Carrollian Amplitudes from Holographic Correlators

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Eurostrings 2024 meets Fundamental Physics UK – Southampton, September 3, 2024

How to formulate flat space holography?

- Correspondence between **gravity in asymptotically flat spacetimes** and a **lower-dimensional field theory** without gravity.
- **Bottom-up** approaches to build candidates for **holographic duals**.
- Two proposals for flat space holography in 4d:

  $\Longrightarrow$ **Celestial holography**: the dual theory is a **2d CFT** living on the **celestial sphere** S^2.

  [de Boer-Solodukhin '03] [He-Mitra-Strominger '15] [Kapec-Mitra-Raclariu-Strominger '16] [Cheung-de la Fuente-Sundrum '16] [Pasterski-Shao-Strominger '17] [Pasterski-Shao '17] [Donnay-Puhm-Strominger '18] [Stieberger-Taylor '18] [Pate-Raclariu-Strominger-Yuan '19] [Adamo-Mason-Sharma '21] ...

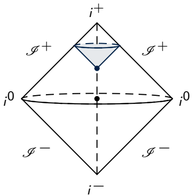
  $\Longrightarrow$ **Carrollian holography**: the dual theory is a **3d Carrollian CFT** living at **null infinity** $\mathscr{I} \simeq \mathbb{R} \times S^2$.

  [Arcioni-Dappiaggi '03] [Dappiaggi-Moretti-Pinamonti '06] [Barnich-Compère '07] [Bagchi '10] [Barnich '12] [Bagchi-Detournay-Fareghbal-Simon '12] [Barnich-Gomberoff-Gonzalez '12] [Bagchi-Basu-Grumiller-Riegler '15] [Ciambelli-Marteau-Petkou-Petropoulos-Siampos '18] [Donnay-Fiorucci-Herfray-Ruzziconi '22] ...
- Take-away messages from this talk:

  $\Longrightarrow$ The two proposals are **related** [Donnay-Fiorucci-Herfray-Ruzziconi '22] [Bagchi-Banerjee-Basu-Dutta '22].

  $\Longrightarrow$ Define **Carrollian amplitudes** to encode the bulk $\mathcal{S}$-matrix at $\mathscr{I}$ [Mason-Ruzziconi-Yelleshpur Srikant '23].

  $\Longrightarrow$ Carrollian amplitudes from the flat limit of **holographic correlators** in AdS [Alday-Nocchi-Ruzziconi-Yelleshpur Srikant '24].
- Ultimate goal: **Flat limit of AdS/CFT**.



Carrollian geometry

- "Carroll" refers to the limit $c \to 0$ where c is the speed of light [Lévy-Leblond '65] (opposite to the usual Galilean limit $c \to \infty$).
- Inönü-Wigner contraction: Poincaré$_d \xrightarrow{c\to 0} \mathfrak{Carr}_d$ and $SO(d,2) \xrightarrow{c\to 0}$ (Global) $\mathfrak{CCarr}_d$.
- Metric degenerates to spatial metric in the limit $c \to 0$:

$$ds^2 = \eta_{ab} dx^a dx^b = -c^2 dt^2 + \delta_{ij} dx^i dx^j \quad \xrightarrow{c\to 0} \quad ds^2 = \delta_{ij} dx^i dx^j$$

- Inverse metric degenerates to temporal bi-vector:

$$-c^2 \eta^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -c^2 \delta^{ij} \end{pmatrix} \quad \xrightarrow{c\to 0} \quad n^a n^b \text{ with } n^a = \delta^a_t$$

- Carrollian geometry: degenerate metric q_{ab} and vector field n^a such that $q_{ab} n^b = 0$. [Henneaux '79] [Duval-Gibbons-Horvathy-Zhang '14]
- Why is it relevant at null infinity?

$\Longrightarrow$ $\mathscr{I}$ being a null hypersurface, the induced metric is degenerate!

$\Longrightarrow$ Conformal Carrollian structure at $\mathscr{I}$ induced by conformal compactification [Penrose '63] [Geroch '77] [Ashtekar '14]

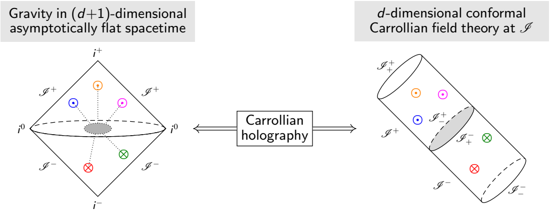
$$q_{ab} \sim \omega^2 q_{ab}, \quad n^a \sim \omega^{-1} n^a$$

$\Longrightarrow$ BMS symmetries = Asymptotic symmetries in flat space [Bondi-van der Burg-Metzner '62] [Sachs '62]
= Conformal symmetries of the Carrollian structure at $\mathscr{I}$ ($\mathfrak{CCarr}_d$).

$$\mathcal{L}_{\bar\xi} q_{ab} = 2\alpha q_{ab} , \qquad \mathcal{L}_{\bar\xi} n^a = -\alpha n^a$$

- Isomorphism: $\boxed{\mathfrak{bms}_{d+1} \simeq \mathfrak{CCarr}_d.}$ [Duval-Gibbons-Horvathy '14] and Poincaré$_{d+1} \simeq$ Global $\mathfrak{CCarr}_d$.

Carrollian holography



- Asymptotic symmetries of the bulk theory = global symmetries in the dual theory.
- The dual theory is a d-dimensional Carrollian CFT (= theory exhibiting conformal Carroll/BMS spacetime symmetries)
  $\Longrightarrow$ Can be constructed by taking $c \to 0$ limit of standard Lorentzian CFTs, see e.g.
  [Schild '77] [Isberg-Lindstrom-Sundborg-Theodoridis '94] [Barnich-Gomberoff-Gonzalez '12] [Duval-Gibbons-Horvathy '14] [Bagchi-Mehra-Nandi '19] ...
- Carrollian holography follows a similar pattern than AdS/CFT correspondence: $(d+1)$-dimensional bulk / d-dimensional boundary duality.
  $\Longrightarrow$ Naturally arises from a flat limit procedure ($\ell \to \infty$).
  $\Longrightarrow$ The flat limit in the bulk induces a Carrollian limit ($c \to 0$) at the boundary.

[Bagchi '10] [Barnich-Gomberoff-Gonzalez '12] [Ciambelli-Marteau-Petkou-Petropoulos-Siampos '18] [Compère-Fiorucci-Ruzziconi '19] [Campoleoni-Delfante-Pekar-Petropoulos-Rivera Betancour '23]

Can we encode the bulk $\mathcal{S}$-matrix into boundary Carrollian CFT correlators?

Bondi coordinates

- Flat Bondi coordinates $\{u, r, z, \bar{z}\}$ $(u, r \in \mathbb{R},\, z \in \mathbb{C})$:

$$X^\mu = u\,\partial_z\partial_{\bar z} q^\mu(z,\bar z) + r\, q^\mu(z,\bar z), \qquad q^\mu(z,\bar z) \equiv \frac{1}{\sqrt{2}}\Big(1+z\bar z,\, z+\bar z,\, -i(z-\bar z),\, 1-z\bar z\Big).$$

- Minkowski metric:

$$ds^2 = -2du dr + 2r^2 dz d\bar z.$$

- Induced Carrollian structure at future/past null infinity $\mathscr{I}^\pm = \{r \to \pm\infty\}$:

$$ds^2_{\mathscr{I}} = q_{ab}dx^a dx^b = 0du^2 + 2dzd\bar z, \qquad n^a\partial_a = \partial_u$$

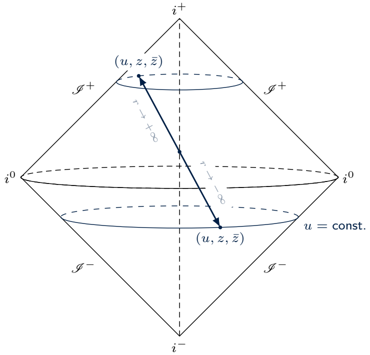
with $x^a = (u, z, \bar z)$ the boundary coordinates. [Penrose '63] [Geroch '77] [Ashtekar '14]

- BMS/conformal Carroll symmetries: [Bondi-van der Burg-Metzner '62] [Sachs '62]

$$\bar\xi^a\partial_a = \left[\mathcal{T} + \frac{u}{2}(\partial\mathcal{Y} + \bar\partial\bar{\mathcal{Y}})\right]\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar\partial$$

with defining property: $\mathcal{L}_{\bar\xi} q_{ab} = 2\alpha q_{ab}$, $\mathcal{L}_{\bar\xi} n^a = -\alpha n^a$, $\alpha = \frac{1}{2}(\partial\mathcal{Y} + \bar\partial\bar{\mathcal{Y}})$.

  1. $\mathcal{T} = \mathcal{T}(z,\bar z)$ is the supertranslation parameter;
  2. $\mathcal{Y} = \mathcal{Y}(z)$, $\bar{\mathcal{Y}} = \bar{\mathcal{Y}}(\bar z)$ are the superrotation parameters satisfying the conformal Killing equation. [Barnich-Troessaert '10]



Carrollian primaries

- Conformal Carrollian primary field $\Phi_{(k,\bar{k})}(u,z,\bar{z})$ [Donnay-Fiorucci-Herfray-Ruzziconi '22] [Nguyen-West '23]:

$$\delta_{\xi}\Phi_{(k,\bar{k})} = \left[\left(\mathcal{T} + \frac{u}{2}(\partial\mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}})\right)\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + k\,\partial\mathcal{Y} + \bar{k}\,\bar{\partial}\bar{\mathcal{Y}}\right]\Phi_{(k,\bar{k})}, \qquad (k,\bar{k})\text{: Carrollian weights.}$$

  (analogue of primary field in CFT)

- Carrollian correlators living at $\mathscr{I}$: $\langle \Phi_{(k_1,\bar{k}_1)}(u_1,z_1,\bar{z}_1)\Phi_{(k_2,\bar{k}_2)}(u_2,z_2,\bar{z}_2)\ldots\rangle$.

- Relation with the bulk?

  $\Longrightarrow$ Strategy: start form a spin-s ($s = 0,1,2,\ldots$) massless field in flat space, $\phi^{(s)}_{\mu_1\ldots\mu_s}(X)$, and push it to $\mathscr{I}$.

- Carrollian operators = boundary value of bulk operators:

  [Arcioni-Dappiaggi '03] [Dappiaggi-Moretti-Pinamonti '05] [Donnay-Fiorucci-Herfray-Ruzziconi '22]

$$\Phi^{\epsilon=+1}_{(k,\bar{k})}(u,z,\bar{z}) = \lim_{r\to+\infty}\left(r^{1-s}\phi^{(s)}_{z\ldots z}(u,r,z,\bar{z})\right) \text{ at } \mathscr{I}^+, \qquad \Phi^{\epsilon=-1}_{(k,\bar{k})}(u,z,\bar{z}) = \lim_{r\to-\infty}\left(r^{1-s}\phi^{(s)\dagger}_{z\ldots z}(u,r,z,\bar{z})\right) \text{ at } \mathscr{I}^-.$$

$\Longrightarrow$ This implies $k = \frac{1+\epsilon J}{2}$, $\bar{k} = \frac{1-\epsilon J}{2}$ with $\epsilon = \pm 1$ for outgoing/incoming and J the helicity.

Carrollian holography identification

- n-point massless scattering amplitude:

$$\mathcal{A}_n\left(\{p_1^\mu\}_{J_1}^{\epsilon_1},\dots,\{p_n^\mu\}_{J_n}^{\epsilon_n}\right) \quad \text{where} \quad p^\mu(\omega,w,\bar{w}) = \epsilon\frac{\omega}{\sqrt{2}}\left(1+w\bar{w}, w+\bar{w}, -i(w-\bar{w}), 1-w\bar{w}\right).$$

- Carrollian correlators = scattering amplitudes in position space at $\mathscr{I}$:

$$\begin{aligned}\langle\Phi^{\epsilon_1}_{(k_1,\bar{k}_1)}(u_1,z_1,\bar{z}_1)\dots\Phi^{\epsilon_n}_{(k_n,\bar{k}_n)}(u_n,z_n,\bar{z}_n)\rangle &= \prod_{i=1}^n\left(\int_0^{+\infty}\frac{d\omega_i}{2\pi}\,e^{i\epsilon_i\omega_i u_i}\right)\mathcal{A}_n\left(\{\omega_1,z_1,\bar{z}_1\}_{J_1}^{\epsilon_1},\dots,\{\omega_n,z_n,\bar{z}_n\}_{J_n}^{\epsilon_n}\right)\\ &= \mathcal{C}_n\left(\{u_1,z_1,\bar{z}_1\}_{J_1}^{\epsilon_1},\dots,\{u_n,z_n,\bar{z}_n\}_{J_n}^{\epsilon_n}\right).\end{aligned}$$

$\Longrightarrow$ Amplitudes in position space at $\mathscr{I}$ = Carrollian amplitudes. [Donnay-Fiorucci-Herfray-Ruzziconi '22] [Mason-Ruzziconi-Yelleshpur Srikant '23]

- Extrapolate dictionary for Carrollian holography:

$$\langle\Phi^{\epsilon_1}_{(k_1,\bar{k}_1)}(u_1,z_1,\bar{z}_1)\dots\Phi^{\epsilon_n}_{(k_n,\bar{k}_n)}(u_n,z_n,\bar{z}_n)\rangle = \lim_{r\to\epsilon\infty}\langle r^{1-s_1}\phi^{(s_1)}(u_1,r_1,z_1,\bar{z}_1)\dots r^{1-s_n}\phi^{(s_n)}(u_n,r_n,z_n,\bar{z}_n)\rangle.$$

Carrollian Ward identities

- Consistent with the (global) conformal Carrollian Ward identities:

$$\sum_{i=0}^{n}\Big[\Big(\mathcal{T}(z_i,\bar z_i)+\frac{u_i}{2}(\partial_{z_i}\mathcal{Y}(z_i)+\partial_{\bar z_i}\bar{\mathcal{Y}}(\bar z_i))\Big)\partial_{u_i}+\mathcal{Y}(z_i)\partial_{z_i}+\bar{\mathcal{Y}}(\bar z_i)\partial_{\bar z_i}$$
$$+k_i\partial_{z_i}\mathcal{Y}(z_i)+\bar k_i\partial_{\bar z_i}\bar{\mathcal{Y}}(\bar z_i)\Big]\langle\Phi_{(k_1,\bar k_1)}(u_1,z_1,\bar z_1)\dots\Phi_{(k_n,\bar k_n)}(u_n,z_n,\bar z_n)\rangle=0$$

where

$$\mathcal{T}(z,\bar z)=1,z,\bar z,z\bar z,\qquad \mathcal{Y}(z)=1,z,z^2,\qquad \bar{\mathcal{Y}}(\bar z)=1,\bar z,\bar z^2$$

$\Longrightarrow$ The low-point correlation functions are completely fixed by the conformal Carrollian symmetries.

- In particular, for the 2-point function [Chen-Liu-Zheng, '21]:

$$\langle\Phi_{(k_1,\bar k_1)}(u_1,z_1,\bar z_1)\Phi_{(k_2,\bar k_2)}(u_2,z_2,\bar z_2)\rangle=\begin{cases}\dfrac{\alpha}{(u_1-u_2)^{k_1+k_2+\bar k_1+\bar k_2-2}}\delta^{(2)}(z_1-z_2)\delta_{k_1+k_2,\bar k_1+\bar k_2} & \text{(Electric branch)}\\[2ex] \dfrac{\beta}{(z_1-z_2)^{k_1+k_2}(\bar z_1-\bar z_2)^{\bar k_1+\bar k_2}}\delta_{k_1,k_2}\delta_{\bar k_1,\bar k_2} & \text{(Magnetic branch)}\end{cases}$$

$\Longrightarrow$ Electric branch relevant for massless scattering.

[Donnay-Fiorucci-Herfray-Ruzziconi '22] [Bagchi-Banerjee-Basu-Dutta '22] [Mason-Ruzziconi-Yelleshpur Srikant '23]

Modified Mellin transform

- If $\Phi_{(k,\bar{k})}(u,z,\bar{z})$ is a Carrollian primary, then $\partial_u^m \Phi_{(k,\bar{k})}(u,z,\bar{z})$ is also a Carrollian primary with shifted weights $(k+\frac{m}{2}, \bar{k}+\frac{m}{2})$.
- Correlators of primary-descendants:

$$\mathcal{C}_n^{m_1\cdots m_n}\left(\{u_1,z_1,\bar{z}_1\}_{J_1}^{\epsilon_1},\dots,\{u_n,z_n,\bar{z}_n\}_{J_n}^{\epsilon_n}\right) = \partial_{u_1}^{m_1}\dots\partial_{u_n}^{m_n}\mathcal{C}_n\left(\{u_1,z_1,\bar{z}_1\}_{J_1}^{\epsilon_1},\dots,\{u_n,z_n,\bar{z}_n\}_{J_n}^{\epsilon_n}\right)$$
$$= \prod_{i=1}^{n}\left(\int_0^{+\infty}\frac{d\omega_i}{2\pi}\,(i\epsilon\omega_i)^{m_i}\,e^{i\epsilon_i\omega_i u_i}\right)\mathcal{A}_n\left(\{\omega_1,z_1,\bar{z}_1\}_{J_1}^{\epsilon_1},\dots,\{\omega_n,z_n,\bar{z}_n\}_{J_n}^{\epsilon_n}\right)$$
$$= \langle\partial_{u_1}^{m_1}\Phi_{(k_1,\bar{k}_1)}^{\epsilon_1}(u_1,z_1,\bar{z}_1)\dots\partial_{u_n}^{m_n}\Phi_{(k_n,\bar{k}_n)}^{\epsilon_n}(u_n,z_n,\bar{z}_n)\rangle$$

$\Longrightarrow$ Related to the Modified Mellin transform used to regularize Mellin transform of graviton amplitudes.

[Banerjee '18] [Banerjee-Ghosh-Pandey-Saha '20]

- Redundant to encode the $\mathcal{S}$-matrix, but can be useful to get rid of IR divergences.

Two-point Carrollian amplitude

- Two-point amplitude (one incoming and one outgoing particle):

$$\mathcal{A}_2(\{\omega_1, z_1, \bar{z}_1\}^-_{J_1}, \{\omega_2, z_2, \bar{z}_2\}^+_{J_2}) = \kappa^2_{J_1,J_2}\, \pi\, \frac{\delta(\omega_{12})}{\omega_1}\, \delta^{(2)}(z_{12})\, \delta_{J_1,J_2},$$

where $\omega_{12} = \omega_1 - \omega_2$ and $z_{12} = z_1 - z_2$.

- Two-point Carrollian amplitude [Liu-Long '22] [Donnay-Fiorucci-Herfray-Ruzziconi '22]:

$$\begin{aligned}\mathcal{C}_2^{m_1,m_2}(\{u_1, z_1, \bar{z}_1\}^-_{J_1}, \{u_2, z_2, \bar{z}_2\}^+_{J_2}) &= \langle \partial^{m_1}_{u_1} \Phi^{\epsilon_1}_{(k_1,\bar{k}_1)}(u_1, z_1, \bar{z}_1)\, \partial^{m_2}_{u_2} \Phi^{\epsilon_n}_{(k_n,\bar{k}_n)}(u_n, z_n, \bar{z}_n)\rangle \\ &= \lim_{\varepsilon \to 0^+} \frac{\kappa^2_{J_1,J_2}}{4\pi} \frac{(-1)^{m_1}\Gamma(m_1+m_2)}{(u_{12}+i\varepsilon)^{m_1+m_2}}\, \delta^{(2)}(z_{12})\, \delta_{J_1,J_2}\end{aligned}$$

$\implies$ Standard solution of the Carrollian Ward identities (electric branch) for operators with fixed Carrollian weights $k + \bar{k} = 1 + m$. ✓

Four-point contact diagram

- Useful example for later: four-point contact diagram $\mathcal{A}_{4,c} = \kappa_4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4)$ for a massless scalar field.
- Use the following parametrization for the momentum conserving δ-function:

$$\delta^{(4)}(p_1+p_2+p_3+p_4) = \frac{1}{4\omega_4 \, |z_{24}\bar{z}_{13}|^2} \delta\left(\omega_1 + z\left|\frac{z_{24}}{z_{12}}\right|^2 \epsilon_1\epsilon_4\omega_4\right) \times \delta\left(\omega_2 - \frac{1-z}{z}\left|\frac{z_{34}}{z_{23}}\right|^2 \epsilon_2\epsilon_4\omega_4\right) \delta\left(\omega_3 + \frac{1}{1-z}\left|\frac{z_{14}}{z_{13}}\right|^2 \epsilon_3\epsilon_4\omega_4\right) \delta(z-\bar{z})$$

Carrollian amplitude:

$$\mathcal{C}^{m_1 m_2 m_3 m_4}_{4,c} = \frac{\kappa_4}{(2\pi)^4} (-1)^{m_1+m_3} \times \boxed{\delta(z-\bar{z})} \times \Theta(-z\epsilon_1\epsilon_4)\,\Theta((1-z)\,z\epsilon_2\epsilon_4)\,\Theta((z-1)\epsilon_3\epsilon_4) \times \frac{|z_{14}|^{2m_3}\,|z_{24}|^{2m_1-2}\,|z_{34}|^{2m_2}}{|z_{12}|^{2m_1}\,|z_{13}|^{2m_3+2}\,|z_{23}|^{2m_2}} \frac{z^{m_1-m_2}(1-z)^{m_2-m_3}\,\Gamma\left(\sum_{i=1}^4 m_i\right)}{\left(u_4 - u_1 z\left|\frac{z_{24}}{z_{12}}\right|^2 + u_2\frac{1-z}{z}\left|\frac{z_{34}}{z_{23}}\right|^2 - u_3\frac{1}{1-z}\left|\frac{z_{14}}{z_{13}}\right|^2\right)^{\sum_{i=1}^4 m_i}},$$

where

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad |z_{ij}|^2 = z_{ij}\bar{z}_{ij}\,.$$

[Mason-Ruzziconi-Yelleshpur Srikant '23] [Alday-Nocchi-Ruzziconi-Yelleshpur Srikant '24]

$\Longrightarrow$ One remaining $\delta(z-\bar{z})$ involving the celestial cross-ratios.

- Similar results for the four-point exchange diagrams, as well as n-point gluon and graviton MHV amplitudes.

The flat limit of AdS

- Holographic correlators are CFT correlators computed via AdS Witten diagrams.
  $\Longrightarrow$ Carrollian amplitudes are natural objects obtained in the flat limit $\ell \to \infty$.
  (Related works: [Pipolo de Gioia-Raclariu '22] [Bagchi-Dhivakar-Dutta '23])
- Bondi coordinates $X = (u, r, z, \bar{z})$ also exist in AdS: $ds^2_{AdS} = -\frac{r^2}{\ell^2} du^2 - 2dudr + 2r^2 dz d\bar{z}$
- Relation to Poincaré coordinates $X = (\rho, x^0, x^1, x^2)$:

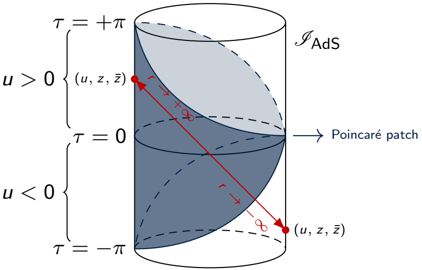
$$ds^2_{AdS} = \frac{\ell^2}{\rho^2}\left(d\rho^2 - (dx^0)^2 + (dx^1)^2 + (dx^2)^2\right) .$$

$$\rho = \frac{\ell}{r} , \quad x^0 = -\frac{\ell}{r} + \frac{u}{\ell} , \quad x^1 = \frac{w + \bar{w}}{\sqrt{2}} , \quad x^2 = \frac{w - \bar{w}}{\sqrt{2}i} .$$

1. Flat limit in the bulk:
   $ds^2_{AdS} = -\frac{r^2}{\ell^2} du^2 - 2dudr + 2r^2 dz d\bar{z} \quad \Longrightarrow \quad ds^2_{Flat} = -2dudr + 2r^2 dz d\bar{z}$
2. Carrollian limit at the boundary:
   $ds^2_{\mathscr{I}_{AdS}} = -\frac{1}{\ell^2} du^2 + 2dz d\bar{z} \quad \Longrightarrow \quad ds^2_{\mathscr{I}} = 0du^2 + 2dz d\bar{z}$
   $\Longrightarrow$ This observation extends to asymptotically AdS solutions of Einstein equations.
   [Barnich-Gomberoff-Gonzalez '12] [Poole-Skenderis-Taylor '18] [Compère-Fiorucci-Ruzziconi '19]

$$\boxed{\text{Flat limit in the bulk } (\ell \to \infty)} \xleftrightarrow{\frac{1}{\ell} \equiv c} \boxed{\text{Carrollian limit at the boundary } (c \to 0)}$$

- Does this correspondence work at the level of the amplitudes?



Flat limit of bulk propagators in Bondi coordinates

- Bulk-to-bulk propagator $\mathcal{G}_{BB}^{AdS,\Delta}(X_1, X_2)$ for a massive scalar field in AdS$_4$ solves

$$(\Box_{X_1} + M^2)\mathcal{G}_{BB}^{AdS,\Delta}(X_1, X_2) = \frac{1}{\sqrt{-g}}\delta^{(4)}(X_{12}), \qquad \Box = \Big[\frac{2}{r^2}\partial_z\partial_{\bar{z}} + \underbrace{\frac{r}{\ell^2}(4\partial_r + r\partial_r^2)}_{\to 0 \text{ if } \ell\to\infty} - \frac{2}{r}\partial_u - 2\partial_u\partial_r\Big], \qquad M^2 = \underbrace{\frac{\Delta(3-\Delta)}{\ell^2}}_{\to 0 \text{ if } \ell\to\infty}$$

- Feynmann propagator $\mathcal{G}_{BB}^{Flat,\Delta}(X_1, X_2)$ of a massless scalar field in flat space solves

$$\Box_{X_1}\mathcal{G}_{BB}^{Flat}(X_1, X_2) = \frac{1}{\sqrt{-g}}\delta^{(4)}(X_{12}), \qquad \Box = \Big[\frac{2}{r^2}\partial_z\partial_{\bar{z}} - \frac{2}{r}\partial_u - 2\partial_u\partial_r\Big]$$

$\Longrightarrow$ The Bondi gauge naturally implements the flat limit: $\boxed{\mathcal{G}_{BB}^{Flat} = \lim_{\ell\to\infty}\mathcal{G}_{BB}^{AdS,\Delta}}$ (Δ disappears)

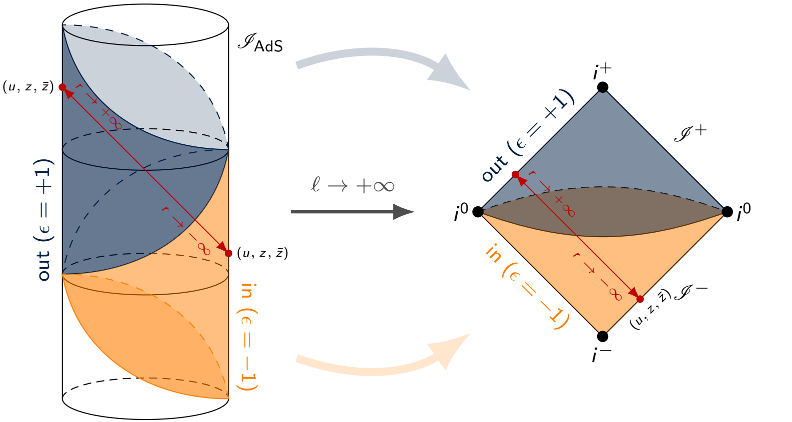
- Bulk-to-boundary propagator in AdS$_4$: $\mathcal{G}_{Bb,+}^{AdS,\Delta}(x_1; X_2) = \ell \lim_{r_1\to+\infty}\left(\frac{r_1}{\ell}\right)^{\Delta}\mathcal{G}_{BB}^{AdS,\Delta}(X_1, X_2)$, $x_1 = (u_1, z_1, \bar{z}_1)$
- Bulk-to-boundary propagator in flat space: $\mathcal{G}_{Bb,+}^{Flat}(x_1; X_2) = \lim_{r_1\to+\infty} r_1\,\mathcal{G}_{BB}^{Flat,\Delta}(X_1, X_2) = \left\langle \Phi^{\epsilon=+1}_{(\frac{1}{2},\frac{1}{2})}(x)\phi(X)\right\rangle$

$\Longrightarrow$ Flat limit of bulk-to-boundary propagator: $\boxed{\mathcal{G}_{Bb,+}^{Flat,m} = \partial_{u_1}^m\mathcal{G}_{Bb,+}^{Flat}(x_1; X_2) = \alpha(\Delta)^{-1}\lim_{\ell\to\infty}\mathcal{G}_{Bb,+}^{AdS,\Delta}}$ $(m = \Delta - 1 = 0, 1, 2, \dots)$

[Alday-Nocchi-Ruzziconi-Yelleshpur Srikant '24]

- Outgoing ($\epsilon = +1$) / incoming ($\epsilon = -1$) bulk-to-boundary propagator obtained by taking $r_1 \to +\infty$ / $r_2 \to -\infty$.

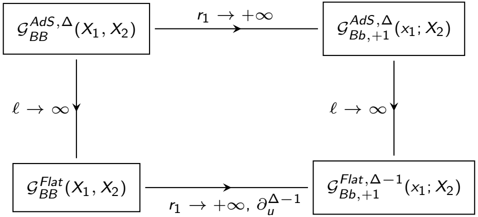
Incoming/outgoing states

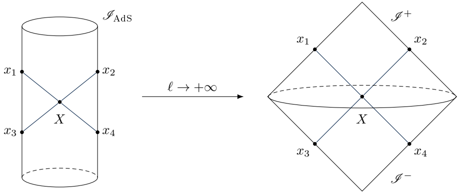


[Alday-Nocchi-Ruzziconi-Yelleshpur Srikant '24]

From AdS Witten to flat space Feynman diagrams

- Summary:





- Example with the four-point contact diagram (scalars):

Boundary: Bulk:

$$\langle \mathcal{O}^{\epsilon_1}_{\Delta_1}(x_1)\mathcal{O}^{\epsilon_2}_{\Delta_2}(x_2)\mathcal{O}^{\epsilon_3}_{\Delta_3}(x_3)\mathcal{O}^{\epsilon_4}_{\Delta_4}(x_4)\rangle = \int_{AdS} d^4X \mathcal{G}^{AdS,\Delta_1}_{Bb,\epsilon_1}(x_1, X)\mathcal{G}^{AdS,\Delta_2}_{Bb,\epsilon_2}(x_2, X)\mathcal{G}^{AdS,\Delta_3}_{Bb,\epsilon_3}(x_3, X)\mathcal{G}^{AdS,\Delta_4}_{Bb,\epsilon_4}(x_4, X)$$

$$\Big\downarrow c \to 0 \quad ? \qquad\qquad \Big\downarrow \ell \to \infty \quad \checkmark$$

$$\langle \partial^{m_1}_{u_1}\Phi^{\epsilon_1}(x_1)\,\partial^{m_2}_{u_2}\Phi^{\epsilon_2}(x_2)\,\partial^{m_3}_{u_3}\Phi^{\epsilon_3}(x_3)\,\partial^{m_4}_{u_4}\Phi^{\epsilon_4}(x_4)\rangle = \int_{Flat} d^4X \mathcal{G}^{Flat,m_1}_{Bb,\epsilon_1}(x_1, X)\mathcal{G}^{Flat,m_2}_{Bb,\epsilon_2}(x_2, X)\mathcal{G}^{Flat,m_3}_{Bb,\epsilon_3}(x_3, X)\mathcal{G}^{Flat,m_4}_{Bb,\epsilon_4}(x_4, X)$$

where $m_i = \Delta_i - 1$ (analogue for exchange diagrams).

$\Longrightarrow$ We recover the Feynman rules for Carrollian amplitudes in the $\ell \to \infty$ limit!

$\Longrightarrow$ Can we reproduce this result by an intrinsic Carrollian limit in the boundary CFT?

Carrollian limit of holographic correlators

- Lorentzian CFT two-point function can be either obtained:
  $\Longrightarrow$ by analytic continuation from Euclidean signature.
  $\Longrightarrow$ directly from the bulk-to-boundary propagator:

$$\langle \mathcal{O}^{+1}_{\Delta}(x_1)\mathcal{O}^{-1}_{\Delta}(x_2)\rangle = \ell \lim_{r_2\to-\infty}\left(\frac{r_2}{\ell}\right)^{\Delta} \mathcal{G}^{AdS,\Delta}_{Bb,+1}(x_1,X_2) = \tilde{C}_2(\Delta)\frac{1}{(-\frac{1}{\ell^2}u_{12}+2|z_{12}|^2-i\varepsilon)^{\Delta}} .$$

- Formal correspondence between the limits: $c_{\text{boundary}} \leftrightarrow \frac{1}{\ell_{\text{bulk}}}$.
- Behaviour of the CFT 2-point function in Lorentzian signature in the Carrollian limit:

$$\langle T\{\mathcal{O}_\Delta(x_1)\,\mathcal{O}_\Delta(x_2)\}\rangle = \frac{\tilde{C}_2(\Delta)}{(-c^2u_{12}^2+2|z_{12}|^2+i\varepsilon)^{\Delta}} \xrightarrow{c\to0} c^0\underbrace{\frac{\tilde{C}_2(\Delta)}{2^\Delta|z_{12}|^{2\Delta}}}_{\text{Magnetic}} + c^{2-2\Delta}\underbrace{\frac{\tilde{C}_2(\Delta)}{2(\Delta-1)}\frac{\delta^{(2)}(z_{12})}{(-u_{12}^2+i\varepsilon)^{\Delta-1}}}_{\text{Electric}} .$$

- $\Delta = 1,2,3,\ldots \Longrightarrow$ Electric branch is leading in the limit $c\to0$ and is found on the support of the δ-function ($z_{12}=0=\bar{z}_{12}$).
- From CFT primary to Carrollian CFT primary: $\alpha(\Delta)\,\partial_u^{\Delta-1}\Phi(x) = \mathcal{O}_\Delta(x)$, $\alpha(\Delta)\sim c^{1-\Delta}$ [Alday-Nocchi-Ruzziconi-Yelleshpur Srikant '24]

$$\lim_{c\to0}\frac{\langle T\{\mathcal{O}_\Delta(x_1)\,\mathcal{O}_\Delta(x_2)\}\rangle}{\alpha(\Delta)^2} = \langle\partial_{u_1}^{\Delta-1}\Phi(x_1)\,\partial_{u_2}^{\Delta-1}\Phi(x_2)\rangle = \mathcal{C}_2^{\Delta-1,\Delta-1} \qquad \checkmark$$

Lorentzian four-point function

- Holographic four-point contact diagram (in Euclidean signature):

$$\langle \mathcal{O}_{\Delta_1}(x_1)\,\mathcal{O}_{\Delta_2}(x_2)\,\mathcal{O}_{\Delta_3}(x_3)\,\mathcal{O}_{\Delta_4}(x_4)\rangle_E^c = \kappa_4 \int_{AdS_4} d^4X \prod_{i=1}^{4} G_{Bb}^{AdS,\Delta_i}(x_i, X)$$

$$\equiv \kappa_4 \mathcal{N}_4 \frac{\left(x_{14}^2\right)^{\frac{1}{2}\Sigma_\Delta - \Delta_1 - \Delta_4}\left(x_{34}^2\right)^{\frac{1}{2}\Sigma_\Delta - \Delta_3 - \Delta_4}}{\left(x_{13}^2\right)^{\frac{1}{2}\Sigma_\Delta - \Delta_4}\left(x_{24}^2\right)^{\Delta_2}} \bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4}(U,V) \;,$$

where

$$\mathcal{N}_4 = \frac{\pi^{\frac{3}{2}}}{2}\Gamma\left(\frac{\Sigma_D - 3}{2}\right)\prod_{i=1}^{4}\frac{\tilde{C}_2(\Delta_i)}{\Gamma(\Delta_i)} \;, \qquad \Sigma_\Delta = \sum_{i=1}^{4}\Delta_i \;, \qquad U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \;, \qquad V = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \;.$$

(see e.g. [Bissi-Sinha-Zhou 22'])

- Example:

$$\bar{D}_{1,1,1,1}(U,V) = \frac{1}{Z - \bar{Z}}\left[2\,\mathrm{Li}_2(Z) - 2\,\mathrm{Li}_2(\bar{Z}) + \log Z\bar{Z}\log\left(\frac{1-Z}{1-\bar{Z}}\right)\right] \;, \qquad U = Z\bar{Z} \;, \qquad V = (1-Z)(1-\bar{Z}) \;.$$

- Analytic continuation from Euclidean to Lorentzian signature: different possibilities [Gary-Giddings-Penedones '09]
- Some analytic continuations lead to singularities in $Z - \bar{Z}$, e.g. $\bar{D}_{1,1,1,1} \to \bar{D}_{1,1,1,1} + \frac{4\pi^2}{Z-\bar{Z}} + \frac{2\pi i}{Z-\bar{Z}}\log\frac{1-\bar{Z}}{1-Z}$

$\Longrightarrow$ **Bulk-point singularity** [Maldacena-Simmons-Duffin-Zhiboedov '17]

- Leading singularity:

$$\bar{D}_{\Delta_1,\Delta_2,\Delta_3,\Delta_4} \xrightarrow{Z\to\bar{Z}} \hat{\Phi}^{ls}_{\Delta_1\Delta_2\Delta_3\Delta_4} = \frac{f_{\Delta_1\Delta_2\Delta_3\Delta_4}(Z)}{(Z-\bar{Z})^{\Sigma_\Delta - 3}} \;, \qquad \left(e.g., \hat{\Phi}^{ls}_{1111} = \frac{4\pi^2}{Z-\bar{Z}}\right) .$$

Carrollian limit of the four-point function

- In Bondi coordinates:

$$(Z-\bar{Z})^2 = \left((1+U-V)^2-4U\right) \xrightarrow{c\to 0} (z-\bar{z})^2 + \mathcal{O}(c^2)\ .$$

[Alday-Nocchi-Ruzziconi-Yelleshpur Srikant '24]

- On the support $z=\bar{z}$, we have $(Z-\bar{Z})^2 \sim c^2 \implies$ The Carrollian limit on the support $z=\bar{z}$ isolates the bulk-point singularity.

$$\lim_{c\to 0} c^{\Sigma_\Delta -4}\hat{\Phi}^{ls}_{\Delta_1\Delta_2\Delta_3\Delta_4} = \mathcal{R}\delta(z-\bar{z})\,,$$

$$\mathcal{R} = \mathcal{K}\left(\frac{|z_{23}|^2}{|z_{34}|^2\,|z_{24}|^2}\right)^{\frac{4-\Sigma_\Delta}{2}} \frac{z^{2-\Delta_1-\Delta_2}(1-z)^{\Delta_1+\Delta_4-2}}{\left(u_4 - z\left|\frac{z_{24}}{z_{12}}\right|^2 u_1 + \frac{1-z}{z}\left|\frac{z_{34}}{z_{23}}\right|^2 u_2 - \frac{1}{1-z}\left|\frac{z_{14}}{z_{13}}\right|^2 u_3\right)^{\Sigma_\Delta -4}}\,.$$

- The choice of analytic continuation fixes the Θ functions (non-trivial Carrollian electric branch on the support of Θ)

$$\lim_{c\to 0}\frac{\langle T\{\mathcal{O}_1(x_1)\,\mathcal{O}_1(x_2)\,\mathcal{O}_1(x_3)\,\mathcal{O}_1(x_4)\}\rangle^c}{\alpha(\Delta_1)\,\alpha(\Delta_2)\,\alpha(\Delta_3)\,\alpha(\Delta_4)} = \langle \partial_{u_1}^{\Delta_1-1}\Phi(x_1)\,\partial_{u_2}^{\Delta_2-1}\Phi(x_2)\,\partial_{u_3}^{\Delta_3-1}\Phi(x_3)\,\partial_{u_4}^{\Delta_4-1}\Phi(x_4)\rangle^c$$

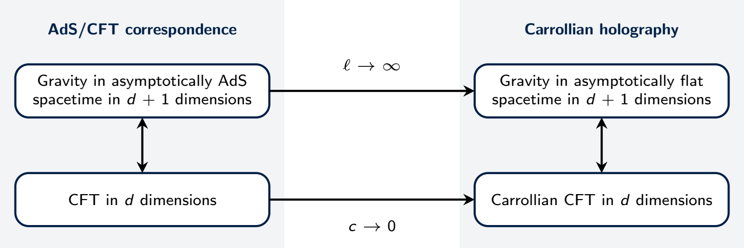
$$= \mathcal{C}_{4,c}^{\Delta_1-1,\Delta_2-1,\Delta_3-1,\Delta_4-1} \qquad \checkmark$$

- Remark: The Carrollian limit of four-point exchange diagrams can be achieved following similar steps as above.

Summary and perspectives

- Carrollian holography is a useful path:
  - $\Longrightarrow$ $\mathcal{S}$-matrix can be encoded in terms of Carrollian CFT correlators ✓
  - $\Longrightarrow$ Naturally related to AdS/CFT via the flat limit / Carrollian limit ✓
  - $\Longrightarrow$ This correspondence relates holographic correlators to Carrollian amplitudes. ✓

- Perspectives:
  - $\Longrightarrow$ Flat/Carrollian limit of AdS/CFT: how far can we go?
  - $\Longrightarrow$ Top-down models for Carrollian holography?



Thank you!

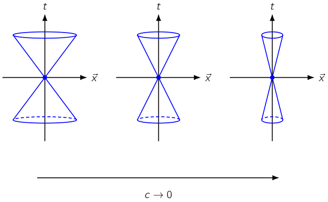
Carrollian algebra

- "Carroll" refers to the limit $c \to 0$ where c is the speed of light [Lévy-Leblond '65].
  $\Longrightarrow$ Opposite to the usual Galilean limit ($c \to \infty$).
- Carrollian limit of the Poincaré algebra:
  $\Longrightarrow$ Translations $H = \partial_t$, $P_i = \partial_i$ and rotations $J_{ij} = x_i \partial_j - x_j \partial_i$ are unchanged.
  $\Longrightarrow$ Boosts are affected: $B_i = c^2 t \partial_i - x_i \partial_t \xrightarrow{c \to 0} B_i = -x_i \partial_t$.
- Carrollian boosts shift time but do not affect space:

$$t' = t - \vec{b}.\vec{x}, \qquad \vec{x}' = \vec{x}$$

  $\Longrightarrow$ Space becomes absolute (see diagram).
  $\Longrightarrow$ Opposite to the usual Galilean limit ($c \to \infty$) where time becomes absolute:

$$t' = t, \qquad \vec{x}' = \vec{x} - \vec{v}t$$



- Carrollian algebra (Ionu-Wigner contraction of Poincaré algebra when $c \to 0$):

$$[B_i, H] = 0, \quad [B_i, B_j] = 0, \quad [B_i, P_j] = \delta_{ij} H, \quad [B_k, J_{ij}] = \delta_{k[i} B_{j]}, \quad [P_k, J_{ij}] = \delta_{k[i} P_{j]}$$

- (Global) conformal Carrollian algebra (Ionu-Wigner contraction of the conformal algebra $SO(d,2)$ when $c \to 0$):
  $\Longrightarrow$ Add the dilatation: $D = (t\partial_t + x^i \partial_i)$, and the Carrollian special conformal generators: $K = x^2 \partial_u$ and $K_i = x^2 \partial_i - 2x_i x^j \partial_j - 2x_i t \partial_t$.

Relation between Carrollian CFT and celestial CFT

- Celestial amplitudes obtained by Mellin transform [de Boer-Solodukhin '03] [Pasterski-Shao-Strominger '17] [Pasterski-Shao '17]:

$$\mathcal{M}_n\left(\{\Delta_1,z_1,\bar{z}_1\}^{\epsilon_1}_{J_1},\dots,\{\Delta_n,z_n,\bar{z}_n\}^{\epsilon_n}_{J_n}\right)=\prod_{i=1}^n\left(\int_0^{+\infty}d\omega_i\,\omega_i^{\Delta_i-1}\right)\mathcal{A}_n\left(\{\omega_1,z_1,\bar{z}_1\}^{\epsilon_1}_{J_1},\dots,\{\omega_n,z_n,\bar{z}_n\}^{\epsilon_n}_{J_n}\right)$$
$$\equiv\langle\mathcal{O}^{\epsilon_1}_{\Delta_1,J_1}(z_1,\bar{z}_1)\dots\mathcal{O}^{\epsilon_n}_{\Delta_n,J_n}(z_n,\bar{z}_n)\rangle$$

- Relation between Carrollian and celestial amplitudes [Donnay-Fiorucci-Herfray-Ruzziconi '22]:

$$\mathcal{M}_n\left(\{\Delta_1,z_1,\bar{z}_1\}^{\epsilon_1}_{J_1},\dots,\{\Delta_n,z_n,\bar{z}_n\}^{\epsilon_n}_{J_n}\right)=\prod_{i=1}^n\left((-i\epsilon_i)^{\Delta_i}\Gamma[\Delta_i]\int_{-\infty}^{+\infty}\frac{du_i}{(u_i-i\epsilon_i\varepsilon)^{\Delta_i}}\right)\mathcal{C}_n\left(\{u_1,z_1,\bar{z}_1\}^{\epsilon_1}_{J_1},\dots,\{u_n,z_n,\bar{z}_n\}^{\epsilon_n}_{J_n}\right)$$

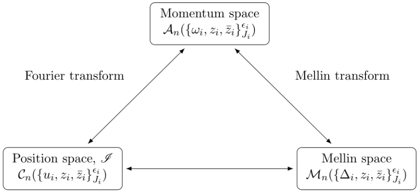
- Relation between Carrollian and celestial operators:

$$\boxed{\mathcal{O}^{\epsilon}_{\Delta_i,J_i}(z_i,\bar{z}_i)=(-i\epsilon_i)^{\Delta_i}\Gamma[\Delta_i]\int_{-\infty}^{+\infty}\frac{du_i}{(u_i-i\epsilon_i\varepsilon)^{\Delta_i}}\,\Phi^{\epsilon}_{(k_i,\bar{k}_i)}(u_i,z_i,\bar{z}_i)}$$

$\Longrightarrow$ Exchange between time and conformal dimension.

$\Longrightarrow$ Three scattering bases $(\omega,\ u,\ \Delta)$ [Donnay-Pasterski-Puhm '22] [Freidel-Pranzetti-Raclariu '22].

$\Longrightarrow$ Extrapolate dictionary in celestial holography [Pasterski-Puhm-Trevisani '21].



Modified Mellin transform

- If $\Phi_{(k,\bar{k})}(u,z,\bar{z})$ is a Carrollian primary, then $\partial_u \Phi_{(k,\bar{k})}(u,z,\bar{z})$ is also a Carrollian primary with shifted weights $(k+\frac{1}{2}, \bar{k}+\frac{1}{2})$.
- Remark: the leading order of the Peeling is $\partial_u^{|J|}\Phi$. For gravity, $\Psi_4^0 = \partial_u^2 C_{zz} \implies$ Transforms as primary under the full $\mathfrak{CCarr}_4$ algebra. [Mason-Ruzziconi-Yelleshpur Srikant '23]
- Descendants:

$$\begin{aligned}
\mathcal{C}_n^{m_1\cdots m_n}\left(\{u_1,z_1,\bar{z}_1\}_{J_1}^{\epsilon_1},\dots,\{u_n,z_n,\bar{z}_n\}_{J_n}^{\epsilon_n}\right) &= \partial_{u_1}^{m_1}\dots\partial_{u_n}^{m_n}\mathcal{C}_n\left(\{u_1,z_1,\bar{z}_1\}_{J_1}^{\epsilon_1},\dots,\{u_n,z_n,\bar{z}_n\}_{J_n}^{\epsilon_n}\right) \\
&= \prod_{i=1}^{n}\left(\int_0^{+\infty}\frac{d\omega_i}{2\pi}\,(i\epsilon\omega_i)^{m_i}e^{i\epsilon_i\omega_i u_i}\right)\mathcal{A}_n\left(\{\omega_1,z_1,\bar{z}_1\}_{J_1}^{\epsilon_1},\dots,\{\omega_n,z_n,\bar{z}_n\}_{J_n}^{\epsilon_n}\right) \\
&= \langle \partial_{u_1}^{m_1}\Phi_{(k_1,\bar{k}_1)}^{\epsilon_1}(u_1,z_1,\bar{z}_1)\dots\partial_{u_n}^{m_n}\Phi_{(k_n,\bar{k}_n)}^{\epsilon_n}(u_n,z_n,\bar{z}_n)\rangle
\end{aligned}$$

- In particular, $\tilde{\mathcal{C}}_n \equiv \mathcal{C}_n^{1\dots1}$.
- Redundant to encode the $\mathcal{S}$-matrix, but can be useful to get rid of IR divergences (see example of the 2-point function).
- Remark: analytic continuation $m_i = \delta_i - 1$ ($\delta_i \in \mathbb{C}$) gives

$$\prod_{i=1}^{n}\left(\int_0^{+\infty}\frac{d\omega_i}{2\pi}\,(i\epsilon\omega_i)^{\delta_i-1}e^{i\epsilon_i\omega_i u_i}\right)\mathcal{A}_n\left(\{\omega_1,z_1,\bar{z}_1\}_{J_1}^{\epsilon_1},\dots,\{\omega_n,z_n,\bar{z}_n\}_{J_n}^{\epsilon_n}\right),$$

$\implies$ Modified Mellin transform used to regularize Mellin transform of graviton amplitudes. [Banerjee '18] [Banerjee-Ghosh-Pandey-Saha '20]

$\implies$ Clarifies the link with the alternative approach to Carrollian amplitudes. [Bagchi-Banerjee-Basu-Dutta '22]

Massless scattering in flat space

- Strategy: start form the bulk operators, and deduce the boundary operators at $\mathscr{I}$. [Ashtekar '81] [Arcioni-Dappiaggi '03] [Strominger '17] [Donnay-Fiorucci-Herfray-Ruzziconi '22]
- Consider a spin-s ($s = 0, 1, 2, \ldots$) massless field in flat space:

$$\phi_I^{(s)}(X) = \frac{K_{(s)}}{16\pi^3} \sum_{\alpha=\pm} \int \omega \, d\omega \, d^2 w \left[a_\alpha^{(s)}(\omega, w, \bar{w}) \varepsilon_I^{*\alpha}(w, \bar{w}) \, e^{i\omega q^\mu X_\mu} + a_\alpha^{(s)}(\omega, w, \bar{w})^\dagger \, \varepsilon_I^{\alpha}(w, \bar{w}) \, e^{-i\omega q^\mu X_\mu} \right]$$

with $I = (\mu_1 \mu_2 \ldots \mu_s)$ and

$$p^\mu(\omega, w, \bar{w}) = \omega \, q^\mu(w, \bar{w}), \quad q^\mu(w, \bar{w}) = \frac{1}{\sqrt{2}} \Big(1 + w\bar{w}, w + \bar{w}, -i(w - \bar{w}), 1 - w\bar{w} \Big),$$

$$\varepsilon^{\pm}_{\mu_1 \ldots \mu_s}(\vec{q}) = \varepsilon^{\pm}_{\mu_1}(\vec{q}) \varepsilon^{\pm}_{\mu_2}(\vec{q}) \ldots \varepsilon^{\pm}_{\mu_s}(\vec{q}), \qquad \varepsilon^{+}_{\mu}(\vec{q}) = \partial_w q_\mu = \frac{1}{\sqrt{2}} \left(-\bar{w}, 1, -i, -\bar{w} \right), \qquad \varepsilon^{-}_{\mu}(\vec{q}) = [\varepsilon^{+}_{\mu}(\vec{q})]^* .$$

- Taking $r \to \infty$ (stationary phase approximation), we find the boundary values:

$$\bar{\phi}^{(s)}_{z \ldots z}(u, z, \bar{z})^{\text{out}} = \lim_{r \to +\infty} \left(r^{1-s} \phi^{(s)}_{z \ldots z}(u, r, z, \bar{z}) \right) = - \frac{iK_{(s)}}{8\pi^2} \int_0^{+\infty} d\omega \left[a_+^{(s)\,\text{out}}(\omega, z, \bar{z}) e^{-i\omega u} - a_-^{(s)\,\text{out}}(\omega, z, \bar{z})^\dagger e^{i\omega u} \right] \quad \text{at } \mathscr{I}^+,$$

$$\bar{\phi}^{(s)}_{z \ldots z}(u, z, \bar{z})^{\text{in}} = \lim_{r \to -\infty} \left(r^{1-s} \phi^{(s)}_{z \ldots z}(u, r, z, \bar{z}) \right) = - \frac{iK_{(s)}}{8\pi^2} \int_0^{+\infty} d\omega \left[a_+^{(s)\,\text{in}}(\omega, z, \bar{z}) e^{-i\omega u} - a_-^{(s)\,\text{in}}(\omega, z, \bar{z})^\dagger e^{i\omega u} \right] \quad \text{at } \mathscr{I}^-.$$

$\Longrightarrow$ Insertion operators for a massless scattering between $\mathscr{I}^-$ (in) and $\mathscr{I}^+$ (out).

Collinear limit and Carrollian OPE

- Collinear limit of two outgoing particles ($\epsilon_1 = \epsilon_2 = +1$):

$$\mathcal{A}_n\left(1^{J_1}, 2^{J_2}, 3^{J_3}, \dots, n^{J_n}\right) \xrightarrow{1||2} \sum_J \mathcal{A}_3\left(1^{J_1}, 2^{J_2}, -P^{-J}\right) \frac{1}{\langle 12\rangle[21]} \mathcal{A}_{n-1}\left(P^J, 3^{J_3}, \dots, n^{J_n}\right)$$

where J is the helicity of the exchanged particle.

- In the limit $z_{12} \to 0$, we obtain the Carrollian OPE block [Mason-Ruzziconi-Yelleshpur Srikant '23]:

$$\begin{aligned}
&\Phi_{J_1}(u_1, z_1, \bar{z}_1)\,\Phi_{J_2}(u_2, z_2, \bar{z}_2) \\
&\quad\sim -\frac{\kappa_{J_1,J_2,-J}}{2\pi}\frac{\bar{z}_{12}^p}{z_{12}}\int_0^1 dt\, t^{J_2-J-1}(1-t)^{J_1-J-1}\left(\frac{\partial}{\partial u}\right)^p \Phi_J(u, z_2, \bar{z}_2 + t\bar{z}_{12})|_{u=u_2+tu_{12}} \\
&\quad\sim -\frac{\kappa_{J_1,J_2,-J}}{2\pi z_{12}}\sum_{m,n=0}^{\infty} B(J_2-J+m+n, J_1-J)\frac{\bar{z}_{12}^{p+m}u_{12}^n}{m!n!}\left(\frac{\partial}{\partial \bar{z}_2}\right)^m\left(\frac{\partial}{\partial u_2}\right)^{p+n}\Phi_J(u_2, z_2, \bar{z}_2)
\end{aligned}$$

with implicit sum on $p = J_1 + J_2 - J - 1$ with range determined by

$$p \geq 0, \qquad |J_1 + J_2 - p - 1| \leq 2 \quad \text{and} \quad |J_1| \leq 2, \qquad |J_2| \leq 2.$$

- Invariance under global $\mathfrak{CCarr}_3$ explicitly checked ✓
- Using the Carroll/celestial correspondence, we recover the celestial OPE block

$$\mathcal{O}_{\Delta_1,J_1}(z_1,\bar{z}_1)\,\mathcal{O}_{\Delta_2,J_2}(z_2,\bar{z}_2) \sim -\kappa_{J_1,J_2,-J}\frac{\bar{z}_{12}^p}{z_{12}}\int_0^1 dt\, t^{2\bar{h}_1+p-1}(1-t)^{2\bar{h}_2+p-1}\mathcal{O}_{\Delta_1+\Delta_2+p-1,J}.$$

[Fan-Fotopoulos-Taylor '19] [Pate-Raclariu-Strominger-Yuan '19]

Three-point Carrollian amplitude

- The three-point amplitude generically vanishes in Lorentzian signature $\Longrightarrow$ Go to split $(2,2)$ signature.

$$p_i^\mu = \epsilon_i q_i^\mu = \epsilon_i \omega_i \left(1 + z_i \bar{z}_i, z_i + \bar{z}_i, z_i - \bar{z}_i, 1 - z_i \bar{z}_i\right).$$

  Here $(z_i, \bar{z}_i)$ are coordinates on a Poincaré patch of $\mathcal{LT}_2$ and $\epsilon_i = \pm 1$ labels the Poincaré patches.

- Using spinor-helicity notations, $p_{\alpha\dot{\alpha}} \equiv \sigma^\mu_{\alpha\dot{\alpha}} p_\mu = \kappa_\alpha \tilde{\kappa}_{\dot{\alpha}}$ and $[ij] = \tilde{\kappa}_{i\dot{\alpha}} \tilde{\kappa}_j^{\dot{\alpha}}$, at tree-level, the three-point amplitude reads as

$$\mathcal{A}_3(1^{J_1}, 2^{J_2}, 3^{J_3}) = \kappa_{J_1,J_2,J_3} [12]^{J_1+J_2-J_3} [23]^{J_2+J_3-J_1} [31]^{J_3+J_1-J_2} \delta^{(4)}(p_1 + p_2 + p_3), \text{ if } J_1 + J_2 + J_3 > 0$$

  A similar expression exists for $J_1 + J_2 + J_3 + 2 < 0$.

- Three-point Carrollian amplitude (still determined by Ward identities):

  [Banerjee-Ghosh-Pandey-Saha '20] [Salzer '23] [Mason-Ruzziconi-Yelleshpur Srikant '23]

$$\tilde{\mathcal{C}}_3 = \kappa_{J_1,J_2,J_3} \frac{-i\epsilon_1\epsilon_2\epsilon_3\, \delta(z_{12})\, \delta(z_{23})}{4(2\pi)^3} \Theta\left(-\frac{\bar{z}_{13}}{\bar{z}_{23}}\epsilon_1\epsilon_2\right) \Theta\left(\frac{\bar{z}_{12}}{\bar{z}_{23}}\epsilon_1\epsilon_3\right) |\bar{z}_{12}|^{J_1+J_2} |\bar{z}_{23}|^{J_2+J_3} |\bar{z}_{31}|^{J_3+J_1}$$

$$\times (\text{sign } \bar{z}_{12})^{J_1+J_2-J_3+1} (\text{sign } \bar{z}_{23})^{J_2+J_3-J_1+1} (\text{sign } \bar{z}_{13})^{J_1+J_3-J_2+1} \frac{(i\,\epsilon_1 \text{ sign}(\bar{z}_{23}))^{J_1+J_2+J_3+2}\, \Gamma(J_1 + J_2 + J_3 + 2)}{(\bar{z}_{23} u_1 - \bar{z}_{13} u_2 + \bar{z}_{12} u_3 + i\epsilon_1 \text{sign}(\bar{z}_{23})\varepsilon)^{J_1+J_2+J_3+2}}$$

  for $J_1 + J_2 + J_3 + 2 > 0$ (similarly for $J_1 + J_2 + J_3 + 2 > 0$).

- Using Carroll/celestial correspondence ($\bar{h}_k = \frac{\Delta_k - J_k}{2}$):

$$\mathcal{M}_3 = \frac{(-i)^{J_1+J_2+J_3}\, \pi}{2} \kappa_{J_1,J_2,J_3} \delta(z_{12})\, \delta(z_{23})\, \Theta\left(-\frac{\bar{z}_{13}}{\bar{z}_{23}}\epsilon_1\epsilon_2\right) \Theta\left(\frac{\bar{z}_{12}}{\bar{z}_{23}}\epsilon_1\epsilon_3\right) \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{12}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}$$

$$\times (\text{sign } \bar{z}_{12})^{J_1+J_2-J_3} (\text{sign } \bar{z}_{23})^{J_2+J_3-J_1} (\text{sign } \bar{z}_{13})^{J_1+J_3-J_2} (\epsilon_1)^{\Delta_1} (\epsilon_2)^{\Delta_2} (\epsilon_3)^{\Delta_3} \delta(\Delta_1 + \Delta_2 + \Delta_3 + J_1 + J_2 + J_3 - 4).$$

  [Pasterski-Shao-Strominger '17]

Four-point Carrollian amplitude

- At tree-level the 4-point gluon MHV amplitude is given by

$$\mathcal{A}_4\left(1^{+1},2^{-1},3^{-1},4^{+1}\right)=\kappa^2_{1,1,-1}\frac{\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}=\kappa^2_{1,1,-1}\frac{\omega_2\omega_3}{\omega_1\omega_4}\frac{z_{23}^3}{z_{12}z_{34}z_{41}},$$

- Applying the Fourier transform yields the corresponding Carrollian amplitude (very similar computation for gravitons):
  [Banerjee-Ghosh-Pandey-Saha '20] [Mason-Ruzziconi-Yelleshpur Srikant '23]

$$\tilde{\mathcal{C}}_4\left(1^{+1},2^{-1},3^{-1},4^{+1}\right)=\frac{\kappa^2_{1,1,-1}}{(2\pi)^4}\frac{z_{34}^2\bar{z}_{14}^4\bar{z}_{34}^2}{z^3(1-z)z_{13}^3z_{24}\bar{z}_{13}^5\bar{z}_{24}^3}\delta\left(z-\bar{z}\right)\Theta\left(-z\left|\frac{z_{24}}{z_{12}}\right|^2\epsilon_1\epsilon_4\right)\Theta\left(\frac{1-z}{z}\left|\frac{z_{34}}{z_{23}}\right|^2\epsilon_2\epsilon_4\right)$$
$$\Theta\left(-\frac{1}{1-z}\left|\frac{z_{14}}{z_{13}}\right|^2\epsilon_3\epsilon_4\right)\times\frac{3!}{\left(u_4-u_1z\left|\frac{z_{24}}{z_{12}}\right|^2+u_2\frac{1-z}{z}\left|\frac{z_{34}}{z_{23}}\right|^2-u_3\frac{1}{1-z}\left|\frac{z_{14}}{z_{13}}\right|^2\right)^4},$$

- Using Carroll/celestial correspondence:

$$\mathcal{M}_4=\prod_{i=1}^4\left(\int_{-\infty}^{+\infty}du_i\,(-i\epsilon_i)^{\Delta_i}\Gamma(\Delta_i-1)\,u_i^{1-\Delta_i}\right)\tilde{\mathcal{C}}_4$$
$$=\prod_{i=1}^4(-i\epsilon_i)^{\Delta_i}z^{-\frac{1}{3}}\left(1-z\right)^{\frac{5}{3}}\prod_{i<j}z_{ij}^{\frac{h}{3}-h_i-h_j}\bar{z}_{ij}^{\frac{\bar{h}}{3}-\bar{h}_i-\bar{h}_j}(-1)^{\Delta_2+\Delta_4+1}2\pi\delta\left(\Delta_1+\Delta_2+\Delta_3+\Delta_4-4\right)$$
$$\delta\left(z-\bar{z}\right)\Theta\left(-z\left|\frac{z_{24}}{z_{12}}\right|^2\epsilon_1\epsilon_4\right)\Theta\left(\frac{1-z}{z}\left|\frac{z_{34}}{z_{23}}\right|^2\epsilon_2\epsilon_4\right)\Theta\left(-\frac{1}{1-z}\left|\frac{z_{14}}{z_{13}}\right|^2\epsilon_3\epsilon_4\right).$$

[Pasterski-Shao-Strominger '17]

n-point MHV Carrollian amplitude

- Colour ordered MHV gluon amplitude (with n + 1 identified with 1):

$$\mathcal{A}_n\left(1^-,2^-,3^+,\dots,n^+\right)=\kappa^{n-2}_{1,1,-1}\frac{\langle 12\rangle^4}{\prod_{j=1}^n\langle jj+1\rangle}=\kappa^{n-2}_{1,1,-1}\frac{\omega_1\omega_2}{\prod_{j=3}^n\omega_j}\frac{z_{12}^3}{\prod_{j=2}^n z_{jj+1}}$$

(similar formula for gravitons)

- Use the decomposition of the delta distribution [Schreiber-Volovich-Zlotnikov '17]:

$$\delta^{(4)}\left(\sum_{i=1}^n p_i\right)=\frac{1}{|\mathcal{U}_{1234}|}\prod_{I=1}^4\delta\left(\omega_I-\omega_I^*\right),\quad \text{with}\quad \omega_I^*=-\frac{1}{\mathcal{U}_{1234}}\sum_{i=5}^n\omega_i\mathcal{U}_{Ii}$$

where

$$\mathcal{U}_{1234}=\det\left(q_1^\mu,\dots,q_4^\mu\right),\qquad \mathcal{U}_{Ii}=\mathcal{U}_{1234}|_{I\to i},\quad I=1,2,3,4;\, i=5,\dots,n.$$

- Applying the Fourier transform [Mason-Ruzziconi-Yelleshpur Srikant '23]:

$$\tilde{\mathcal{C}}_n\left(1^-,2^-,3^+,\dots,n^+\right)=\frac{\kappa^{n-2}_{1,1,-1}}{(2\pi)^n\,|\mathcal{U}_{1234}|}\frac{z_{12}^3}{\prod_{j=2}^n z_{jj+1}}\frac{\partial^4}{\partial u_1^2\,\partial u_2^2}I_n$$

where the integral I_n can be computed explicitly:

$$I_n=\int\prod_{j=5}^n d\omega_j e^{i\omega_j L_j}\prod_{I=1}^4\Theta\left(\omega_I^*\right)=(-1)^{n-4}\prod_{j=5}^n\frac{1}{L_j}$$

with $L_j=\left(\epsilon_j u_j-\sum_{J=1}^4\epsilon_J u_J\frac{\mathcal{U}_{Jj}}{\mathcal{U}_{1234}}\right)$

- Non-trivial dynamical constraints on the dual Carrollian CFT.
- Similar expression for the MHV graviton amplitude (same integral I_n).
- Surprisingly simpler than its celestial counterparts involving Aomoto-Gelfand hypergeometric function. [Schreiber-Volovich-Zlotnikov '17]

Celestial symmetries and twistor space

- Action of $Lw_{1+\infty}$ at $\mathscr{I}$? $\implies$ Can be deduced from Carrollian OPEs.
- Definite the (outgoing) soft operators [Guevara-Himwich-Pate-Strominger '21]

$$H_J^k \equiv \lim_{\Delta \to k} (\Delta - k)\Gamma(\Delta - 1)(-i)^\Delta \int_{-\infty}^{\infty} du\, u^{1-\Delta} \partial_u \Phi_J(u, z, \bar{z}) , \qquad k = 1, 0, -1, -2, \ldots$$

- Action of $Lw_{1+\infty}$ on gravitons [Mason-Ruzziconi-Yelleshpur Srikant '23]:

$$H_2^k(z_1, \bar{z}_1)\, \partial_{u_2} \Phi_2(u_2, z_2, \bar{z}_2) \sim -\frac{\kappa_{2,2,-2}}{z_{12}} \sum_{m=0}^{1-k} \frac{\bar{z}_{12}^{m+1}}{m!} \frac{(-iu_2)^{1-k-m}}{(1-k-m)!} \left(-i\partial_{u_2}\right)^{2-m} \partial_{\bar{z}_2}^m \Phi_2(u_2, z_2, \bar{z}_2) .$$

$\implies$ Local for $k \geq -1$ corresponding to universal soft theorems.

- The action of $Lw_{1+\infty}$ on twistor space is local and has a geometric interpretation (Poisson diffeomorphisms). [Adamo-Mason-Sharma '21] [Bu-Heuveline-Skinner '22] [Mason '22]
- Natural to relate Carrollian CFT at $\mathscr{I}$ and twistor space:

$$\Phi_J(u, \lambda, \tilde{\lambda}) = \partial_u^{|J|} \int d^2\mu\, \delta(u - [\mu\tilde{\lambda}]) f(\lambda_\alpha, \mu^{\dot{\alpha}})$$

where we introduced homogeneous coordinates $(u, \lambda_\alpha, \tilde{\lambda}_{\dot{\alpha}}) = \left(\lambda_0 \tilde{\lambda}_{\dot{0}} u_B, \lambda_0 z_\alpha , \tilde{\lambda}_{\dot{0}} \bar{z}_{\dot{\alpha}}\right)$.

$\implies$ Upcoming work: what is the structure preserved by $Lw_{1+\infty}$ at $\mathscr{I}$?