Elliptic integrable models and their spectra from the superconformal indices

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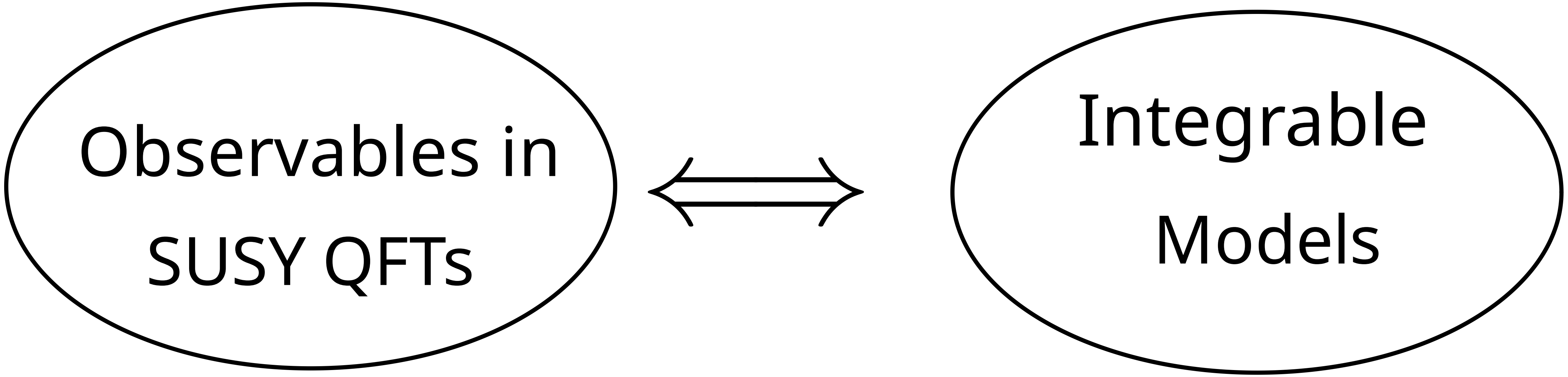
Southampton

based on works with S.Razamat, H.-C. Kim and B.Nazzal

2407.08776, 2305.09718, 2106.08335

General Setting

Main idea is in the relation

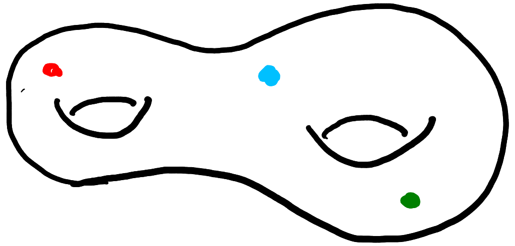


In the spirit of BPS/CFT (Nekrasov-Shatashvili et al.)

Particular Setting

6d SCFT

compactification
on the punctured
Riemann surface



Allows one to write index for any, even non-Lagrangian compactification (index bootstrap)

Superconformal index (SCI) of 4d IR theory

6d (2,0) theory

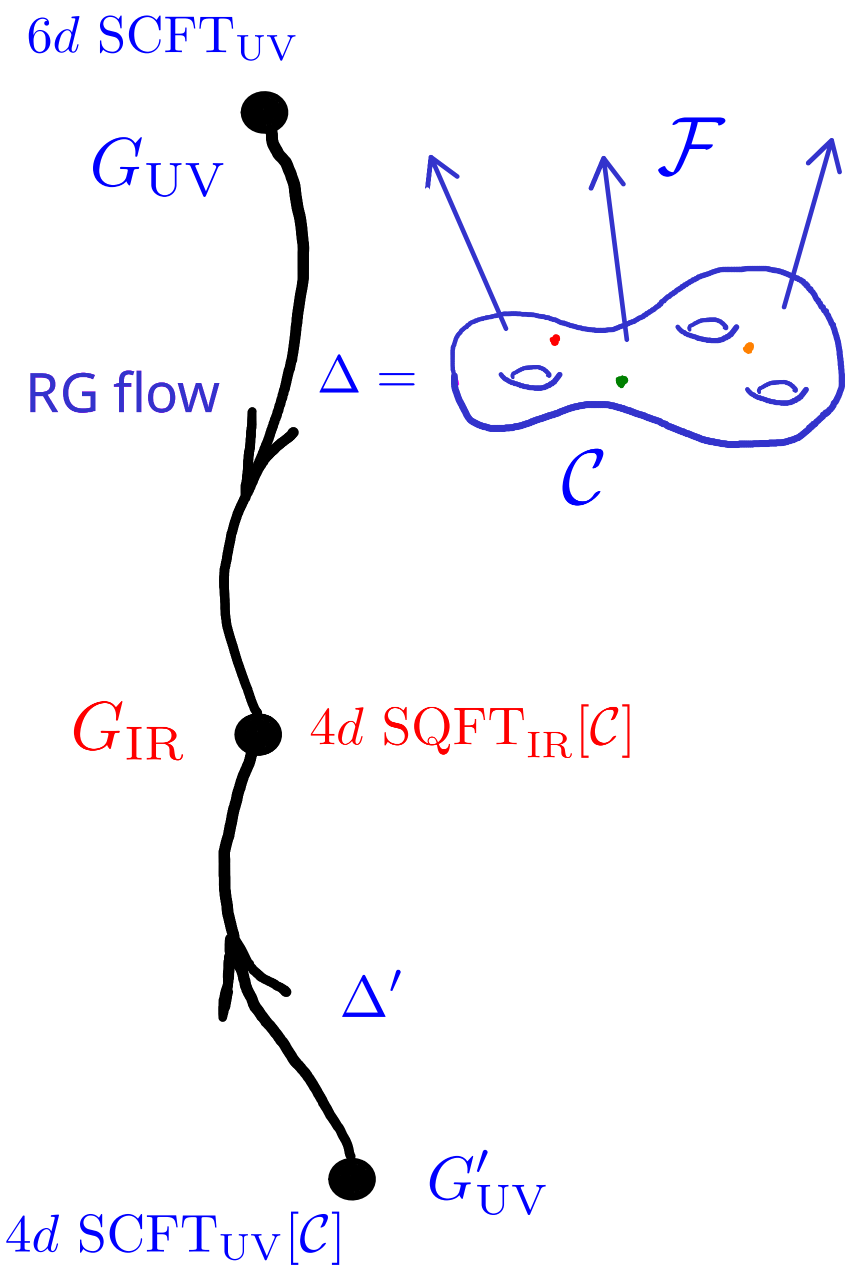
Introduce codim.-2 defect in 4d

Integrable analytic finite-difference operator $A\Delta O$

Ruijsenaars-Schneider model

Gaiotto, Rastelli, Razamat 1207.3577

$4d\ \mathcal{N}=1$ SQFTs from $6d$ SCFTs



- Generalities: talk of Shlomo. Razamat, Sabag, Sela, Zafrir 2203.06880 and many more

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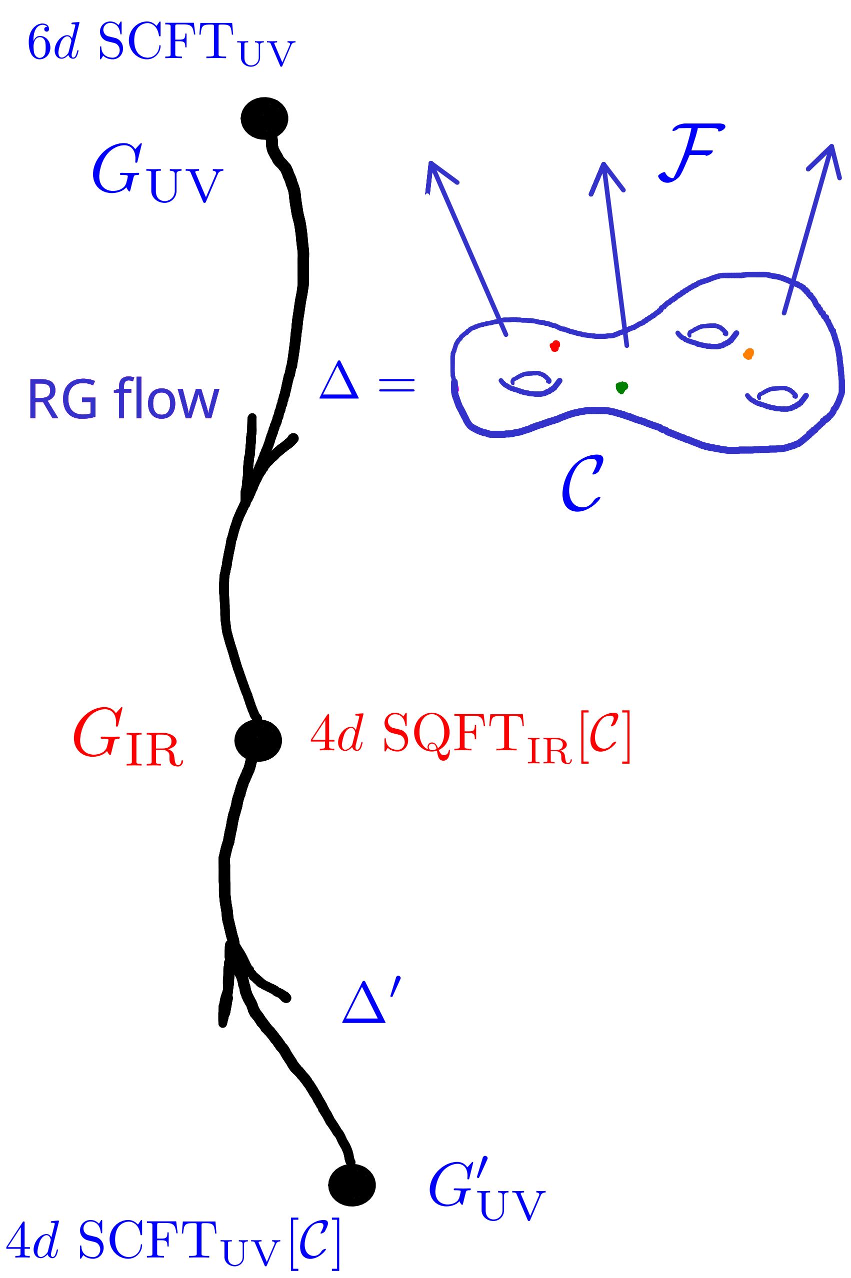

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- Punctures: characterized by the global symmetry G_p and a set of operators M_i charged under $G_{M_i} \subseteq G_p$

  We assume 6d SCFT has 5d lagrangian QFT compactification.

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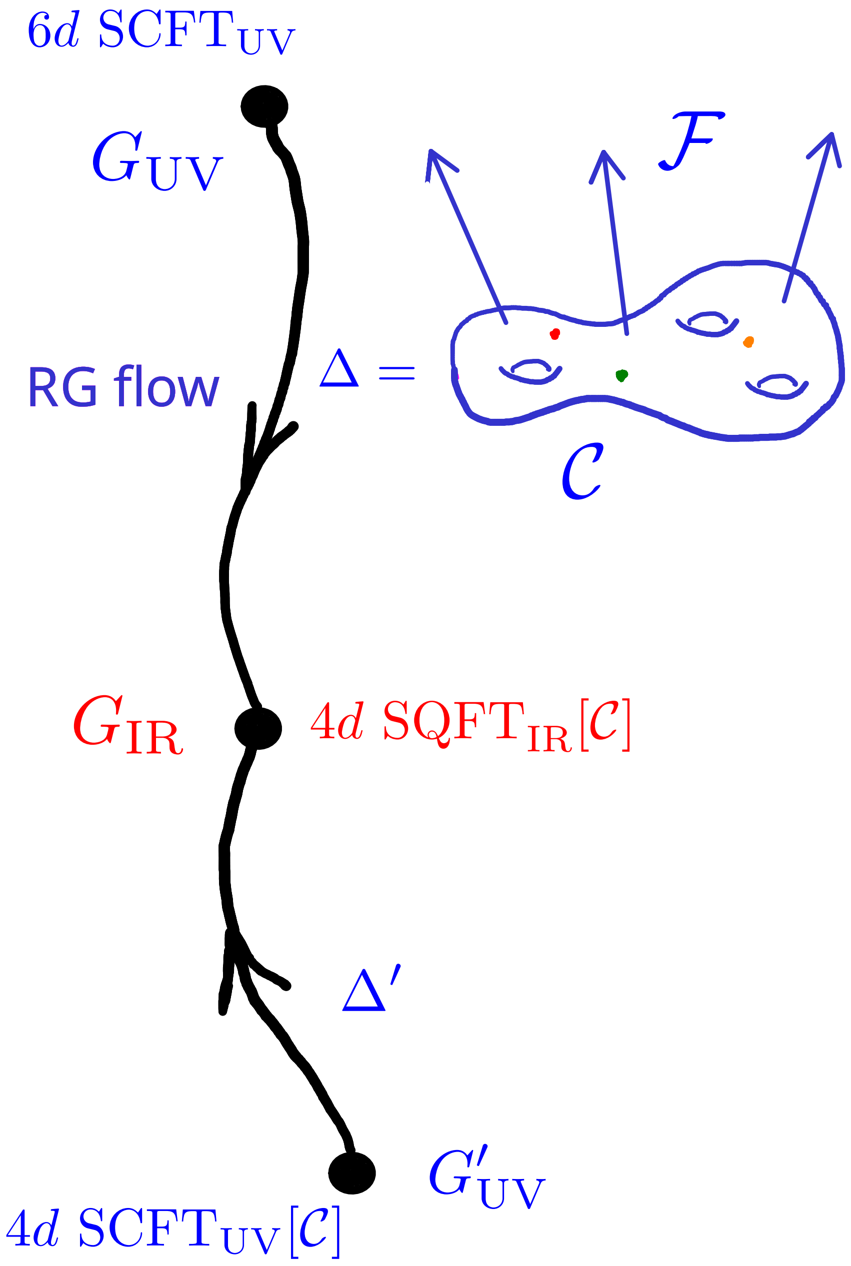

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- Gluing two punctures:
  1) Identify operators of two punctures $W \sim M \cdot M'$
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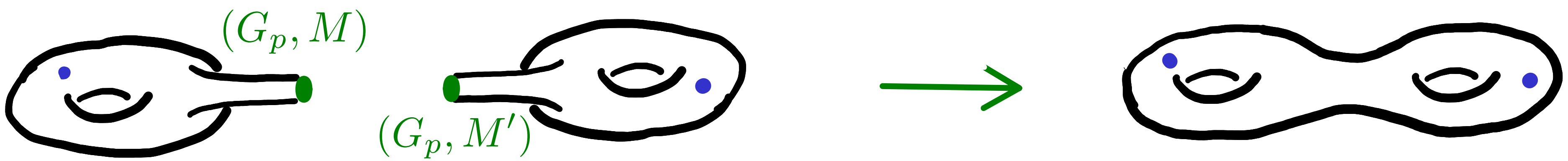

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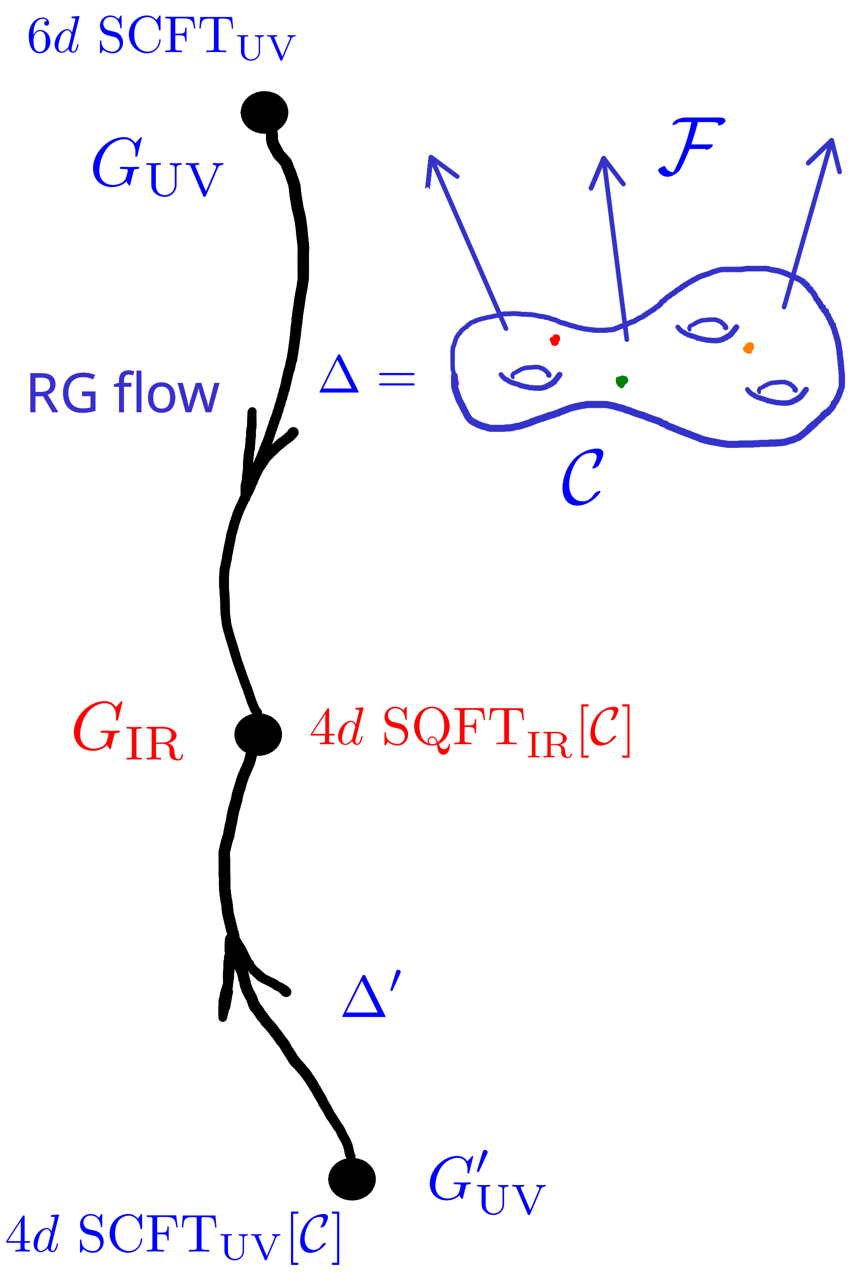

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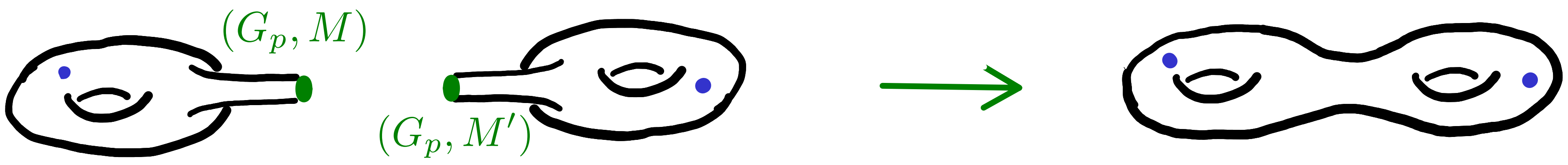


- Punctures can be closed partially or completely by giving VEV $\langle \partial_+^k \partial_-^l M_i \rangle \neq 0$
  Global symmetry is broken to a subgroup

Superconformal Index

- Given $4d$ $\mathrm{SQFT}_{\mathrm{IR}}[\mathcal{C}]$ put it on $S^3 \times \mathbb{R}$ and compute superconformal index

$$\mathcal{I}(p,q,\{u\}) = \mathrm{tr}_{S^3}\left[(-1)^F q^{j_2-j_1+\frac{1}{2}R} p^{j_2+j_1+\frac{1}{2}R} \prod_{i=1}^{G_{4d}} u_i^{Q_i}\right]$$

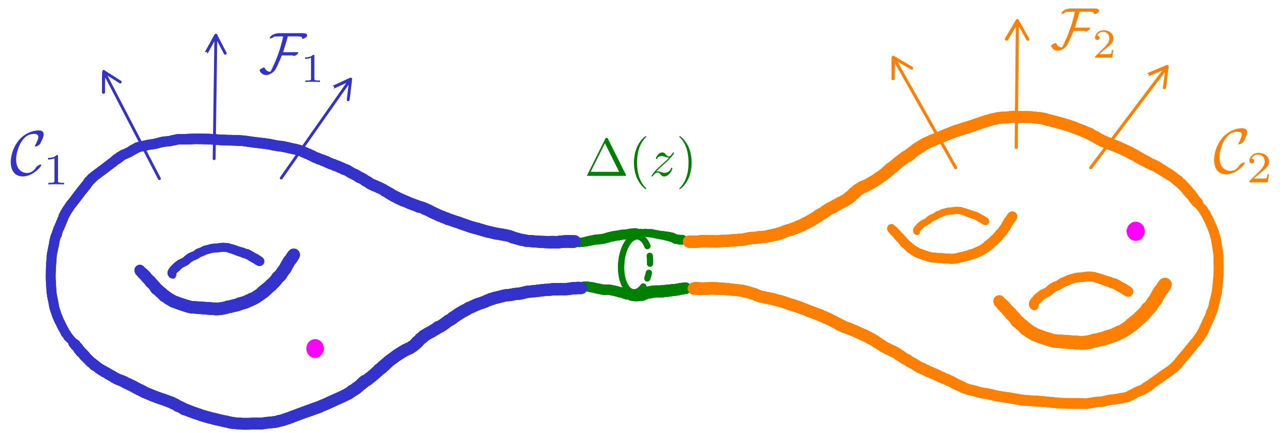
- $(j_1, j_2) \rightarrow$ charges under Cartans of $Spin(4)$, corresponding fugacities are p and q .
- $R \rightarrow$ charges of $U(1)$ R-symmetry
- $Q_i \rightarrow$ charges of the Cartan of $4d$ global symmetry G_{4d} corresponding fugacities are u_i
  - → Cartans of $6d$ global symmetry G_{6d}
  - → Cartans of punctures global symmetry G_p

- Using SUSY localization or UV free theory calculation can be reduced to matrix integral in case Lagrangian description is known.

Puncture Operations for Indices

- Gluing:

$$\mathcal{I}\left[\mathcal{C}_1 \oplus \mathcal{C}_2, \mathcal{F}_1 + \mathcal{F}_2\right] = \oint \prod_{i=1}^{\text{rank } G_p} \frac{dz_i}{2\pi i z_i} \Delta(z;u;q,p) \cdot \mathcal{I}\left[\mathcal{C}_1, \mathcal{F}_1\right] \cdot \mathcal{I}\left[\mathcal{C}_2, \mathcal{F}_2\right]$$



$\Delta(z;u;q,p)$ is gluing measure. Depends on the details of gluing and compactification.

Simplest example: vector multiplet contribution $\prod_{i\neq j}^{\text{rank} G_p} \frac{1}{\Gamma\left(z_i/z_j;p,q\right)}$.

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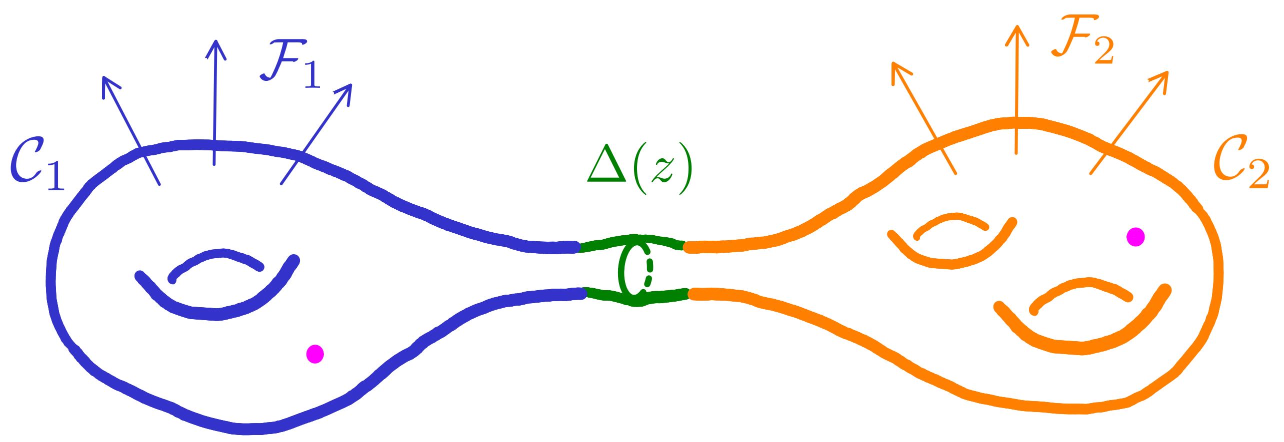

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- Closing punctures: Operator $\hat{O}$ of charge '-1' under $U(1)_u$ global symetry

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VEV $\langle \partial_{12}^m \partial_{34}^n \hat{O} \rangle \neq 0$ triggers RG flow . Index of IR theory: $\mathcal{I}_{IR} \sim \text{Res}_{u\to p^m q^n U_{\hat{O}}} \mathcal{I}_{UV}(u)$ $m,n \in \mathbb{Z}_{\geq 0}$

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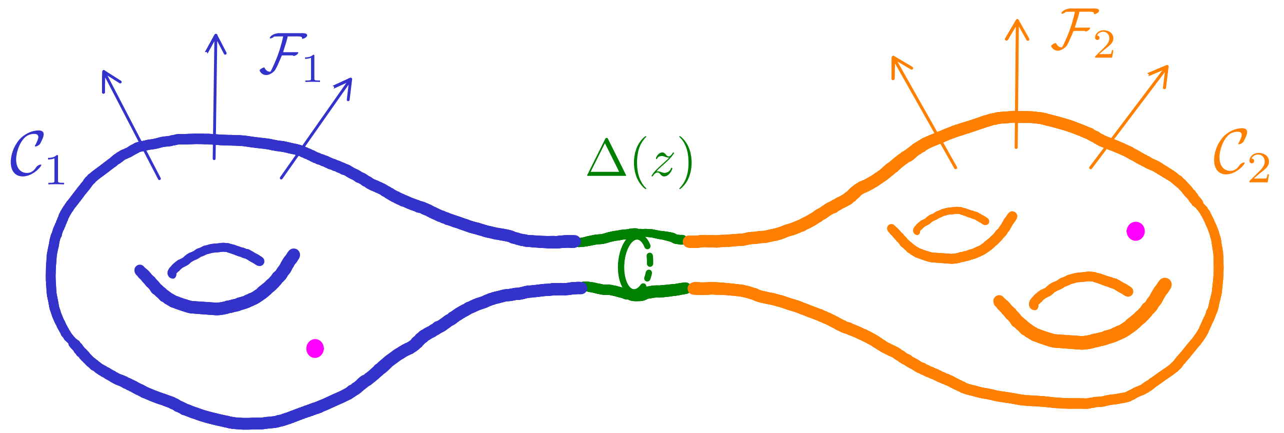

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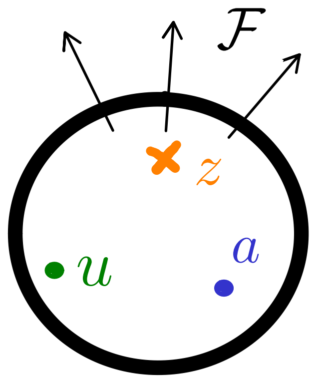
- When $m \neq 0$ and/or $n \neq 0$ we physically introduce codimension-two defect Gadde, Gukov 1305.0266

Integrable Operators from Indices



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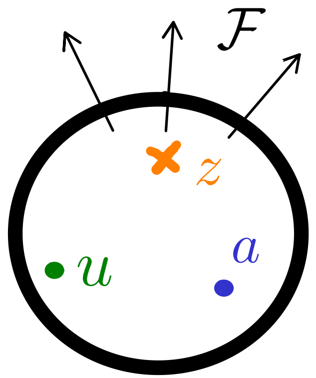

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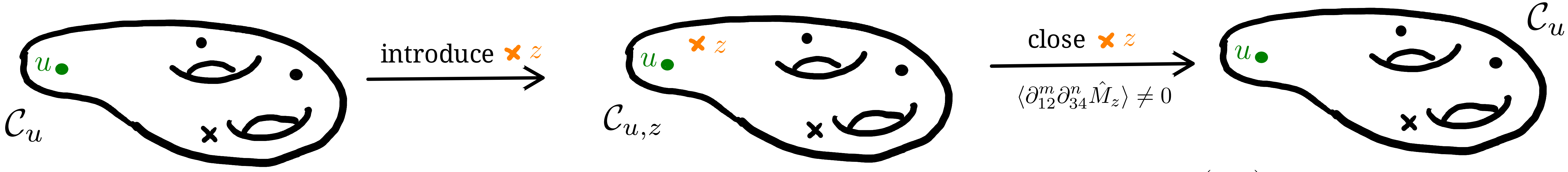

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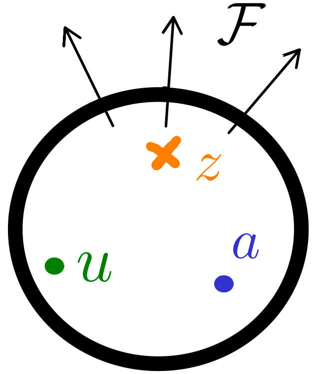

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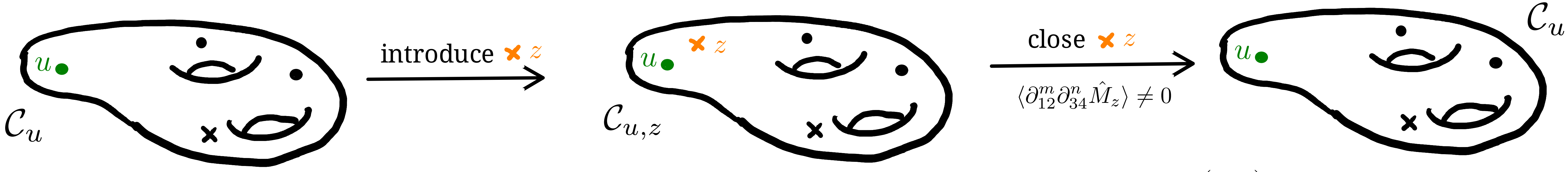

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- **Index with (m,n) defect:** $\qquad$ $\mathcal{H}_u^{(m,n)}$ - tower of finite difference operators $(A\Delta Os)$

$$\mathcal{I}_{IR}^{(m,n)} \sim \mathrm{Res}_{z\to p^m q^n U_{\hat{M}_z}} \mathcal{I}\left[\mathcal{C}_{u,z}\right](u,z) \quad \Longrightarrow \quad \boxed{\mathcal{I}_{IR}^{(m,n)} \sim \mathcal{H}_u^{(m,n)} \cdot \mathcal{I}\left[\mathcal{C}_{g,s}[u],\mathcal{F}\right]}$$

Gaiotto, Rastelli, Razamat 1207.3577 $\qquad$ Nazzal, AN, Razamat 2106.08335

Properties of Operators

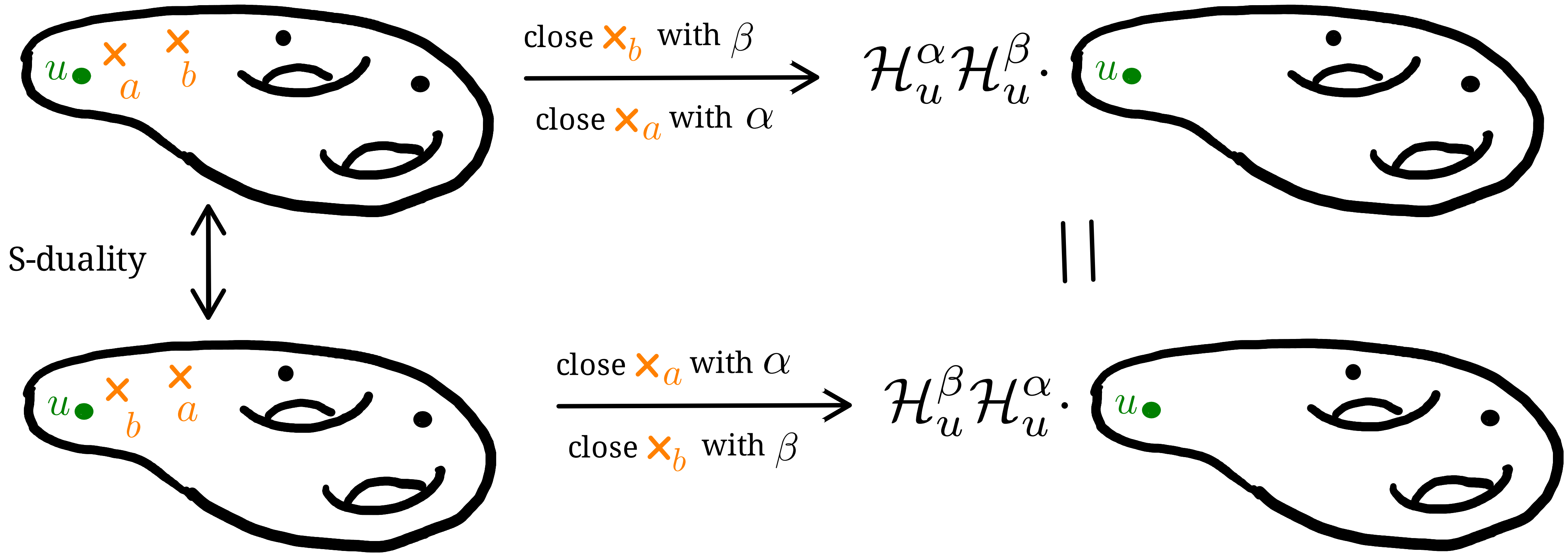
- Two properties follow by construction from 4d S-duality:

(I) $A\Delta Os$ all commute with each other

$$\left[\mathcal{H}_u^{\alpha}, \mathcal{H}_u^{\beta}\right] = 0$$

α, β - labels of operators including (m,n) and more (puncuture closure)

$\Longrightarrow$ integrability?



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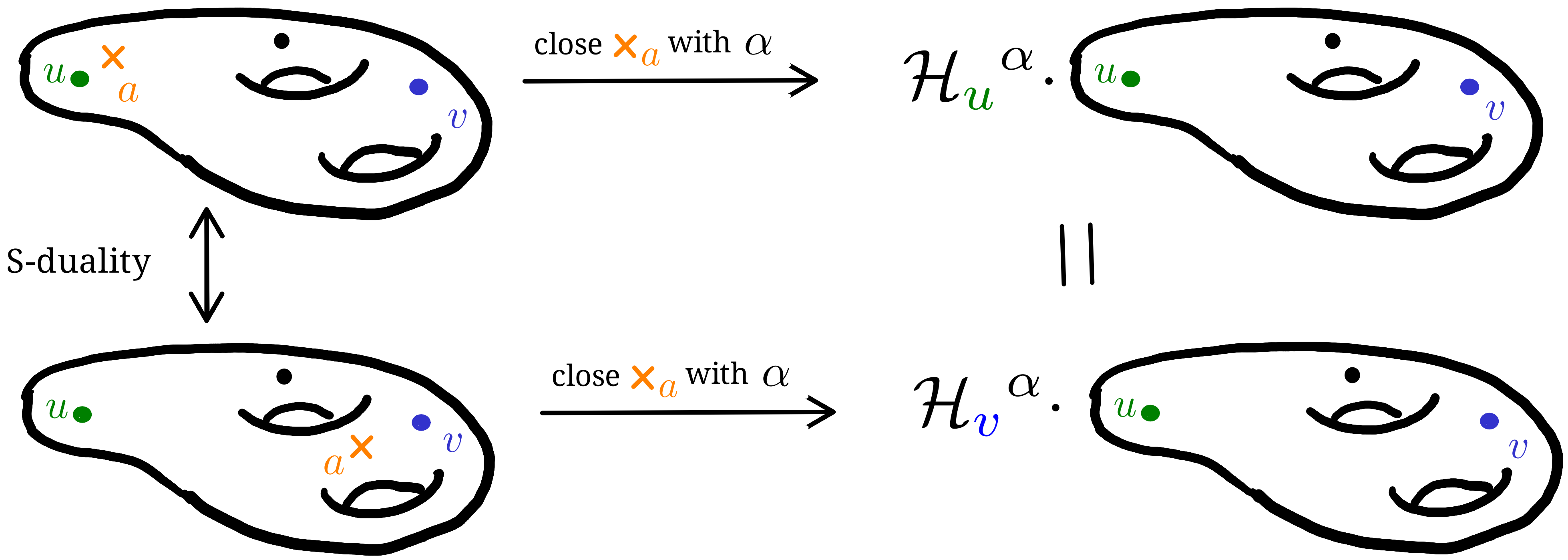
(II) Indices of 4d $\mathcal{N}=1$ theories obtained in certain compactifications are Kernel Functions of the corresponding operators:

$$\mathcal{H}_u^\alpha \cdot \mathcal{I}_{g,s}[u,v,...] = \mathcal{H}_v^\alpha \cdot \mathcal{I}_{g,s}[u,v,...] = ...$$



Indices from Spectrum

- Assume spectrum of $\mathcal{H}_x^\alpha$ is known:

$$\boxed{\mathcal{H}_{\mathbf{x}}^\alpha \cdot \psi_\lambda(\mathbf{x}) = E_{\alpha,\lambda}\psi_\lambda(\mathbf{x})}$$

λ - eigenstate label. Depends on Hamiltonian (integer, partition etc.)

Eigenfunctions ψ_λ are orthogonal w.r.t. gluing measure $\Delta(x)$:

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- Natural ansatz for Kernel functions of $\mathcal{H}_x^\alpha$

$$\mathcal{I}(\{\mathbf{x}_j\}) = \sum_\lambda C_\lambda \prod_{j=1}^{s} \psi_\lambda(\mathbf{x}_j)$$

Compactification on Riemmann surface with s punctures of the same type.

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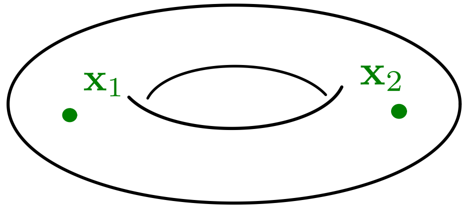
- Assume λ have natural ordering so we can enumerate them: $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$

- Simplest example, two-punctured surface $\mathcal{I}(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{\infty} C_{\lambda_i}\psi_{\lambda_i}(\mathbf{x})\psi_{\lambda_i}(\mathbf{y})$

  C_{λ_i} depends on details of compactification (fluxes, genus)

Spectrum from Indices

Nazzal, AN, Razamat 2305.09718



- Start with two-punctured surface with some non-zero flux and/or non-zero genus

$$\mathcal{I}_1(\mathbf{x},\mathbf{y}) = \sum_{i=0}^{\infty} C_{\lambda_i} \psi_{\lambda_i}(\mathbf{x}_1)\psi_{\lambda_i}(\mathbf{x}_2)$$

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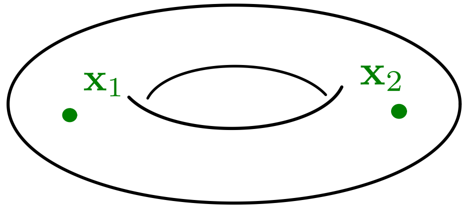

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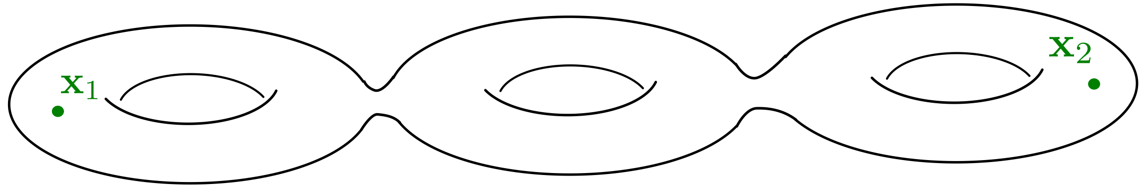
- Glue together n copies

$$\boxed{\mathcal{I}_n(x_1,x_2) = \sum_{i=0}^{\infty} (C_{\lambda_i})^n\, \psi_{\lambda_i}(\mathbf{x}_1)\psi_{\lambda_i}(\mathbf{x}_2)}$$



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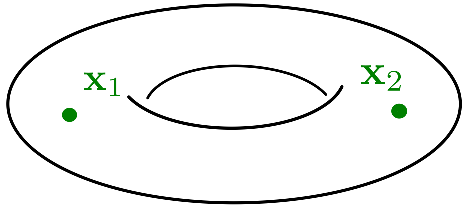

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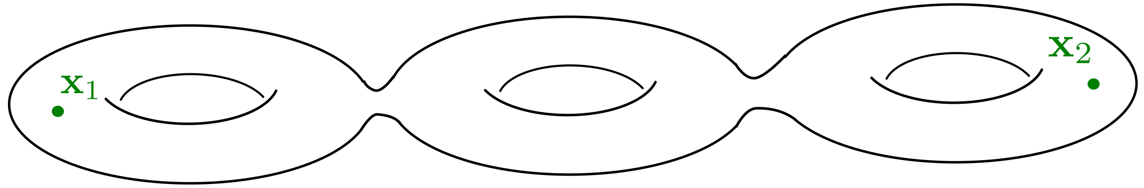

- Consider series in $y = pq$ parameter assuming C_{λ_i} are ordered:

$$C_{\lambda_0} = O(1), \quad C_{\lambda_1} = O\left(y^{n_1}\right), n_1 > 0 \quad C_{\lambda_2} = O\left(y^{n_2}\right), n_2 > n_1, \ldots.$$

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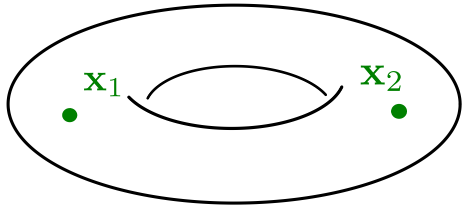

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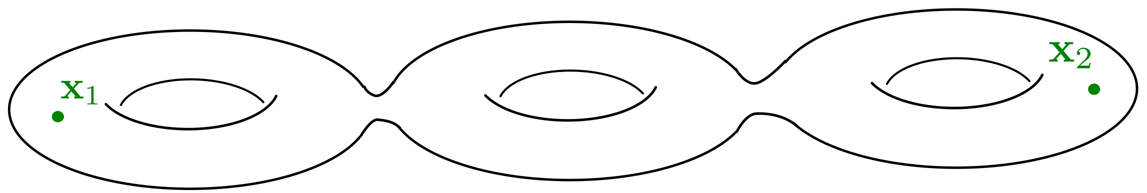

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- Now we can fix C_{λ_0} and ground state wave funciton $\psi_{\lambda_0}(\mathbf{x})$:

In practice: take finite n, results are valid up to a fixed order in $y = pq$

$$C_{\lambda_0} = \lim_{n\to\infty} \frac{\mathcal{I}_{n+1}(\mathbf{x}_1,\mathbf{x}_2)}{\mathcal{I}_n(\mathbf{x}_1,\mathbf{x}_2)}$$

$$\psi_{\lambda_0}(\mathbf{x}) = \lim_{n\to\infty} \left(C_{\lambda_0}\right)^{-n} \mathcal{I}_n(\mathbf{x},\mathbf{1})$$

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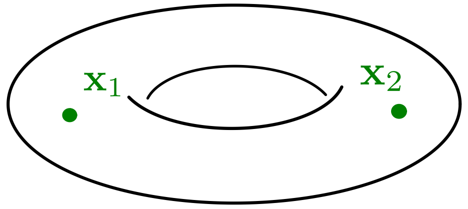

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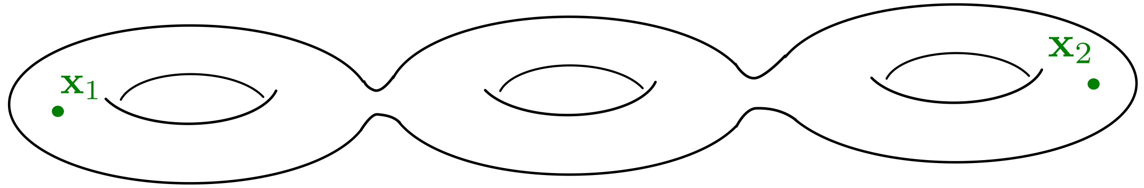

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- Other C_{λ_i} and $\psi_{\lambda_i}(\mathbf{x})$ can be found similarly using results for the lower states.

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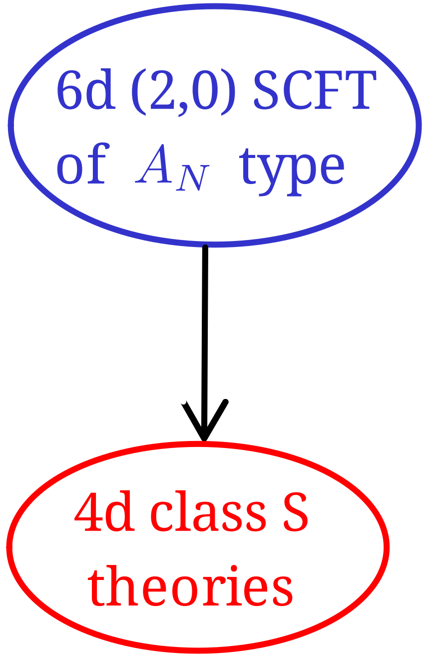
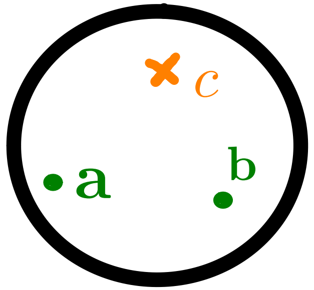
$$\psi_{\lambda_0}(\mathbf{x}) = \lim_{n\to\infty} \left(C_{\lambda_0}\right)^{-n} \mathcal{I}_n(\mathbf{x},\mathbf{1})$$

Example : Ruijsenaars – Schneider Model

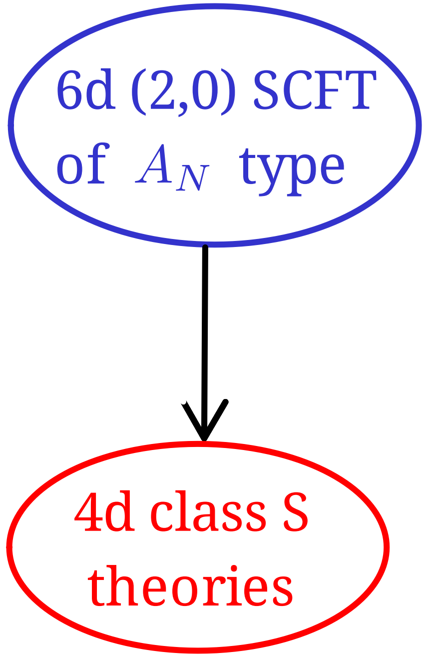


- Punctures with A_N symmetry (5d compactification is $\mathcal{N}=2$ SU(N+1) SYM).
- Trinion theory is free bi-fundamental hypermultiplet:

$$\mathcal{I}_3 = \prod_{i,j=1}^{N+1} \Gamma\left(t^{1/2}(a_i b_j z^{\pm 1})^{\pm 1}\right)$$



Example : Ruijsenaars – Schneider Model



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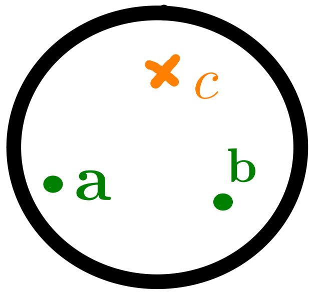

- Associated operators: A_N Ruijsenaars-Schneider system

Gaiotto, Rastelli, Razamat 1207.3577
Ruijsenaars and Schneider '86

A_1 case, only one operator.
(N independent operators in general)

$$\mathcal{H}_{A_1} \cdot \psi(x) = \frac{\theta_p\left(tx^2\right)}{\theta_p\left(x^2\right)}\psi(q^{1/2}x) + \frac{\theta_p\left(tx^{-2}\right)}{\theta_p\left(x^{-2}\right)}\psi(q^{-1/2}x)$$

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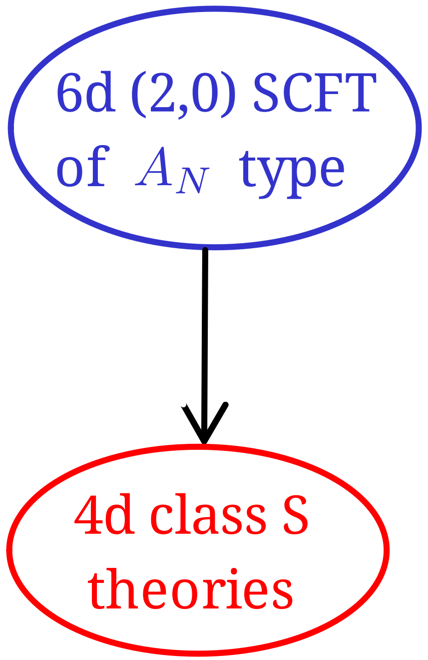

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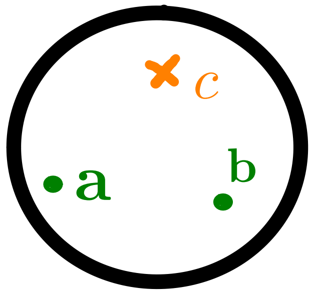

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- Macdonald limit: $p \to 0$ spectrum is known. Eigenfunctions are Macdonald Polynomials

  Used for index calculations. Gadde, Rastelli, Razamat, Yan 1110.3740

- In general case explicit form of the spectrum is not known yet !

A_1 RS Spectrum from Indices

Kim, AN, Razamat 2407.08776; Nazzal, AN, Razamat 2305.09718;

- Staring point: two-puncutured sphere (tube) with flux

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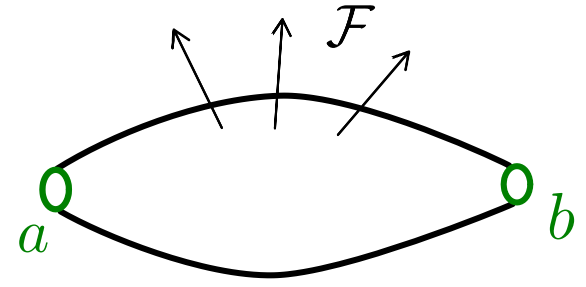

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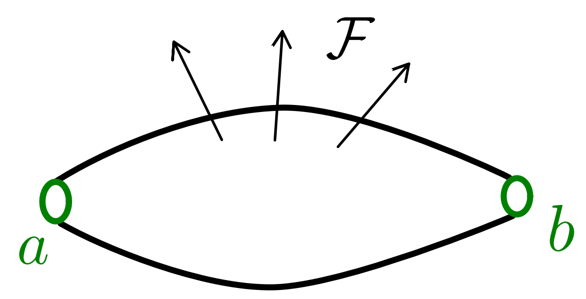

- Gluing measure:

$$\Delta(x) = \frac{(p;p)_\infty (q;q)_\infty}{2} \frac{\Gamma\left(\sqrt{pq}t^{-1}x^{\pm 2}\right)\Gamma\left(\sqrt{pq}t^{-1}\right)}{\Gamma\left(x^{\pm 2}\right)}$$

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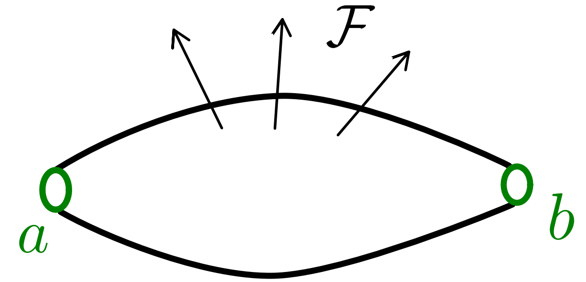

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Expansion is up to $(pq)^{\frac{7}{2}}$ for $\psi_0(x)$ and up to $(pq)^{\frac{3}{2}}$ for $\psi_1(x)$

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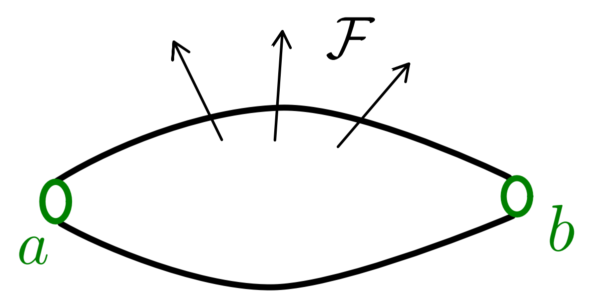

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$$\mathcal{H}_{A_1}\cdot\psi_n(x) = E_n\psi_n(x)$$

Substituting eigenfunctions

$$E_0 = 1-p+(t+\frac{1}{t})\sqrt{pq}-pq+(t+\frac{1}{t})p\sqrt{pq}-p^2+\ldots$$

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- Alternative approach for A_{N-1} RS: ramified instanton partition function Bullimore, Kim, Koroteev 1412.6081;
  - Theory: $5d\ \mathcal{N}=1^*\ SU(N)\ SYM$ on the Ω background $S^1\times\mathbb{R}^4_{\epsilon_1,\epsilon_2}$

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- For $\rho = [1,1]$ defect in $SU(2)$ SYM introduce coordinate x: $Q_1 = x; \quad Q_2 = Q/x;$

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Eigenvalue is the expectation value of the Wilson loop in fundamental representation

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A-model: $\mu_{1,2}$ solve BAE

(instanton corrections in p)

S - duality ⟷

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- Check:

$$\left. \frac{\psi_n^{ind.}(x)}{D_{[1,1]}(x, \mu_{1,2})} \right|_{\mu_2 = \mu_1^{-1} = \sqrt{tq^n}} = const; \quad E_0^{ind.} = \langle W_{\square}(\mu_{1,2}) \rangle \big|_{\mu_2 = \mu_1^{-1} = \sqrt{t}}$$

Kim, AN, Razamat 2407.08776

More Results

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- Minimal (D,D) conformal matter $\Longrightarrow$ novel A_N and C_2 generalizations of van Diejen.

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Conclusions and Outlook

- We proposed a new method for deriving the perturbative spectrum of a large class of relativistic elliptic integrable models.

- The method has been tested on various models: Rujsenaars-Schneider, van Diejen and some novel models derived in 6d compactifications.

  In all cases at least ground state wavefunction and energy were derived.

- For the cases of A_1 and A_2 Ruijsenaars-Schneider model results were compared with the alternative approach of ramified instantons claculations.

- Many more things to be considered (see previous slide).

Thank You for Attention !