar{\theta}$
- 12 fermions masses
- $8(+2)$ flavor mixing parameters

Most of the SM parameters are in the flavor sector!

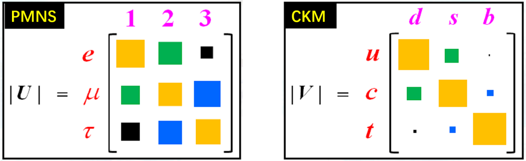
Flavor puzzle

(see talk by Chang, Omar ...)

- What is the origin of the hierarchies in the masses of leptons & quarks ?



- How to understand the flavor mixing patterns of leptons & quarks ?

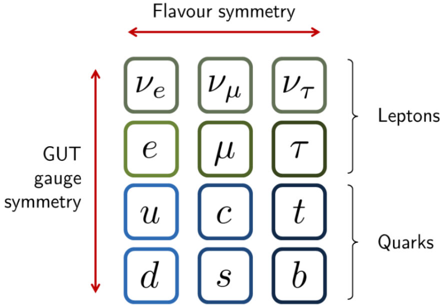


Remarks:

- ◇ In SM, the fermion masses and flavor mixing are determined by Yukawa coupling constants which are completely unconstrained
- ◇ No guiding principles
- ◇ Symmetries?

(see talk by Ramond and Ma)

- **Flavor symmetry**: Horizontal symmetry linking different families.



◇ **Flavor transformation**:

$$\varphi \xrightarrow{g} \rho(g)\varphi$$

with $g \in G_f, \rho(g) \in$ **Rep**(G_f).

◇ **Flavor group G_f** can be Abelian or non-Abelian, continuous Lie groups or discrete groups...

e.g.: (see talk by Chun, Mondragon and Chang)

- Froggatt-Nielsen models: $G_f = \mathrm{U}(1)_{\mathrm{FN}}$. (Froggatt and Nielsen 1979)
- Non-Abelian finite flavor symmetry: $G_f = A_4,\ S_4,\ A_5 \ldots$.(Feruglio and Romanino 2021)

😃 New game: **Modular (flavor) symmetries** $\begin{cases}\text{Hierarchical structure}\\ \text{Fixed flavor structure}\end{cases}$

Toy flavor model—discrete $U(1)_{\text{FN}}(Z_3)$

- Flavor group $G_f = Z_3$
- Charged lepton diagonal, neutrino mass from Weinberg operator.
- One flavon ϕ_ν: nontrivial singlet of Z_3 (singlet of SM)
- The irreps assignments of fields: $\rho_L \sim \text{diag}(1, \omega, \omega^2)\,, \rho_{\phi_\nu} \sim \omega\,, \rho_H \sim 1$

$\Longrightarrow$ neutrino mass Lagrangian

$$\mathcal{W}_\nu = \frac{1}{\Lambda}\left[(HL) f\left(\frac{\phi_\nu}{\Lambda_f}\right)(HL)\right]_{\mathbf{1}} + \ldots$$

$\Longrightarrow$ Neutrino mass matrix

$$M_\nu = \frac{v^2}{\Lambda}\left[\begin{pmatrix} a_{11} & a_{12}\epsilon^2 & a_{13}\epsilon \\ a_{12}\epsilon^2 & a_{22}\epsilon & a_{23} \\ a_{13}\epsilon & a_{23} & a_{33}\epsilon^2 \end{pmatrix} + \epsilon^3 \begin{pmatrix} b_{11} & b_{12}\epsilon^2 & b_{13}\epsilon \\ b_{12}\epsilon^2 & b_{22}\epsilon & b_{23} \\ b_{13}\epsilon & b_{23} & b_{33}\epsilon^2 \end{pmatrix} + \ldots\right]$$

where the VEV of flavon $\langle \phi_\nu / \Lambda_f \rangle = \epsilon \ll 1$.

Mass spectrum: $m_1 : m_2 : m_3 = \mathcal{O}(1) : \mathcal{O}(1) : \mathcal{O}(1)$

Toy flavor model—A_4

- Flavor group $G_f = A_4$
- Charged lepton diagonal, neutrino mass from Weinberg operator.
- One flavon ϕ_ν: triplet of A_4 (singlet of SM)
- The irreps assignments of fields: $\rho_L \sim \mathbf{3}\,,\rho_{\phi_\nu} \sim \mathbf{3}\,,\rho_H \sim \mathbf{1}$

$\Longrightarrow$ neutrino mass Lagrangian

$$\mathcal{W}_\nu = \frac{\alpha}{\Lambda}[(HL)(HL)]_{\mathbf{1}} + \frac{\beta}{\Lambda_f}[(HL)\phi_\nu(HL)]_{\mathbf{1}} + \dots$$

$\Longrightarrow$ Neutrino mass matrix

$$M_\nu = \frac{v_u^2}{\Lambda}\left[\alpha\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} + \frac{\beta}{\Lambda_f}\begin{pmatrix}2a & -c & -b\\ -c & 2b & -a\\ -b & -a & 2c\end{pmatrix} + \dots\right]$$

where the VEV alignment of flavon $\langle\phi_\nu\rangle = (a, b, c)^T$.

Take $\langle\phi_\nu\rangle = (1, 1, 1)^T$:

$$\text{Tri-bimaximal:}\quad U_{\text{PMNS}} = \begin{pmatrix}\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

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1. Flavor puzzle and flavor symmetry
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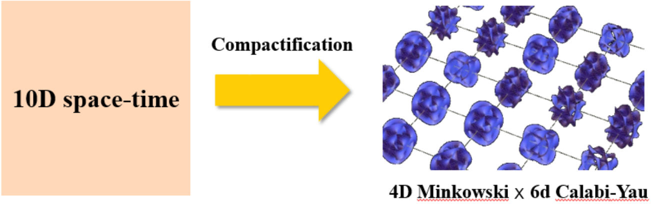
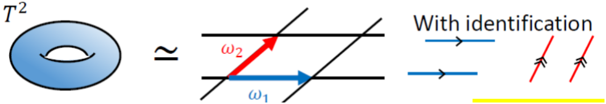
(Feruglio 2017)

In string theory, extra dimension compactification leads to target-space **Modular symmetery** (see talk by Saul Ramos-Sanchez)



$$S = \int d^4x d^6y \mathcal{L}_{10D} \Rightarrow \int d^4x \mathcal{L}_{eff}(\varphi, \tau_i)$$

Example: Torus compactification ($6D \to 4D$):



Modular symmetry

- The shape of a torus is characterized by complex structure modulus

$$\tau = \omega_1/\omega_2, \quad \text{Im}(\tau) > 0$$

which is in the complex upper half plane $\mathcal{H} = \langle \tau \in \mathbb{C} \,|\, \text{Im}(\tau) > 0 \rangle$.

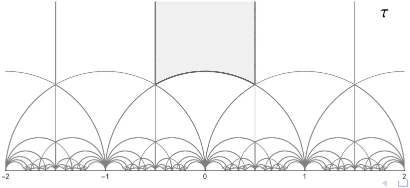
- The lattice (torus) is left invariant by modular transformations

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \implies \tau \to \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$

$\implies$ **(Target-space) Modular symmetry!**

- The inequivalent moduli vacua: Fundamental domain $\mathcal{D} = \mathcal{H}/\mathrm{SL}(2, \mathbb{Z})$



Modular forms

- $\mathrm{SL}(2,\mathbb{Z})$ has two generators satisfying $S^4=(ST)^3=1\,,S^2T=TS^2$

$$S=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\,,\qquad T=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

They act on modulus by: $\tau\xrightarrow{S}-1/\tau,\;\;\tau\xrightarrow{T}\tau+1$

- Principal congruence subgroup $\Gamma(N)\trianglelefteq\mathrm{SL}(2,\mathbb{Z})$

$$\Gamma(N)=\left\{\gamma\in\mathrm{SL}(2,\mathbb{Z})\;\middle|\;\gamma\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\mod N\right\}$$

The quotients are finite modular groups: (Feruglio 2017; Liu and Ding 2019)

  - ◇ $\Gamma_N\equiv\mathrm{SL}(2,\mathbb{Z})/\pm\Gamma(N)\cong\mathrm{PSL}(2,\mathbb{Z}_N):\;\;\Gamma_2\cong S_3\,,\;\Gamma_3\cong A_4\,,\;\Gamma_4\cong S_4\,,\;\Gamma_5\cong A_5\,...$
  - ◇ $\Gamma'_N\equiv\mathrm{SL}(2,\mathbb{Z})/\Gamma(N)\cong\mathrm{SL}(2,\mathbb{Z}_N):\;\;\Gamma'_2\cong S_3\,,\;\Gamma'_3\cong T'\,,\;\Gamma'_4\cong S'_4\,,\;\Gamma'_5\cong A'_5\,...$

- **Modular forms (automorphic forms):** holomorphic functions in $\mathcal{H}\cup i\infty$ and

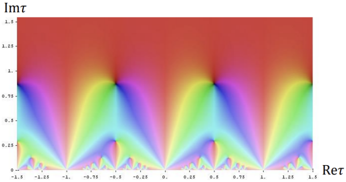
$$f_i(\gamma\tau)=(c\tau+d)^k f_i(\tau)\qquad\forall\gamma\in\Gamma(N)$$

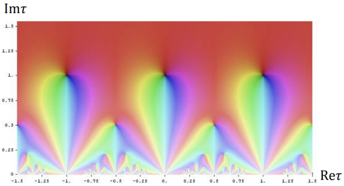
- Modular forms can be regarded as the generalization of periodic functions

$$f_i(\tau+1)=f_i(\tau)\,,\quad f_i(-1/\tau)=(-\tau)^k f_i(\tau)$$

Modular forms

- Some examples of modular forms
  - Dedekind eta function $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ with $q = e^{2\pi i \tau}$
  - Eisenstein series $E_k(\tau) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}} (m\tau + n)^{-k}$ with $\gcd(m,n) = 1$
- Complex Arg plot for E_4 and E_6





- Modular forms span a finite dimensional vector space $\mathcal{M}_k(\Gamma(N))$, which is actually the representation space of finite modular group $\Gamma_N^{(\prime)}$:

$$f_i(\gamma\tau) = (c\tau + d)^k \rho_{ij}(\gamma) f_j(\tau)\,, \qquad \forall \gamma \in \mathrm{SL}(2,\mathbb{Z})$$

where $\rho(\gamma) \in \textbf{Rep}(\Gamma_N^{(\prime)})$ $\quad (\ker(\rho) = \Gamma(N), \mathrm{Im}(\rho) \cong \mathrm{SL}(2,\mathbb{Z}) / \ker(\rho))$

As the low energy effective field theory of string compactification, the $\mathcal{N}=1$ SUSY theory is a non-linear σ-model which contains moduli superfield τ and matter superfields φ_I.

- The action: (Ferrara et al. 1989; Feruglio 2017)

$$\mathcal{S}=\int d^4x d^2\theta d^2\bar{\theta}\ \mathcal{K}(\varphi_I,\bar{\varphi}_I;\tau,\bar{\tau})+\int d^4x d^2\theta\ \mathcal{W}(\varphi_I,\tau)+\text{h.c.}$$

where

- Kähler potential (in minimal form):

$$\mathcal{K}(\varphi_I,\bar{\varphi}_I;\tau,\bar{\tau})=-h\Lambda^2\log(-i\tau+i\bar{\tau})+\sum_I(-i\tau+i\bar{\tau})^{-k_I}|\varphi_I|^2$$

- Superpotential (in power series of φ_I):

$$\mathcal{W}(\varphi_I,\tau)=\sum_n Y_{I_1\ldots I_n}(\tau)\ \varphi_{I_1}\ldots\varphi_{I_n}$$

- The **modular (flavor) transformation** of superfields are non-linear (Lauer, Mas, and Nilles 1989; Feruglio 2017) :

$$\begin{cases} \tau \to \gamma\tau = \dfrac{a\tau + b}{c\tau + d}\,, \\ \varphi_I \to (c\tau + d)^{-k_I}\rho_I(\gamma)\varphi_I\,, \end{cases} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$$

where

- k_I: modular weight of matter field φ_I
- ρ_I: unitary irreps of Γ_N(or Γ'_N) (Feruglio 2017; Liu and Ding 2019)

- Modular invariance of $\mathcal{W}$ requires Yukawa couplings

$$Y_{I_1\ldots I_n}(\tau) \xrightarrow{\gamma} Y_{I_1\ldots I_n}(\gamma\tau) = (c\tau + d)^{k_Y}\rho_Y(\gamma)Y_{I_1\ldots I_n}(\tau)\,,$$

with $k_Y = k_{I_1} + \ldots + k_{I_n}\,, \quad \rho_Y \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_n} \ni \mathbf{1}.$

$\Longrightarrow$ Yukawa couplings are **modular forms** ! $Y_{I_1\ldots I_n}(\tau) \in \mathcal{M}_{k_Y}(\Gamma(N))$

Bottom-up flavor model building

- Freedom of model building: $\varphi_I,\ k_I,\ \rho_I\ (\Gamma_N^{(\prime)})$
- For the given k_Y and ρ_Y, the modular forms space is finite-dimensional: Only a finite number of possible Yukawa couplings !
  e.g.: (Liu and Ding 2019)

N	$\dim\mathcal{M}_k(\Gamma(N))$	Γ_N (Γ'_N)	Modular forms multiplets			
			$k=1$	$k=2$	$k=3$	$k\geq 4$
2	$k/2+1\,(k\in\texttt{even})$	S_3 (S_3)	$-$	$Y_{\mathbf{2}}^{(2)}$	$-$	...
3	$k+1$	A_4 (T')	$Y_{\mathbf{2}}^{(1)}$	$Y_{\mathbf{3}}^{(2)}$	$Y_{\mathbf{2}}^{(3)}, Y_{\mathbf{2}''}^{(3)}$	...
4	$2k+1$	S_4 (S'_4)	$Y_{\hat{\mathbf{3}}'}^{(1)}$	$Y_{\mathbf{2}}^{(2)}, Y_{\mathbf{3}}^{(2)}$	$Y_{\hat{\mathbf{1}}'}^{(3)}, Y_{\hat{\mathbf{3}}}^{(3)}, Y_{\hat{\mathbf{3}}'}^{(3)}$	...
5	$5k+1$	A_5 (A'_5)	$Y_{\mathbf{6}}^{(1)}$	$Y_{\mathbf{3}}^{(2)}, Y_{\mathbf{3}'}^{(2)}, Y_{\mathbf{5}}^{(2)}$	$Y_{\mathbf{4}'}^{(3)}, Y_{\mathbf{6}I}^{(3)}, Y_{\mathbf{6}II}^{(3)}$	...
...	...	...	...	...	...	...

(...; Kobayashi, Tanaka, and Tatsuishi 2018; Feruglio 2017; Penedo and Petcov 2019; Novichkov et al. 2019a; Ding, King, and Liu 2019; Liu and Ding 2019; Liu, Yao, and Ding 2021; Novichkov, Penedo, and Petcov 2021a; Wang, Yu, and Zhou 2021; Yao, Liu, and Ding 2020)

- All higher-dimensional operators in τ are completely determined
- No additional flavons other than the modulus τ
- Modular symmetry is spontaneously broken by modulus VEV $\langle\tau\rangle$ which is treated as a free parameter in bottom-up approach.

An example model —— Group theory

- Finite modular group: $\Gamma_3 \cong A_4$ $= \langle S, T \mid S^2 = (ST)^3 = T^3 = 1 \rangle$
- Irreps: $\mathbf{1},\ \mathbf{1}',\ \mathbf{1}'',\ \mathbf{3}$
- Irreps matrices of generators ($\omega = e^{2\pi i/3}$):

$$\begin{aligned}
\mathbf{1} &: \quad \rho(S) = 1\,, \qquad \rho(T) = 1 \\
\mathbf{1}' &: \quad \rho(S) = 1\,, \qquad \rho(T) = \omega \\
\mathbf{1}'' &: \quad \rho(S) = 1\,, \qquad \rho(T) = \omega^2
\end{aligned}$$

$$\mathbf{3} \ : \quad \rho(S) = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix},\ \rho(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

- Tensor product:

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3}_a \oplus \mathbf{3}_s$$

- There are only one triplet modular forms at lowest weight 2:

$$Y_{\mathbf{3}}^{(2)}(\tau) \equiv \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} \frac{i}{2\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)}\right) \\ \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega^2\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega'\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right) \\ \frac{-i}{\pi}\left(\frac{\eta'(\tau/3)}{\eta(\tau/3)} + \omega\frac{\eta'((\tau+1)/3)}{\eta((\tau+1)/3)} + \omega^2\frac{\eta'((\tau+2)/3)}{\eta((\tau+2)/3)}\right) \end{pmatrix}$$

- Modular forms of higher weight can be constructed by tensor product:

$$k=4: \quad \begin{cases} Y_{\mathbf{1}}^{(4)} = Y_1^2 + 2Y_2Y_3 \\ Y_{\mathbf{1}'}^{(4)} = Y_3^2 + 2Y_1Y_2 \\ Y_{\mathbf{3}}^{(4)} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix} \end{cases}$$

$$k=6: \quad \ldots$$

An example model

In this toy model, we assume (Feruglio 2017)

- Charged lepton diagonal, neutrino mass from Weinberg operator.
- Based on $\Gamma_3 \cong A_4$.
- The assignments of weights and irreps: $\rho_L \sim \mathbf{3}\,,\ \rho_{H_u} \sim \mathbf{1}\,;\, k_L = 1\,,\ k_{H_u} = 0\,.$

$\Longrightarrow$ Modular invariant superpotential $\mathcal{W}_\nu = \frac{1}{\Lambda}[(H_u L) Y_{\mathbf{3}}^{(2)}(\tau)(H_u L)]_{\mathbf{1}}$

$\Longrightarrow$ Neutrino mass matrix

$$M_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}$$

Remarks:

- Modular flavor model is simple, complexity is hidden in modular forms.
- It looks like traditional A_4 model, but each entry are correlated.
- **3** free parameters $\Lambda, \mathrm{Re}(\tau), \mathrm{Im}(\tau)$ determine **9** observables:
  3 neutrino masses + **3** lepton mixing angles + **3** CP phases

- Unfortunately, this model is unrealistic due to its prediction θ_{13} too small:

NuFIT 5.2 (2022)

without SK atmospheric data

	Inverted Ordering ($\Delta\chi^2 = 2.3$)	
	bfp $\pm 1\sigma$	3σ range
$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
$\sin^2\theta_{23}$	$0.578^{+0.016}_{-0.021}$	$0.412 \to 0.623$
$\theta_{23}/^\circ$	$49.5^{+0.9}_{-1.2}$	$39.9 \to 52.1$
$\sin^2\theta_{13}$	$0.02219^{+0.00060}_{-0.00057}$	$0.02047 \to 0.02396$
$\theta_{13}/^\circ$	$8.57^{+0.12}_{-0.11}$	$8.23 \to 8.90$
$\delta_{\rm CP}/^\circ$	286^{+27}_{-32}	$192 \to 360$
$\frac{\Delta m^2_{21}}{10^{-5}\ {\rm eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
$\frac{\Delta m^2_{3\ell}}{10^{-3}\ {\rm eV}^2}$	$-2.498^{+0.032}_{-0.025}$	$-2.581 \to -2.408$

- Best-fit value for τ:

$$\langle\tau\rangle = 0.0111 + 0.9946i$$

- Model predictions:

$$\sin^2\theta_{12} = 0.295$$
$$\sin^2\theta_{23} = 0.651$$
$$\sin^2\theta_{13} = 0.00447 \quad ☹$$
$$\delta_{CP} = 279^\circ$$
$$\frac{\Delta m^2_{sol}}{\Delta m^2_{atm}} = 0.0292$$

The original $\mathrm{SL}(2,\mathbb{Z})$ modular invariant theory has been extended to

- Include the odd weight modular forms, and $\Gamma_N \mapsto \Gamma'_N$. (Liu and Ding 2019)
- Include the rational weight modular forms, and $\Gamma'_N \mapsto \widetilde{\Gamma_N}$. (Liu et al. 2020)
- Reformulate in VVMFs, and $\Gamma_N \mapsto \Gamma/\text{Nor}$ (Liu and Ding 2022)
- Combine with the CP symmetry: $\tau \xrightarrow{CP} -\tau^*$ (Baur et al. 2019; Novichkov et al. 2019b)
- Eclectic flavor symmetry: tradition flavor $\cup$ modular flavor (Nilles, Ramos-Sánchez, and Vaudrevange 2020; Nilles, Ramos-Sanchez, and Vaudrevange 2020)

Minimal model building:

- The minimal lepton models (6 real input parameters). (Ding, Liu, and Yao 2023; Ding et al. 2023)
- The minimal quark models (8 real input parameters). (Ding et al. 2023)
- The minimal lepton + quark models (14 real input parameters) (Ding et al. 2023)

Other applications:

- Applying to solve Strong CP problem. (Feruglio, Strumia, and Titov 2023; Petcov and Tanimoto 2024; Penedo and Petcov 2024)
- Modular inflation. (Ding, Jiang, and Zhao 2024; King and Wang 2024)
- ...

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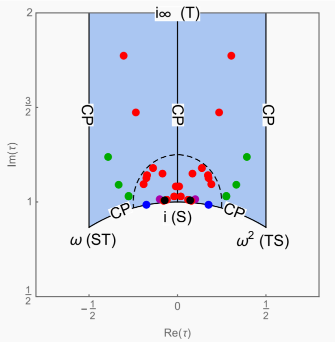
Motivation

- Modular flavor models are elegant and simple, but also complicated.
  "Make things as simple as possible, but no simpler.———-Albert Einstein"
- At present only numerical analysis, lack of (semi-)analytical understanding.
- The best-fit values of modulus shows a specific distribution : (Feruglio 2023a)

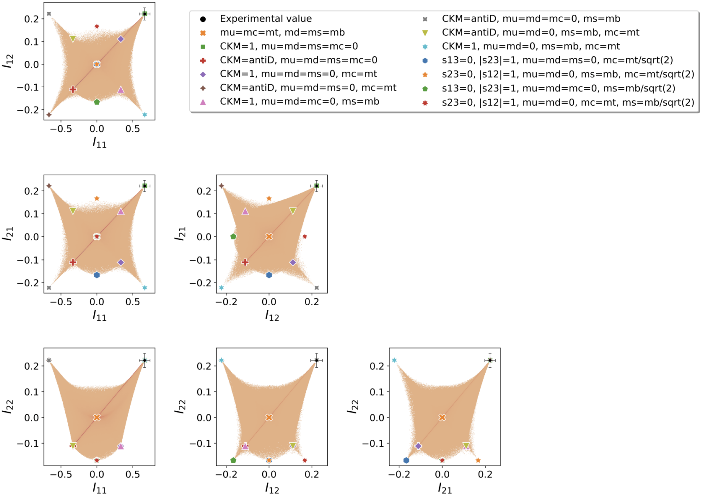


- Are there any model-independent features?

Hint from data

- There are no non-trivial residual flavor symmetries at low energy.
- Flavor observables close to a "special point" (in terms of flavor invariants)?

(Bento, Silva, and Trautner 2024)



Mass matrix pattern near the fixed point

- The modular form has q-expansion, and the first few terms are accurate enough for numerical calculation.

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + 84q^4 + \dots \\ -6q^{1/3}(1 + 7q + 8q^2 + 18q^3 + 14q^4 + \dots) \\ -18q^{2/3}(1 + 2q + 5q^2 + 4q^3 + 8q^4 + \dots) \end{pmatrix} \xrightarrow{\tau \to i\infty} \begin{pmatrix} 1 \\ -6q^{1/3} \\ -18q^{2/3} \end{pmatrix}$$

where $q = e^{2\pi i\tau} = e^{-2\pi \mathrm{Im}\tau} e^{2\pi i \mathrm{Re}\tau}$ (therefore $\tau \to i\infty \Rightarrow q \to 0$.)

- Neutrino mass matrix :

$$M_\nu = \begin{pmatrix} 2Y_1 & -Y_3 & -Y_2 \\ -Y_3 & 2Y_2 & -Y_1 \\ -Y_2 & -Y_1 & 2Y_3 \end{pmatrix} \frac{v_u^2}{\Lambda} \xrightarrow{q \to 0} \begin{pmatrix} 2 & -18q^{2/3} & -6q^{1/3} \\ -18q^{2/3} & -12q^{1/3} & -1 \\ -6q^{1/3} & -1 & -36q^{2/3} \end{pmatrix} \frac{v_u^2}{\Lambda} + \dots$$

- Define the new variable $\epsilon := q^{1/3} = e^{2\pi i\tau/3}$, then

$$M_\nu = \frac{v_u^2}{\Lambda} \left[\begin{pmatrix} 2 & -18\epsilon^2 & -6\epsilon \\ -18\epsilon^2 & -12\epsilon & -1 \\ -6\epsilon & -1 & -36\epsilon^2 \end{pmatrix} + \epsilon^3 \begin{pmatrix} 24 & 36\epsilon^2 & 42\epsilon \\ 36\epsilon^2 & 84\epsilon & -12 \\ 42\epsilon & -12 & -72\epsilon^2 \end{pmatrix} + \cdots \right]$$

It really mimics a traditional Z_3 flavor model in the vicinity of $i\infty$!
(Notice that all the coefficients are fixed, not arbitrary, even not $\mathcal{O}(1)$)

Question: What is the origin of the patterns of mass matrices?
Answer: The linear sub-symmetry in modular flavor symmetry.

- Let us focus on the sub-symmetry $Z^T = \langle T \rangle \in \mathrm{SL}(2, \mathbb{Z})$:

$$\begin{cases} \tau \xrightarrow{\gamma} \frac{a\tau+b}{c\tau+d} \\ \varphi_I \xrightarrow{\gamma} (c\tau+d)^{-k_I} \rho_I(\gamma)\, \varphi_I \end{cases} \supset \begin{cases} \tau \xrightarrow{T} \tau+1 \\ \varphi_I \xrightarrow{T} \rho_I(T)\, \varphi_I \end{cases}$$

- Change the field variable $\tau \mapsto u = e^{2\pi i \tau/3}; \varphi_I \mapsto \varphi_I$:

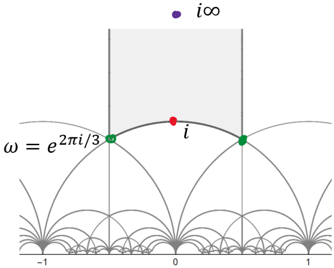
$$\begin{cases} u = e^{2\pi i \tau/3} \xrightarrow{T} \omega\, u \\ \varphi_I \xrightarrow{T} \rho_I(T) \varphi_I \end{cases}$$

The original sub-symmetry Z^T becomes a linear flavor symmetry

Fields	L_1	L_2	L_3	u
$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	$(2, -1/2)$	$(2, -1/2)$	$(2, -1/2)$	$(1, 0)$
$Z_3^T \subset \mathrm{SL}(2, \mathbb{Z})$	1	ω	ω^2	ω

There are two other sub-symmetries in $\mathrm{SL}(2,\mathbb{Z})$ that can be linearized.

- Fixed points in moduli space and their stabilizer:



- Fixed points (critical points) τ_0 are invarinat under their stabilizers G_0:

$$\gamma_0 \tau_0 = \tau_0 \,, \qquad \gamma_0 \in G_0$$

Fixed Point τ_0	i	ω	$i\infty$
Stabilizer G_0	Z_4^S	$Z_3^{ST} \times Z_2^{S^2}$	$Z^T \times Z_2^{S^2}$
Order of G_0	4	6	∞

- Field redefinition and linearization of G_0: (Feruglio et al. 2021; Novichkov, Penedo, and Petcov 2021b)

  - $u := \dfrac{\tau - \tau_0}{\tau - \bar{\tau}_0}$ $\qquad \Phi_I := (1-u)^{k_I}\varphi_I \,,$ $\qquad$ for $\tau_0 = i, \omega$.
  - $u := e^{2\pi i \tau/N}$, $\qquad \Phi_I := \varphi_I \,,$ $\qquad$ for $\tau_0 = i\infty$.

$$\longrightarrow \quad \begin{cases} u \xrightarrow{\gamma_0} e^{i\theta_0}\, u \\ \Phi_I \xrightarrow{\gamma_0} \Omega_I(\gamma_0)\, \Phi_I \end{cases}$$

with

$$\theta_0 = \begin{cases} \frac{1}{2} 2\pi & \text{for } \gamma_0 = S \\ \frac{2}{3} 2\pi & \text{for } \gamma_0 = ST \\ \frac{1}{N} 2\pi & \text{for } \gamma_0 = T \end{cases}, \qquad \Omega_I = \begin{cases} i^{k_I} \rho_I(S) & \text{for } \gamma_0 = S \\ \omega^{k_I} \rho_I(ST) & \text{for } \gamma_0 = ST \\ \rho_I(T) & \text{for } \gamma_0 = T \end{cases}$$

Remarks: Ω_I are still Reps of $\mathrm{SL}(2,\mathbb{Z})$, determined by (k_I, ρ_I). It is actually the Rep decomposition of $\mathrm{SL}(2,\mathbb{Z})$ under G_0.

- We can expand M around the fixed point $u = 0$:

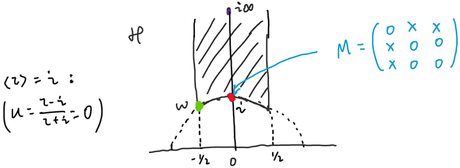
$$M_{ij}(u) = m_{ij}^{(0)} + m_{ij}^{(1)} u + m_{ij}^{(2)} u^2 + m_{ij}^{(3)} u^3 + \cdots .$$

- Invariance of mass matrices under G_0 (where $\Omega := \Omega_{H_u} \Omega_L$): e.g.:

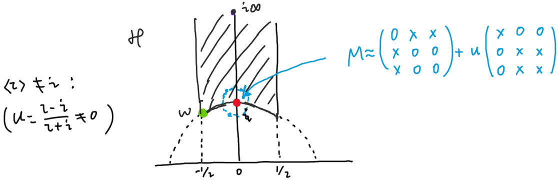
$$M_\nu(u) \xrightarrow{\gamma_0} M_\nu(e^{i\theta_0} u) = \Omega^* M_\nu(u) \Omega^\dagger ,$$

Universal scaling behavior near critical point

- The form of Ω_I determine the hierarchy pattern of mass matrix M.
- In the vicinity of critical point, namely $0 \approx |u| \ll 1$, only the leading term dominates : **Power law** (scaling behavior)
  - At the critical point $u = 0$ (i.e.,$\tau = \tau_0$), linear symmetry G_0 is unbroken: G_0-symmetric phase, where the flavor data will be unrealistic.



  - Deviate a little bit from the critical point, $u \approx 0$, linear symmetry G_0 is broken: G_0-broken phase, where realistic flavor data can be reproduced.



From the representations of $\mathrm{SL}(2,\mathbb{Z})$, we can list all possible Ω_I

- In the case of $G_0 = Z^T$ (base point $i\infty$), there are too many possibilities for $\Omega(T)$ and there is no complete classification yet, so we ignore this case.
- In the case of $G_0 = Z_4^S$ and $G_0 = Z_3^{ST} \times Z_2^{S^2}$ (base points i and ω), there are only a few cases of decomposition:
  - Irreducible "triplets" : (Feruglio 2023a; Feruglio 2023b)

$$\Omega(S) = i^s \mathrm{diag}(1,-1,-1) \; ;$$
$$\Omega(ST) = \mathrm{diag}(1,\omega,\omega^2) \; .$$

  - Reducible "doublets + singlets" : (Chen et al. 2024a)

$$\Omega(S) = i^{s_2} \mathrm{diag}(1,-1,0) + i^{s_1} \mathrm{diag}(0,0,1) \; ;$$
$$\Omega(ST) = \omega^{s_2} \mathrm{diag}(1,\omega,0) + \omega^{s_1} \mathrm{diag}(0,0,1) \; .$$

Mass matrix patterns near the critical point i

(s_2, s_1)	$M_{\bar{e}e}$	M_ν	M_ν^{-1}
$(0,0)$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & m_{13}^{(0)} \\ \tilde{u}m_{12} & m_{22}^{(0)} & \tilde{u}m_{23} \\ m_{13}^{(0)} & \tilde{u}m_{23} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & m_{13}^{(0)} \\ \tilde{u}m_{12} & m_{22}^{(0)} & \tilde{u}m_{23} \\ m_{13}^{(0)} & \tilde{u}m_{23} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & m_{13}^{(0)} \\ \tilde{u}m_{12} & m_{22}^{(0)} & \tilde{u}m_{23} \\ m_{13}^{(0)} & \tilde{u}m_{23} & m_{33}^{(0)} \end{pmatrix}$
$(0,1)$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & 0 \\ \tilde{u}m_{12} & m_{22}^{(0)} & 0 \\ 0 & 0 & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & 0 \\ \tilde{u}m_{12} & m_{22}^{(0)} & 0 \\ 0 & 0 & \tilde{u}m_{33} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & 0 \\ \tilde{u}m_{12} & m_{22}^{(0)} & 0 \\ 0 & 0 & \tilde{u}m_{33} \end{pmatrix}$
$(0,2)$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & \tilde{u}m_{13} \\ \tilde{u}m_{12} & m_{22}^{(0)} & m_{23}^{(0)} \\ \tilde{u}m_{13} & m_{23}^{(0)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & \tilde{u}m_{13} \\ \tilde{u}m_{12} & m_{22}^{(0)} & m_{23}^{(0)} \\ \tilde{u}m_{13} & m_{23}^{(0)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & \tilde{u}m_{13} \\ \tilde{u}m_{12} & m_{22}^{(0)} & m_{23}^{(0)} \\ \tilde{u}m_{13} & m_{23}^{(0)} & m_{33}^{(0)} \end{pmatrix}$
$(0,3)$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & 0 \\ \tilde{u}m_{12} & m_{22}^{(0)} & 0 \\ 0 & 0 & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & 0 \\ \tilde{u}m_{12} & m_{22}^{(0)} & 0 \\ 0 & 0 & \tilde{u}m_{33} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & 0 \\ \tilde{u}m_{12} & m_{22}^{(0)} & 0 \\ 0 & 0 & \tilde{u}m_{33} \end{pmatrix}$
$(1,0)$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & 0 \\ \tilde{u}m_{12} & m_{22}^{(0)} & 0 \\ 0 & 0 & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} \tilde{u}m_{11} & m_{12}^{(0)} & 0 \\ m_{12}^{(0)} & \tilde{u}m_{22} & 0 \\ 0 & 0 & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} \tilde{u}m_{11} & m_{12}^{(0)} & 0 \\ m_{12}^{(0)} & \tilde{u}m_{22} & 0 \\ 0 & 0 & m_{33}^{(0)} \end{pmatrix}$
$(1,1)$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & m_{13}^{(0)} \\ \tilde{u}m_{12} & m_{22}^{(0)} & \tilde{u}m_{23} \\ m_{13}^{(0)} & \tilde{u}m_{23} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} \tilde{u}m_{11} & m_{12}^{(0)} & \tilde{u}m_{13} \\ m_{12}^{(0)} & \tilde{u}m_{22} & m_{23}^{(0)} \\ \tilde{u}m_{13} & m_{23}^{(0)} & \tilde{u}m_{33} \end{pmatrix}$	$\boxed{\begin{pmatrix} \tilde{u}m_{11} & m_{12}^{(0)} & \tilde{u}m_{13} \\ m_{12}^{(0)} & \tilde{u}m_{22} & m_{23}^{(0)} \\ \tilde{u}m_{13} & m_{23}^{(0)} & \tilde{u}m_{33} \end{pmatrix}}$
$(1,2)$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & 0 \\ \tilde{u}m_{12} & m_{22}^{(0)} & 0 \\ 0 & 0 & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} \tilde{u}m_{11} & m_{12}^{(0)} & 0 \\ m_{12}^{(0)} & \tilde{u}m_{22} & 0 \\ 0 & 0 & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} \tilde{u}m_{11} & m_{12}^{(0)} & 0 \\ m_{12}^{(0)} & \tilde{u}m_{22} & 0 \\ 0 & 0 & m_{33}^{(0)} \end{pmatrix}$
$(1,3)$	$\begin{pmatrix} m_{11}^{(0)} & \tilde{u}m_{12} & \tilde{u}m_{13} \\ \tilde{u}m_{12} & m_{22}^{(0)} & m_{23}^{(0)} \\ \tilde{u}m_{13} & m_{23}^{(0)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} \tilde{u}m_{11} & m_{12}^{(0)} & m_{13}^{(0)} \\ m_{12}^{(0)} & \tilde{u}m_{22} & \tilde{u}m_{23} \\ m_{13}^{(0)} & \tilde{u}m_{23} & \tilde{u}m_{33} \end{pmatrix}$	$\boxed{\begin{pmatrix} \tilde{u}m_{11} & m_{12}^{(0)} & m_{13}^{(0)} \\ m_{12}^{(0)} & \tilde{u}m_{22} & \tilde{u}m_{23} \\ m_{13}^{(0)} & \tilde{u}m_{23} & \tilde{u}m_{33} \end{pmatrix}}$

where $\tilde{u}m_{ij} = um_{ij}^{(10)} + \bar{u}m_{ij}^{(01)}$.

Mass matrix patterns near the critical point ω

(s_2, s_1)	$M_{\bar{e}e}$	M_ν	M_ν^{-1}
$(0,0)$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & m_{13}^{(0)} \\ \bar{u} m_{12}^{(10)} & m_{22}^{(0)} & \bar{u} m_{23}^{(10)} \\ m_{13}^{(0)} & u m_{23}^{(10)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & m_{13}^{(0)} \\ u m_{12}^{(10)} & \bar{u} m_{22}^{(01)} & u m_{23}^{(10)} \\ m_{13}^{(0)} & u m_{23}^{(10)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \bar{u} m_{12}^{(01)} & m_{13}^{(0)} \\ \bar{u} m_{12}^{(01)} & u m_{22}^{(10)} & \bar{u} m_{23}^{(01)} \\ m_{13}^{(0)} & \bar{u} m_{23}^{(01)} & m_{33}^{(0)} \end{pmatrix}$
$(0,1)$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & u m_{13}^{(10)} \\ \bar{u} m_{12}^{(01)} & m_{22}^{(0)} & m_{23}^{(0)} \\ \bar{u} m_{13}^{(01)} & m_{23}^{(0)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & u m_{13}^{(10)} \\ u m_{12}^{(10)} & \bar{u} m_{22}^{(01)} & \bar{u} m_{23}^{(01)} \\ u m_{13}^{(10)} & \bar{u} m_{23}^{(01)} & \bar{u} m_{33}^{(01)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \bar{u} m_{12}^{(01)} & \bar{u} m_{13}^{(01)} \\ \bar{u} m_{12}^{(01)} & u m_{22}^{(10)} & u m_{23}^{(10)} \\ \bar{u} m_{13}^{(01)} & u m_{23}^{(10)} & u m_{33}^{(10)} \end{pmatrix}$
$(0,2)$ $(1,0)$ $(2,1)$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & \bar{u} m_{13}^{(01)} \\ \bar{u} m_{12}^{(01)} & m_{22}^{(0)} & u m_{23}^{(10)} \\ u m_{13}^{(10)} & \bar{u} m_{23}^{(01)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & \bar{u} m_{13}^{(01)} \\ u m_{12}^{(10)} & \bar{u} m_{22}^{(01)} & m_{23}^{(0)} \\ \bar{u} m_{13}^{(01)} & m_{23}^{(0)} & u m_{33}^{(10)} \end{pmatrix}$	$\begin{pmatrix} m_{11}^{(0)} & \bar{u} m_{12}^{(01)} & u m_{13}^{(10)} \\ \bar{u} m_{12}^{(01)} & u m_{22}^{(10)} & m_{23}^{(0)} \\ u m_{13}^{(10)} & m_{23}^{(0)} & \bar{u} m_{33}^{(01)} \end{pmatrix}$
$(1,1)$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & m_{13}^{(0)} \\ \bar{u} m_{12}^{(01)} & m_{22}^{(0)} & \bar{u} m_{23}^{(01)} \\ m_{13}^{(0)} & u m_{23}^{(10)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} \bar{u} m_{11}^{(01)} & m_{12}^{(0)} & \bar{u} m_{13}^{(01)} \\ m_{12}^{(0)} & u m_{22}^{(10)} & m_{23}^{(0)} \\ \bar{u} m_{13}^{(01)} & m_{23}^{(0)} & \bar{u} m_{33}^{(01)} \end{pmatrix}$	$\begin{pmatrix} u m_{11}^{(10)} & m_{12}^{(0)} & u m_{13}^{(10)} \\ m_{12}^{(0)} & \bar{u} m_{22}^{(01)} & m_{23}^{(0)} \\ u m_{13}^{(10)} & m_{23}^{(0)} & u m_{33}^{(10)} \end{pmatrix}$
$(1,2)$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & u m_{13}^{(10)} \\ \bar{u} m_{12}^{(01)} & m_{22}^{(0)} & m_{23}^{(0)} \\ \bar{u} m_{13}^{(01)} & m_{23}^{(0)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} \bar{u} m_{11}^{(01)} & m_{12}^{(0)} & m_{13}^{(0)} \\ m_{12}^{(0)} & u m_{22}^{(10)} & u m_{23}^{(10)} \\ m_{13}^{(0)} & u m_{23}^{(10)} & u m_{33}^{(10)} \end{pmatrix}$	$\begin{pmatrix} u m_{11}^{(10)} & m_{12}^{(0)} & m_{13}^{(0)} \\ m_{12}^{(0)} & \bar{u} m_{22}^{(01)} & \bar{u} m_{23}^{(01)} \\ m_{13}^{(0)} & \bar{u} m_{23}^{(01)} & \bar{u} m_{33}^{(01)} \end{pmatrix}$
$(2,0)$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & u m_{13}^{(10)} \\ \bar{u} m_{12}^{(01)} & m_{22}^{(0)} & m_{23}^{(0)} \\ \bar{u} m_{13}^{(01)} & m_{23}^{(0)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} u m_{11}^{(10)} & \bar{u} m_{12}^{(01)} & \bar{u} m_{13}^{(01)} \\ \bar{u} m_{12}^{(01)} & m_{22}^{(0)} & m_{23}^{(0)} \\ \bar{u} m_{13}^{(01)} & m_{23}^{(0)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} \bar{u} m_{11}^{(01)} & u m_{12}^{(10)} & u m_{13}^{(10)} \\ u m_{12}^{(10)} & m_{22}^{(0)} & m_{23}^{(0)} \\ u m_{13}^{(10)} & m_{23}^{(0)} & m_{33}^{(0)} \end{pmatrix}$
$(2,2)$	$\begin{pmatrix} m_{11}^{(0)} & u m_{12}^{(10)} & m_{13}^{(0)} \\ \bar{u} m_{12}^{(01)} & m_{22}^{(0)} & \bar{u} m_{23}^{(01)} \\ m_{13}^{(0)} & u m_{23}^{(10)} & m_{33}^{(0)} \end{pmatrix}$	$\begin{pmatrix} u m_{11}^{(10)} & \bar{u} m_{12}^{(01)} & u m_{13}^{(10)} \\ \bar{u} m_{12}^{(01)} & m_{22}^{(0)} & \bar{u} m_{23}^{(01)} \\ u m_{13}^{(10)} & \bar{u} m_{23}^{(01)} & u m_{33}^{(10)} \end{pmatrix}$	$\begin{pmatrix} \bar{u} m_{11}^{(01)} & u m_{12}^{(10)} & \bar{u} m_{13}^{(01)} \\ u m_{12}^{(10)} & m_{22}^{(0)} & u m_{23}^{(10)} \\ \bar{u} m_{13}^{(01)} & u m_{23}^{(10)} & \bar{u} m_{33}^{(01)} \end{pmatrix}$

Viable patterns & Predictions

Assuming the coefficients $m_{ij}^{(ab)}$ are $\mathcal{O}(1)$, There is a viable mass matrix pattern near i and near ω (where $\varepsilon = |u| \ll 1$):

τ	(s_2, s_1)	$M_\nu(0,0)$	mass ordering	$\frac{\Delta m^2_{\text{sol}}}{\Delta m^2_{\text{atm}}}$	$\sin^2\theta_{12}$	$\sin^2\theta_{13}$	$\sin^2\theta_{23}$
$\approx i$	$(1,1)$	singular	NO	$\mathcal{O}(\varepsilon^3)$	$\frac{1}{2}(1+\mathcal{O}(\varepsilon))$	$\mathcal{O}(\varepsilon^2)$	$\mathcal{O}(1)$
$\approx \omega$	$(1,2)$	singular	NO	$\mathcal{O}(\varepsilon^3)$	$\frac{1}{2}(1+\mathcal{O}(\varepsilon))$	$\mathcal{O}(\varepsilon^2)$	$\mathcal{O}(1)$

Remarks:

- The viable reducible case $(1,1)$ around i is exactly the same as the successful pattern found in irreducible case ($s=1$ or 3).
- The viable reducible case $(1,2)$ around ω is almost the same as the behavior of case $(1,1)$ near i. It does not exist in the irreducible case near ω.
- However, the assumption that $m_{ij}^{(ab)}$ are $\mathcal{O}(1)$ is not always true. (Chen et al. 2024a)

Mass hierarchy near i

In the vicinity of critical point i (where $\varepsilon = \left|\frac{\tau - i}{\tau + i}\right|$):

Ω	Ω_c	Hierarchy
$\text{diag}(i, -i, -1)$	$\text{diag}(i, -i, -1)$	$(1, 1, 1)$
$\text{diag}(i, -i, i)$	$\text{diag}(i, -i, -i)$	$(1, 1, 1)$
$\text{diag}(i, -i, 1)$	$\text{diag}(i, -i, 1)$	$(1, 1, 1)$
$\text{diag}(1, -1, -1)$	$\text{diag}(1, -1, -1)$	$(1, 1, 1)$
$\text{diag}(1, -1, i)$	$\text{diag}(1, -1, -i)$	$(1, 1, 1)$
$\text{diag}(1, -1, 1)$	$\text{diag}(1, -1, 1)$	$(1, 1, 1)$
$\text{diag}(i, -i, -i)$	$\text{diag}(i, -i, -i)$	$(\varepsilon, 1, 1)$
$\text{diag}(i, -i, i)$	$\text{diag}(i, -i, i)$	$(\varepsilon, 1, 1)$
$\text{diag}(i, -i, 1)$	$\text{diag}(i, -i, -1)$	$(\varepsilon, 1, 1)$
$\text{diag}(1, -1, -i)$	$\text{diag}(1, -1, -i)$	$(\varepsilon, 1, 1)$
$\text{diag}(1, -1, i)$	$\text{diag}(1, -1, i)$	$(\varepsilon, 1, 1)$
$\text{diag}(1, -1, 1)$	$\text{diag}(1, -1, -1)$	$(\varepsilon, 1, 1)$

Mass hierarchy near ω

In the vicinity of critical point ω (where $\varepsilon = \left|\frac{\tau-\omega}{\tau-\omega^2}\right|$):

Ω	Ω_c	Hierarchy
$\text{diag}(1,\omega,1)$	$\text{diag}(1,\omega,\omega^2)$	$(\varepsilon,1,1)$
$\text{diag}(1,\omega,\omega)$	$\text{diag}(\omega,\omega^2,\omega^2)$	$(\varepsilon,1,1)$
$\text{diag}(1,\omega,\omega)$	$\text{diag}(\omega^2,1,1)$	$(\varepsilon,1,1)$
$\text{diag}(1,\omega,\omega^2)$	$\text{diag}(\omega,\omega^2,\omega)$	$(\varepsilon,1,1)$
$\text{diag}(1,\omega,\omega^2)$	$\text{diag}(\omega^2,1,\omega^2)$	$(\varepsilon,1,1)$
$\text{diag}(\omega,\omega^2,\omega^2)$	$\text{diag}(\omega,\omega^2,\omega^2)$	$(\varepsilon,1,1)$
$\text{diag}(\omega^2,1,1)$	$\text{diag}(\omega^2,1,1)$	$(\varepsilon,1,1)$
$\text{diag}(1,\omega,1)$	$\text{diag}(1,\omega,1)$	$(\varepsilon^2,1,1)$
$\text{diag}(1,\omega,1)$	$\text{diag}(\omega^2,1,\omega^2)$	$(\varepsilon^2,1,1)$
$\text{diag}(1,\omega,\omega)$	$\text{diag}(1,\omega,\omega^2)$	$(\varepsilon^2,1,1)$
$\text{diag}(1,\omega,\omega^2)$	$\text{diag}(\omega,\omega^2,\omega^2)$	$(\varepsilon^2,1,1)$
$\text{diag}(1,\omega,\omega^2)$	$\text{diag}(\omega^2,1,1)$	$(\varepsilon^2,1,1)$
$\text{diag}(\omega,\omega^2,\omega)$	$\text{diag}(\omega,\omega^2,\omega)$	$(\varepsilon^2,1,1)$
$\text{diag}(\omega,\omega^2,\omega)$	$\text{diag}(\omega^2,1,\omega^2)$	$(\varepsilon^2,1,1)$

Ω	Ω_c	Hierarchy
$\text{diag}(1,\omega,1)$	$\text{diag}(\omega^2,1,1)$	$(1,1,1)$
$\text{diag}(1,\omega,\omega)$	$\text{diag}(\omega^2,1,\omega^2)$	$(1,1,1)$
$\text{diag}(1,\omega,\omega^2)$	$\text{diag}(1,\omega,\omega^2)$	$(1,1,1)$
$\text{diag}(\omega,\omega^2,\omega)$	$\text{diag}(\omega,\omega^2,\omega^2)$	$(1,1,1)$
$\text{diag}(1,\omega,1)$	$\text{diag}(\omega,\omega^2,\omega)$	$(\varepsilon,\varepsilon,1)$
$\text{diag}(\omega^2,1,\omega^2)$	$\text{diag}(\omega^2,1,\omega^2)$	$(\varepsilon,\varepsilon,1)$
$\text{diag}(1,\omega,1)$	$\text{diag}(1,\omega,\omega)$	$(\varepsilon^2,\varepsilon,1)$
$\text{diag}(1,\omega,1)$	$\text{diag}(\omega,\omega^2,\omega^2)$	$(\varepsilon^2,\varepsilon,1)$
$\text{diag}(1,\omega,\omega)$	$\text{diag}(\omega,\omega^2,\omega)$	$(\varepsilon^2,\varepsilon,1)$
$\text{diag}(\omega,\omega^2,\omega)$	$\text{diag}(\omega^2,1,1)$	$(\varepsilon^2,\varepsilon,1)$
$\text{diag}(\omega,\omega^2,\omega^2)$	$\text{diag}(\omega^2,1,\omega^2)$	$(\varepsilon^2,\varepsilon,1)$
$\text{diag}(\omega^2,1,1)$	$\text{diag}(\omega^2,1,\omega^2)$	$(\varepsilon^2,\varepsilon,1)$
$\text{diag}(1,\omega,\omega)$	$\text{diag}(1,\omega,\omega)$	$(\varepsilon^2,\varepsilon^2,1)$
$\text{diag}(\omega,\omega^2,\omega^2)$	$\text{diag}(\omega^2,1,1)$	$(\varepsilon^2,\varepsilon^2,1)$

Outline

1. Flavor puzzle and flavor symmetry
2. Modular flavor symmetry
3. Near-Critical Behavior in $\mathrm{SL}(2,\mathbb{Z})$ modular models
4. Near-Critical behavior in $\mathrm{Sp}(4,\mathbb{Z})$ modular models
5. Summary and Outlook

(Ding, Feruglio, and Liu 2021)

- String compactifications down to 4D natrually produce many moduli
- The natural generalization of $\mathrm{SL}(2, \mathbb{Z})$ modular invariance is $\mathrm{Sp}(2g, \mathbb{Z})$ Symplectic modular invariance.
- (non-compact) Moduli space: Siegel upper half plane

$$\mathcal{H}_g = \{\tau \in GL(g, \mathbb{C}) \mid \tau^T = \tau,\ \mathrm{Im}(\tau) > 0\}$$

- (Siegel) symplectic modular group $\mathrm{Sp}(2g, \mathbb{Z})$:

$$\mathrm{Sp}(2g, \mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid \gamma^T J \gamma = J \text{ with } J = \begin{pmatrix} 0 & \mathbb{I}_g \\ -\mathbb{I}_g & 0 \end{pmatrix} \right\}$$

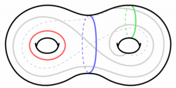
- Action of $\mathrm{Sp}(2g, \mathbb{Z})$ on τ and φ_I:

$$\begin{cases} \tau \to \gamma\tau = (A\tau + B)(C\tau + D)^{-1} \\ \varphi_I \to \det(C\tau + D)^{-k_I} \rho_I(\gamma) \varphi_I \end{cases}$$

Remark: At genus $g = 1$, $\mathrm{Sp}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$.

- Moduli (three components τ_1, τ_2, τ_3):

$$\mathcal{H}_2 = \left\{ \tau = \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} \middle| \, \text{det}(\text{Im}(\tau)) > 0, \ \ \text{tr}(\text{Im}(\tau)) > 0 \right\}$$



  - ◇ $\tau_3 = 0$: $\mathcal{H}_2 \simeq$ 2 factorized tori;
  - ◇ $\tau_3 \neq 0$: $\mathcal{H}_2 \simeq$ generic Riemann surface of genus 2;

- The simplest non-Abelian finite modular group is $\Gamma_{2,2} \cong S_6$, which has **no 3-d irreps to accommodate 3 families of fermions!**

- Restrict to (invariant) subspace Σ of $\mathcal{H}_2$ (Symmetry $\text{Sp}(4, \mathbb{Z}) \mapsto N(H)$):

  - ◇ Two dimensional:
$$\begin{cases} \Sigma_1 : \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix}, & \Gamma_{2,2} \mapsto (S_3 \times S_3) \rtimes Z_2 \\ \Sigma_2 : \begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_1 \end{pmatrix}, & \Gamma_{2,2} \mapsto S_4 \times Z_2 \end{cases}$$
  - ◇ One dimensional:
$$\begin{pmatrix} i & 0 \\ 0 & \tau_2 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \tau_2 \end{pmatrix}, \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_1 \end{pmatrix}, \begin{pmatrix} \tau_1 & 1/2 \\ 1/2 & \tau_1 \end{pmatrix}, \begin{pmatrix} \tau_1 & \tau_1/2 \\ \tau_1/2 & \tau_1 \end{pmatrix}.$$

Fixed points in $\mathcal{H}_2$

(Ding, Feruglio, and Liu 2024)

- Unlike the $\mathrm{SL}(2,\mathbb{Z})$, the fundamental domain $\mathcal{F}_2$ of the $\mathrm{Sp}(4,\mathbb{Z})$ is very complicated and cannot be visualized.
- There six inequivalent fixed points in $\mathcal{F}_2$, four of which are also in the subspace Σ_2:

#	Fixed points τ_0	Residual symmetry in $Sp(4,\mathbb{Z})$	G_0 = Residual symmetry in $N(H)$
1.	$\begin{pmatrix} \zeta & \zeta+\zeta^{-2} \\ \zeta+\zeta^{-2} & -\zeta^{-1} \end{pmatrix}$	Z_{10}	—
2.	$\begin{pmatrix} \eta & \frac{1}{2}(\eta-1) \\ \frac{1}{2}(\eta-1) & \eta \end{pmatrix}$	$GL(2,3)$	$D_4 \quad (\tau\in\Sigma_2)$
3.	$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$(Z_4\times Z_4)\rtimes Z_2$	$\begin{cases} (Z_4\times Z_4)\rtimes Z_2 & (\tau\in\Sigma_1) \\ D_4\circ Z_4 & (\tau\in\Sigma_2) \end{cases}$
4.	$\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$	$(Z_6\times Z_6)\rtimes Z_2$	$\begin{cases} (Z_6\times Z_6)\rtimes Z_2 & (\tau\in\Sigma_1) \\ D_4\times Z_3 & (\tau\in\Sigma_2) \end{cases}$
5.	$\frac{i\sqrt{3}}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$(Z_6\times Z_2)\rtimes Z_2$	$D_4 \quad (\tau\in\Sigma_2)$
6.	$\begin{pmatrix} \omega & 0 \\ 0 & i \end{pmatrix}$	$Z_{12}\times Z_2$	$Z_{12}\times Z_2 \quad (\tau\in\Sigma_1)$

where $\zeta=e^{2\pi i/5}$, $\eta=\frac{1}{3}(1+i2\sqrt{2})$, $\omega=-1/2+i\sqrt{3}/2$. Unlike the case in $\mathrm{SL}(2,\mathbb{Z})$, the residual symmetry G_0 in $\mathrm{Sp}(4,\mathbb{Z})$ are all non-Abelian!

Linearized G_0 symmetry in $\mathrm{Sp}(4, \mathbb{Z})$

- Field redefinition (A generalization from the case of $g = 1$): (Ding, Feruglio, and Liu 2024)

$$u := e^{-i\alpha}(\tau - \tau_0)(\tau - \bar{\tau}_0)^{-1}$$
$$\Phi_I := [\det(1 - e^{i\alpha}u)]^{k_I}\varphi_I\,,$$

where α is a constant.

- The new fields transform linearly under their respective stabilizer G_0.

E.g.: For fixed point #5: $u := \begin{pmatrix} u_1 & u_3 \\ u_3 & u_1 \end{pmatrix} = (\tau - \tau_0)(\tau - \bar{\tau}_0)^{-1}$.

Under the generators a, b of $G_0 = D_4$:

$$u_1 \xrightarrow{a} -u_1\,, \qquad u_1 \xrightarrow{b} +u_1\,, \qquad u_1 \sim \mathbf{1}'_-$$
$$u_3 \xrightarrow{a} +u_3\,, \qquad u_3 \xrightarrow{b} +u_3\,, \qquad u_3 \sim \mathbf{1}_+\,.$$

- The matter fields also transform linearly under G_0 : $\Phi_I \xrightarrow{\gamma_0} \Omega_I(\gamma_0)\,\Phi_I$

Triplet decomposition under G_0

- There are 168 irreducible triplets of $N(H)$
- Triplet decomposition ($\Omega(\gamma_0)$) under respective G_0:

Fixed points	#2	#3	#4	#5
Linearized G_0	D_4	$D_4 \circ Z_4$	$D_4 \times Z_3$	D_4
Triplet decomposition Ω	$\begin{cases} \mathbf{1}_- \oplus \mathbf{1}'_+ \oplus \mathbf{1}'_+ \\ \mathbf{1}_+ \oplus \mathbf{1}'_- \oplus \mathbf{1}'_- \\ \mathbf{1}'_+ \oplus \mathbf{1}_- \oplus \mathbf{1}_- \\ \mathbf{1}'_- \oplus \mathbf{1}_+ \oplus \mathbf{1}_+ \end{cases}$	$\begin{cases} \mathbf{1}_{+++} \oplus \mathbf{1}_{-+-} \oplus \mathbf{1}_{-+-} \\ \mathbf{1}_{-+-} \oplus \mathbf{1}_{+++} \oplus \mathbf{1}_{+++} \\ \mathbf{1}_{+--} \oplus \mathbf{1}_{--+} \oplus \mathbf{1}_{--+} \\ \mathbf{1}_{--+} \oplus \mathbf{1}_{+--} \oplus \mathbf{1}_{+--} \\ \mathbf{1}_{+-+} \oplus \mathbf{1}_{---} \oplus \mathbf{1}_{---} \\ \mathbf{1}_{---} \oplus \mathbf{1}_{+-+} \oplus \mathbf{1}_{+-+} \\ \mathbf{1}_{-++} \oplus \mathbf{1}_{++-} \oplus \mathbf{1}_{++-} \\ \mathbf{1}_{++-} \oplus \mathbf{1}_{-++} \oplus \mathbf{1}_{-++} \end{cases}$	$\begin{cases} \mathbf{1}_{++0} \oplus \mathbf{1}_{++1} \oplus \mathbf{1}_{++2} \\ \mathbf{1}_{+-0} \oplus \mathbf{1}_{+-1} \oplus \mathbf{1}_{+-2} \\ \mathbf{1}_{-+0} \oplus \mathbf{1}_{-+1} \oplus \mathbf{1}_{-+2} \\ \mathbf{1}_{--0} \oplus \mathbf{1}_{--1} \oplus \mathbf{1}_{--2} \end{cases}$	$\begin{cases} \mathbf{1}_- \oplus \mathbf{1}'_+ \oplus \mathbf{1}'_+ \\ \mathbf{1}_+ \oplus \mathbf{1}'_- \oplus \mathbf{1}'_- \\ \mathbf{1}'_+ \oplus \mathbf{1}_- \oplus \mathbf{1}_- \\ \mathbf{1}'_- \oplus \mathbf{1}_+ \oplus \mathbf{1}_+ \end{cases}$

- We do not consider the other reducible representations such as "$2+1$".
- We consider the local $\mathcal{N}=1$ SUSY, then the superpotential is allowed to be modular covariant: $\mathcal{W} \sim$ singlets $\mathbf{r}_s$.

Viable patterns

- Near each critical point, the respective mass matrix patterns depends only on Ω and $\mathbf{r}_s$. (Not sensitive to Ω)
- In the vicinity of critical points # 2, # 3, and # 5, there is a common viable lepton mass matrix pattern:

$$M_{\bar{e}e}(u,\bar{u}) \approx \begin{pmatrix} y_{11}^0 & y_{12}x & y_{13}x \\ y_{12}^{*}x & y_{22}^0 & y_{23}^0 \\ y_{13}^{*}x & y_{23}^0 & y_{33}^0 \end{pmatrix}, \quad M_\nu^{-1}(u,\bar{u}) \approx \begin{pmatrix} x_{11}x & x_{12}^0 & x_{13}^0 \\ x_{12}^0 & x_{22}x & x_{23}x \\ x_{13}^0 & x_{23}x & x_{33}x \end{pmatrix}$$

where $x = u_3$ for #2, and $x = u_1$ for #3 & #5

- The predictions for neutrino oscillation parameters (where $\varepsilon_i = |u_i| \ll 1$)

τ	$\mathbf{r}_s$	$M_\nu(0,0)$	mass ordering	$\frac{\Delta m^2_{\text{sol}}}{\Delta m^2_{\text{atm}}}$	$\sin^2\theta_{12}$	$\sin^2\theta_{13}$	$\sin^2\theta_{23}$
$\approx \begin{pmatrix} \eta & \frac{1}{2}(\eta-1) \\ \frac{1}{2}(\eta-1) & \eta \end{pmatrix}$	$\mathbf{1}'_{-}$	singular	NO	$\mathcal{O}(\varepsilon_3^3)$	$\frac{1}{2}(1+\mathcal{O}(\varepsilon_3))$	$\mathcal{O}(\varepsilon_3^2)$	$\mathcal{O}(1)$
$\approx \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$	$\mathbf{1}_{-+-}$ $\mathbf{1}_{-++}$	singular	NO	$\mathcal{O}(\varepsilon_1^3)$	$\frac{1}{2}(1+\mathcal{O}(\varepsilon_1))$	$\mathcal{O}(\varepsilon_1^2)$	$\mathcal{O}(1)$
$\approx \frac{i\sqrt{3}}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$	$\mathbf{1}'_{-}$	singular	NO	$\mathcal{O}(\varepsilon_1^3)$	$\frac{1}{2}(1+\mathcal{O}(\varepsilon_1))$	$\mathcal{O}(\varepsilon_1^2)$	$\mathcal{O}(1)$

- This viable pattern is the same as the one found in $\mathrm{SL}(2,\mathbb{Z})$.

Outline

1. Flavor puzzle and flavor symmetry
2. Modular flavor symmetry
3. Near-Critical Behavior in $SL(2, \mathbb{Z})$ modular models
4. Near-Critical behavior in $Sp(4, \mathbb{Z})$ modular models
5. **Summary and Outlook**

Summary

- Modular flavor symmetry provides a very attractive solution for flavor puzzle.
- Universal near-critical behavior exist for $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{Sp}(4,\mathbb{Z})$ modular invariant models.
  - Those linearized symmetries determine the behavior of modular models near the corresponding critical points.
  - The near-critical behavior appears as power law for the flavor observables in terms of linearized modulus (i.e., "order parameter").
  - There are only a few different near-critical patterns in a large number of models (universality).
- Near-critical behavior provides a good description of flavor parameters in SM.
- For lepton mass models, we identify a common viable mass matrix pattern in $\mathrm{SL}(2,\mathbb{Z})$ and $\mathrm{Sp}(4,\mathbb{Z})$ modular-invariant models.

Outlook

- The classification of near-critical behavior can be extended to the case of $i\infty$ and also applies to the quark sector (Working in progress...)
- What is the origin of near-critical ? (moduli stabilization)
- Beyond near-criticality: Observables as holomorphic modular invariants (Chen et al. 2024b) (See talk by Xueqi)
- Beyond $\mathrm{SL}(2,\mathbb{Z})$ (even $\mathrm{Sp}(2g,\mathbb{Z})$) modular symmetry. (Working in progress...)
- Get more insight from string theory. (See talk by Saul)
- Is there a "flavor moonshine" ?
- ...

Flavor Moonshine ?

- Monster moonshine: Modular functions ⇔ Monster group $\mathbb{M}$.
  $j(\tau) = (1)\, q^{-1} + 744 + (1 + 196883)\, q + (1 + 196883 + 21296876)\, q^2 + \ldots$
- Flavor parameters are directly related to modular forms. So is there Flavor Moonshine ? (Monster behind Flavor?)
- Three types of "miracles" for mass hierarchy:

Type	Rank$[M_\psi(\varepsilon = 0)]$	Mass hierarchy	Possible asymptotic regions
I	1	$\mathbf{c_1}\varepsilon^m : \mathbf{c_2}\varepsilon^n : \mathbf{c_3}$	$\tau \approx i\infty$
II	2	$\mathbf{c_1}\varepsilon^n : \mathbf{c_2} : \mathbf{c_3}$	$\tau \approx i\infty,\ i$
III	3	$\mathbf{c_1} : \mathbf{c_2} : \mathbf{c_3}$	$\tau \approx i\infty,\ i,\ \omega$

  where $\mathbf{c_i}$ come from (ϵ-expansion coefficients of) modular forms and (C-G coefficients of) finite modular group !
- A dream: Realistic modular flavor model with One-Parameter (τ) ! (Working in progress...)
- Golden Mass Relation or Koide formula from Flavor moonshine? (a hint in talk by Omar)

Flavor observables (RG-invariants) as some modular functions ! (Chen et al. 2024b)

- Feruglio's A_4 model (also see talk by Xueqi)

$$M_e = u\, v_d\, \text{diag}(\alpha, \beta, \gamma)\ ,$$

$$M_\nu = (-i\tau + i\bar{\tau}) \begin{pmatrix} 2Y_1(\tau) & -Y_2(\tau) & -Y_3(\tau) \\ -Y_2(\tau) & 2Y_3(\tau) & -Y_1(\tau) \\ -Y_3(\tau) & -Y_1(\tau) & 2Y_2(\tau) \end{pmatrix} \frac{v_u^2}{\Lambda}$$

$$= U^*_{\text{PMNS}} \text{diag}(m_1, m_2, m_3) U^\dagger_{\text{PMNS}}$$

- The RG invariants $I_{ij}(\tau) := \frac{M_{ii} M_{jj}}{M_{ij}^2}$ are

$$I_{12}(\tau) = -2$$

$$\left.\begin{aligned} I_{13}(\tau) &= -2\left(1 + \tfrac{1}{3} j_3(\tau)\right)^3 \\ I_{23}(\tau) &= 16\left(1 + \tfrac{1}{3} j_3(\tau)\right)^{-3} \end{aligned}\right\} \Rightarrow I_{13} I_{23} = -32$$

where $j_3(\tau) = \eta(\tau/3)^3 \eta(3\tau)^{-3}$ is the Hauptmodul of $\Gamma(3)$.

☺ $I_{23}(\tau) = 432q - 6480q^2 + 73872q^3 - 725328q^4 + 6503328q^5 + \ldots$

Thank you for your attention!

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Backup

"There are five elementary arithmetical operations: addition, subtraction, multiplication, division, and . . . modular forms."———Martin Eichler

Modular forms appear almost everywhere:

- As tools to solve problems in number theory.
- Play a key role in proving Fermat's last theorem.
- As a bridge connecting various math structures: Langlands Program
- Monster moonshine $j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + ...$
- Sphere-packing problem in dimension 8 and 24.
- Everywhere in string theory and mathematical physics
- Feynman integral
- **Flavor physics**
- ...

(Ding, Liu, and Yao 2023)

- Lepton mass model (neutrinos mass originate from the Type-I seesaw)
- The matter fields content and their modular charges:

Fields	$E^c_{1,2,3}$	$L_{1,2,3}$	$N^c_{1,2,3}$	$H_{u,d}$
$\mathrm{SU}(2)_L \times \mathrm{U}(1)_\mathrm{Y}$	$(1,1)$	$(2,-1/2)$	$(1,0)$	$(2,\pm 1/2)$
$\Gamma'_4 \cong S'_4$	$\hat{\mathbf{2}} \oplus \hat{\mathbf{1}}'$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}$
$-k_I$	4	-1	1	0

- Modular invariant superpotential:

$$\mathcal{W}_e = \alpha \left(Y^{(3)}_{\hat{\mathbf{3}}'} E^c_D L\right)_{\mathbf{1}} H_d + \beta \left(Y^{(3)}_{\hat{\mathbf{3}}} E^c_D L\right)_{\mathbf{1}} H_d + \gamma \left(Y^{(3)}_{\hat{\mathbf{3}}} E^c_3 L\right)_{\mathbf{1}} H_d$$

$$\mathcal{W}_\nu = g \left(N^c L\right)_{\mathbf{1}} H_u + \Lambda \left(Y^{(2)}_{\mathbf{2}} \left(N^c N^c\right)_{\mathbf{2},s}\right)_{\mathbf{1}}$$

- Charged lepton and neutrino mass matrices:

$$M_e = \begin{pmatrix} 2\alpha Y^{(3)}_{\hat{\mathbf{3}}',1} & -\alpha Y^{(3)}_{\hat{\mathbf{3}}',3} + \sqrt{3}\beta Y^{(3)}_{\hat{\mathbf{3}},2} & -\alpha Y^{(3)}_{\hat{\mathbf{3}}',2} + \sqrt{3}\beta Y^{(3)}_{\hat{\mathbf{3}},3} \\ -2\beta Y^{(3)}_{\hat{\mathbf{3}},1} & \sqrt{3}\alpha Y^{(3)}_{\hat{\mathbf{3}}',2} + \beta Y^{(3)}_{\hat{\mathbf{3}},3} & \sqrt{3}\alpha Y^{(3)}_{\hat{\mathbf{3}}',3} + \beta Y^{(3)}_{\hat{\mathbf{3}},2} \\ \gamma Y^{(3)}_{\hat{\mathbf{3}},1} & \gamma Y^{(3)}_{\hat{\mathbf{3}},3} & \gamma Y^{(3)}_{\hat{\mathbf{3}},2} \end{pmatrix} v_d$$

$$M_D = g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_u, \quad M_N = \Lambda \begin{pmatrix} 2Y^{(2)}_{\mathbf{2},1} & 0 & 0 \\ 0 & \sqrt{3}Y^{(2)}_{\mathbf{2},2} & -Y^{(2)}_{\mathbf{2},1} \\ 0 & -Y^{(2)}_{\mathbf{2},1} & \sqrt{3}Y^{(2)}_{\mathbf{2},2} \end{pmatrix}.$$

A minimal lepton mass model—predictions

- Light neutrino masses ($M_\nu = -M_D^T M_N^{-1} M_D$):

$$m_1 = \frac{1}{\left|2Y_{2,1}^{(2)}\right|}\frac{g^2 v_u^2}{\Lambda}, \quad m_2 = \frac{1}{\left|Y_{2,1}^{(2)} - \sqrt{3}Y_{2,2}^{(2)}\right|}\frac{g^2 v_u^2}{\Lambda}, \quad m_3 = \frac{1}{\left|Y_{2,1}^{(2)} + \sqrt{3}Y_{2,2}^{(2)}\right|}\frac{g^2 v_u^2}{\Lambda}.$$

- The product of charged lepton masses:

$$m_e m_\mu m_\tau = \det\left[M_e(\tau)\right] = -96\sqrt{6}v_d^3\gamma\left(\beta^2 - 3\alpha^2\right)\eta^{18}(\tau)$$

- The numerical best-fit values of input parameters:

$$\langle\tau\rangle = -0.193773 + 1.08321i, \quad \beta/\alpha = 1.73048\,(\approx\sqrt{3}), \quad \gamma/\alpha = 0.27031$$
$$\alpha v_d = 244.621\,\text{MeV}, \quad g^2 v_u^2/\Lambda = 29.0744\,\text{meV}$$

- The lepton masses and flavor mixing parameters are predicted:

$$\sin^2\theta_{12} = 0.328920, \quad \sin^2\theta_{13} = 0.0218499, \quad \sin^2\theta_{23} = 0.506956, \quad \delta_{CP} = 1.34256\pi$$
$$\alpha_{21} = 1.32868\pi, \quad \alpha_{31} = 0.544383\pi, \quad m_e/m_\mu = 0.00472633, \quad m_\mu/m_\tau = 0.0587566$$
$$m_1 = 14.4007\text{meV}, \quad m_2 = 16.7803\text{meV}, \quad m_3 = 51.7755\text{meV}$$
$$m_\beta = 16.8907\text{meV}, \quad m_{\beta\beta} = 9.25333\text{meV}$$

Irreducible representations of $\mathrm{SL}(2,\mathbb{Z})$

The 1-d, 2-d and 3-d irreps (with finite image) of $\mathrm{SL}(2,\mathbb{Z})$:

1. $\mathrm{SL}(2,\mathbb{Z})$ has 12 one-dimensional irreps

$$\mathbf{1}_p: \qquad \rho_{\mathbf{1}_p}(S) = i^p\,, \qquad \rho_{\mathbf{1}_p}(T) = e^{\frac{i\pi}{6}p}\,,$$

with $p = 0, \ldots, 11$

2. $\mathrm{SL}(2,\mathbb{Z})$ has 54 two-dimensional irreps with finite image, determined by $\rho(T) = \mathtt{diag}(e^{2\pi i r_1}, e^{2\pi i r_2})$ (Mason 2008)
e.g.: $(r_1, r_2) = (3/4, 1/4), (5/6, 1/3), (0, 1/2)\ldots$

3. $\mathrm{SL}(2,\mathbb{Z})$ has > 156 three-dimensional irreps with finite image, also determined by $\rho(T) = \mathtt{diag}(e^{2\pi i r_1}, e^{2\pi i r_2}, e^{2\pi i r_3})$
e.g.: $(r_1, r_2, r_3) = (0, 1/3, 2/3), (0, 1/4, 3/4), (1/7, 2/7, 4/7)\ldots$

The assignments of three generations of matter fields (e.g., e^c, μ^c, τ^c) have three structures: $\mathbf{3}$ or $\mathbf{2}\oplus\mathbf{1}$ or $\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$.

General finite modular groups

The list of finite modular groups Γ/Nor with order < 78:

Normal subgroups			Finite modular groups Γ/Nor	
Index	Label	Additional relators	Group structure	GAP Id
6	$\Gamma(2) \equiv N_{[6,1]}$	T^2	S_3	$[6,1]$
12	$N_{[12,1]}$	S^2T^2	$Z_3 \rtimes Z_4 \cong 2D_3$	$[12,1]$
	$\pm\Gamma(3) \equiv N_{[12,3]}$	S^2, T^3	A_4	$[12,3]$
18	$N_{[18,3]}$	$ST^{-2}ST^2$	$S_3 \times Z_3$	$[18,3]$
24	$\Gamma(3) \equiv N_{[24,3]I}$	T^3	T'	$[24,3]$
	$N_{[24,3]II}$	S^2T^3		
	$\pm\Gamma(4) \equiv N_{[24,12]}$	S^2, T^4	S_4	$[24,12]$
	$N_{[24,13]}$	$S^2, (ST^{-1}ST)^2$	$A_4 \times Z_2$	$[24,13]$
36	$N_{[36,6]}$	$S^3T^{-2}ST^2$	$(Z_3 \rtimes Z_4) \times Z_3$	$[36,6]$
42	$N_{[42,1]I}$	$T^6, (ST^{-1}S)^2TST^{-1}ST^2$	$Z_7 \rtimes Z_6$	$[42,1]$
	$N_{[42,1]II}$	$T^6, ST^{-1}ST(ST^{-1}S)^2T^2$		
48	$N_{[48,28]}$	S^2T^4	$2O$	$[48,28]$
	$N_{[48,29]}$	T^8, ST^4ST^{-4}	$GL(2,3)$	$[48,29]$
	$\Gamma(4) \equiv N_{[48,30]}$	T^4	$A_4 \rtimes Z_4 \cong S_4'$	$[48,30]$
	$N_{[48,31]}$	$(ST^{-1}ST)^2$	$A_4 \times Z_4$	$[48,31]$
	$N_{[48,32]}$	$S^2(ST^{-1}ST)^2$	$T' \times Z_2$	$[48,32]$
	$N_{[48,33]}$	T^{12}, ST^3ST^{-3}	$((Z_4 \times Z_2) \rtimes Z_2) \rtimes Z_3$	$[48,33]$
54	$N_{[54,5]}$	$T^6, (ST^{-1}ST)^3$	$(Z_3 \times Z_3) \rtimes Z_6$	$[54,5]$
60	$\pm\Gamma(5) \equiv N_{[60,5]}$	S^2, T^5	A_5	$[60,5]$
72	$N_{[72,42]}$	T^{12}, ST^4ST^{-4}	$S_4 \times Z_3$	$[72,42]$
	$\pm\Gamma(6) \equiv N_{[72,44]}$	$S^2, T^6, (ST^{-1}STST^{-1}S)^2T^2$	$A_4 \times S_3$	$[72,44]$
...	...	...	...	..

The group presentation: $\Gamma/\mathrm{Nor} = \langle S, T \mid S^4 = (ST)^3 = \text{“relators”} = 1, S^2T = TS^2 \rangle$.

- The general Kähler potential:

$$\mathcal{K}(\tau,\bar\tau) = -\log(-i(\tau-\bar\tau)) + \sum_{\varphi} \varphi^\dagger K_\varphi(\tau,\bar\tau)\varphi$$
$$= -\log\left(2\text{Im}\left(\frac{\tau_0 - u\bar\tau_0}{1-u}\right)\right) + \sum_{\Phi} \Phi^\dagger \hat{K}_\Phi(u,\bar u)\Phi$$

- Move to the canonical basis of Kähler potential (i.e. kinetic terms):

$$\Phi \mapsto \widetilde{\Phi} = Z_\Phi(u,\bar u)\Phi \quad \text{with} \quad \hat{K}_\Phi(u,\bar u) = Z_\Phi(u,\bar u)^{-1\dagger} Z_\Phi(u,\bar u)^{-1}$$

where the matrix Z_Φ transform as $Z_\varphi \xrightarrow{\gamma_0} \Omega_\Phi Z_\phi \Omega_\phi^\dagger$ under G_0.

✓ The mass matrices get the anti-holomorphic contribution:

$$M(u) \mapsto \tilde{M}(u,\bar u) = Z^T_{\Phi^c}(u,\bar u)M(u)Z_\Phi(u,\bar u)$$

✓ The transformation law of new fields $\widetilde{\Phi_I}$ remains unchanged .

- We can expand M around the fixed point $u=0$:

$$M(u,\bar u) = m^{(00)} + m^{(10)}u + m^{(01)}\bar u + m^{(20)}u^2 + m^{(11)}u\bar u + m^{(02)}\bar u^2 + \cdots.$$