Rényi entropies in the $n \to 0$ limit, entanglement temperatures and holography

Based on: arXiv 2309.00044

and to appear

In collaboration with:

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Organization

- A) Entanglement temperatures (ET)
- B) Small n Rényi entropies from ET
- C) Holography
- D) Summary and Future directions

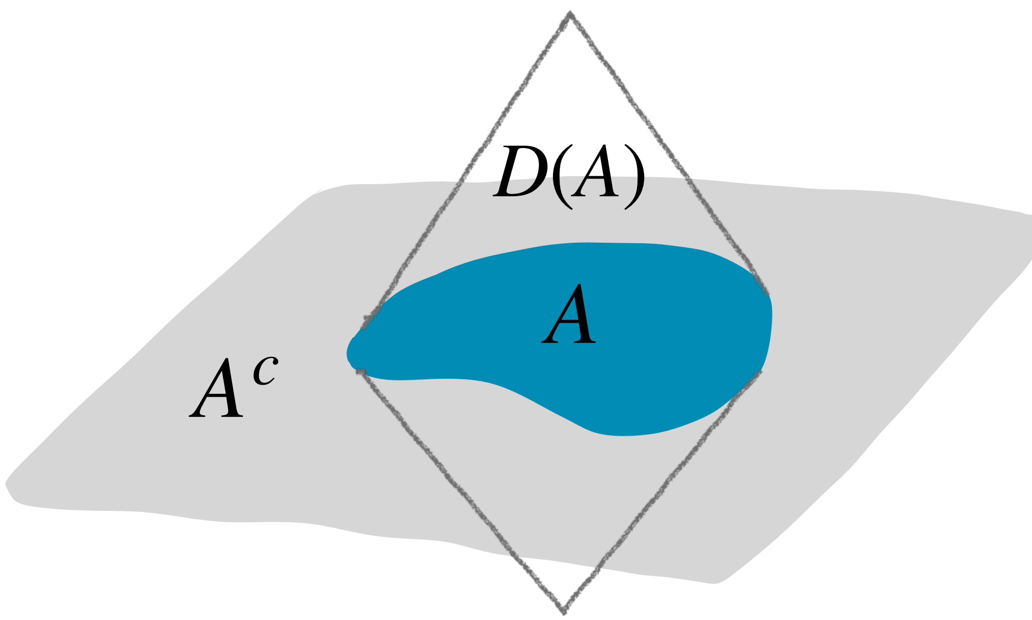
Modular Hamiltonian

Effective description of subsystem

$$\hat{\rho}_A = \frac{e^{-H_A}}{Z} \quad \text{where} \;\; \mathrm{tr}\,\hat{\rho}_A = 1, \;\; \hat{\rho}_A \geq 0$$

$$H_A = -\log \hat{\rho}_A + c \qquad\qquad U_A \sim e^{i\tau H_A}$$

Thermal system H_A with $\beta = 1$



Moments of $\hat{\rho}_A$ or Rényi entropies

$$S_n(A) = \frac{1}{1-n} \log\left(\mathrm{tr}\,\hat{\rho}_A^n\right)$$

$$e^{(1-n)S_n(A)} = \frac{\mathrm{tr}\, e^{-n H_A}}{Z^n} = \int dE\, e^{-nE} \rho(E)$$

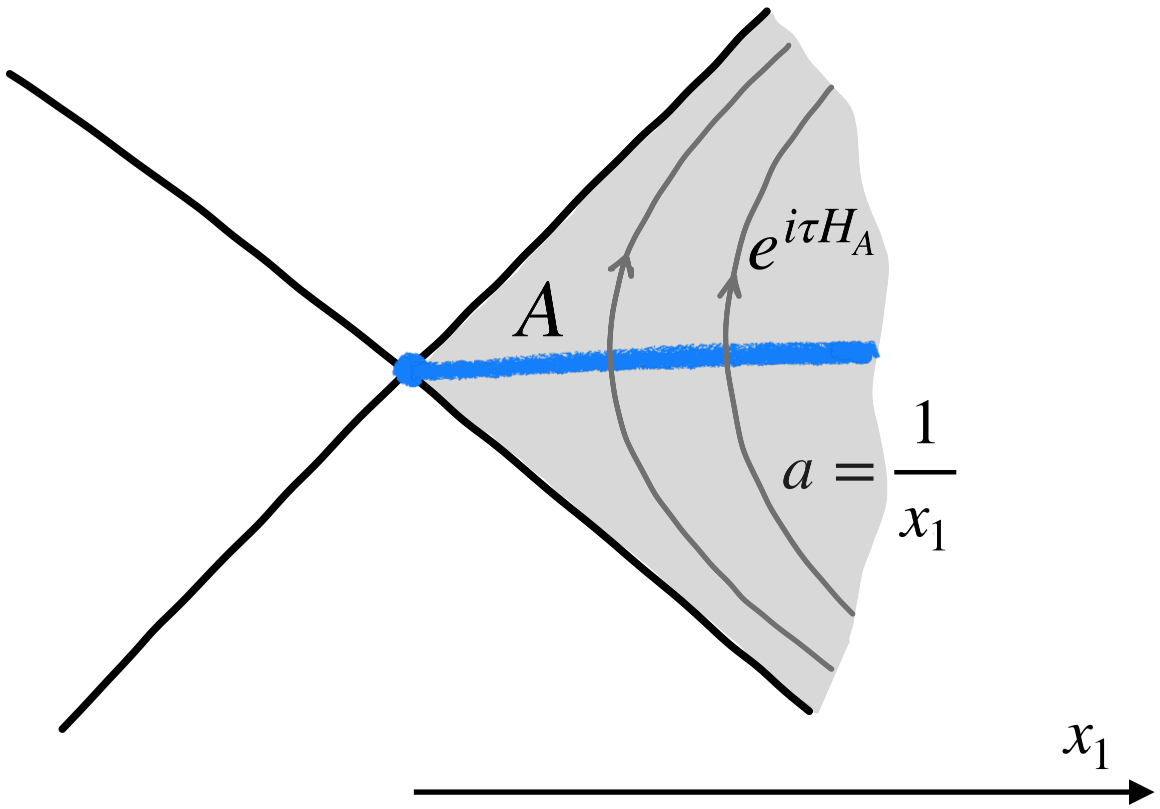
Entanglement entropy

$$S(A) = -\,\mathrm{Tr}_A\, \hat{\rho}_A \log \hat{\rho}_A$$

where $\quad \hat{\rho}_A = \mathrm{Tr}_{\bar{A}}\, |\Omega\rangle\langle\Omega|$

$$S(A) = \lim_{n\to 1} S_n(A)$$

Rindler and Unruh temperature

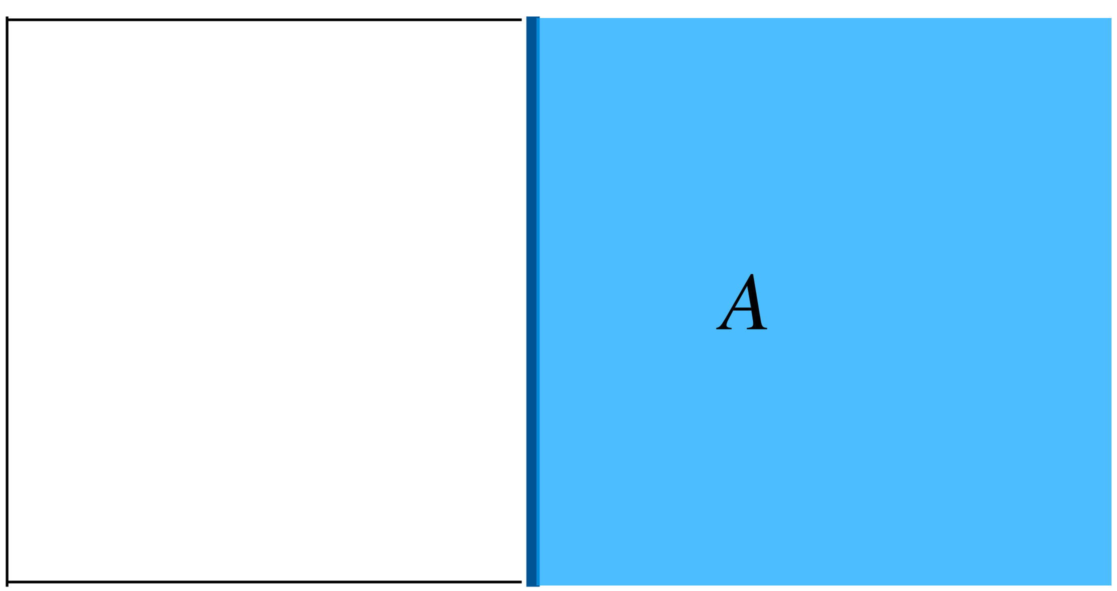


$$\hat{\rho}_A = \frac{e^{-H_A}}{\mathrm{tr} e^{-H_A}}$$

$$H_A = 2\pi \int_A d^{d-1}x \, x_1 \, T_{00}(x)$$

$$U_A = e^{i\tau H_A}$$

$$\boxed{\beta(x) = 2\pi x_1}$$



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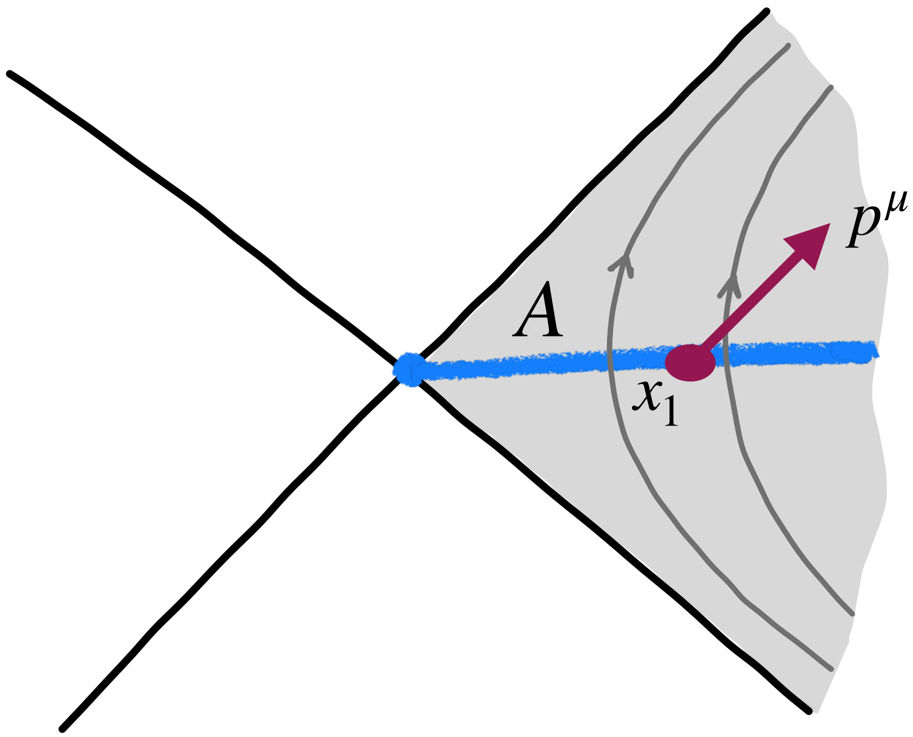

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Relative entropy and **entanglement temperatures**

$$S_{\rm rel}(\rho_A\,|\,\sigma_A) = \mathrm{tr}\left(\rho_A \log\rho_A - \rho_A \log\sigma_A\right) = \Delta\langle H_A^\sigma\rangle - \Delta S_A$$

$$\rho_A = U^\dagger(x,p)\sigma_A U(x,p) \quad \to \quad S_{\rm rel}(\rho_A\,|\,\sigma_A) \approx \beta(x)\, E$$

in general $\boxed{S_{\rm rel}(\rho_A\,|\,\sigma_A) \approx \beta_A(x,\hat p)\, E}$

[Arias, Casini, Huerta, Pontello: 2017]

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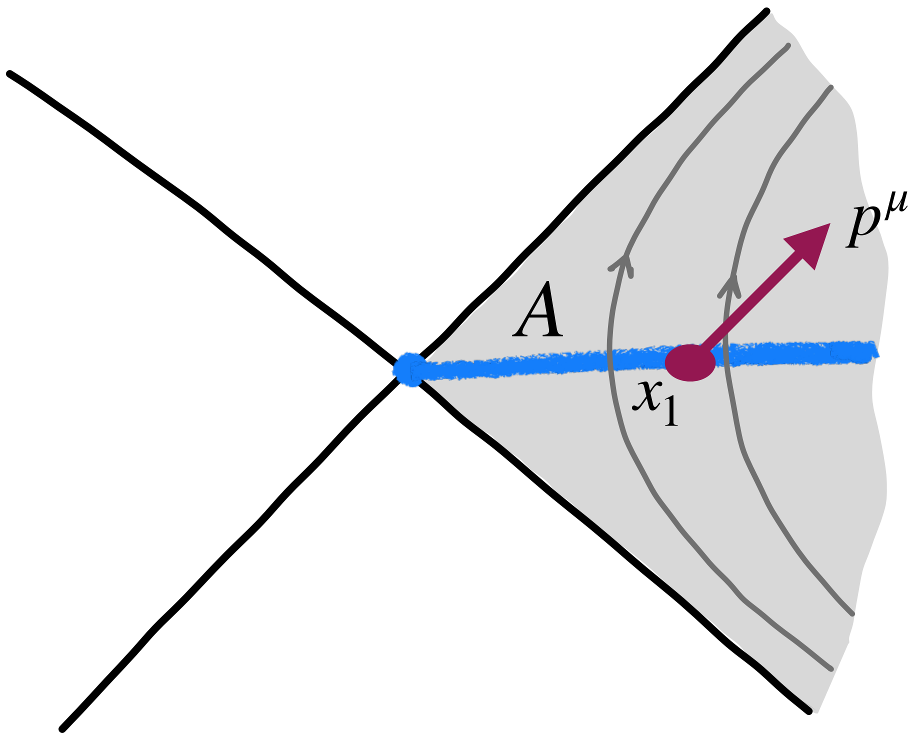

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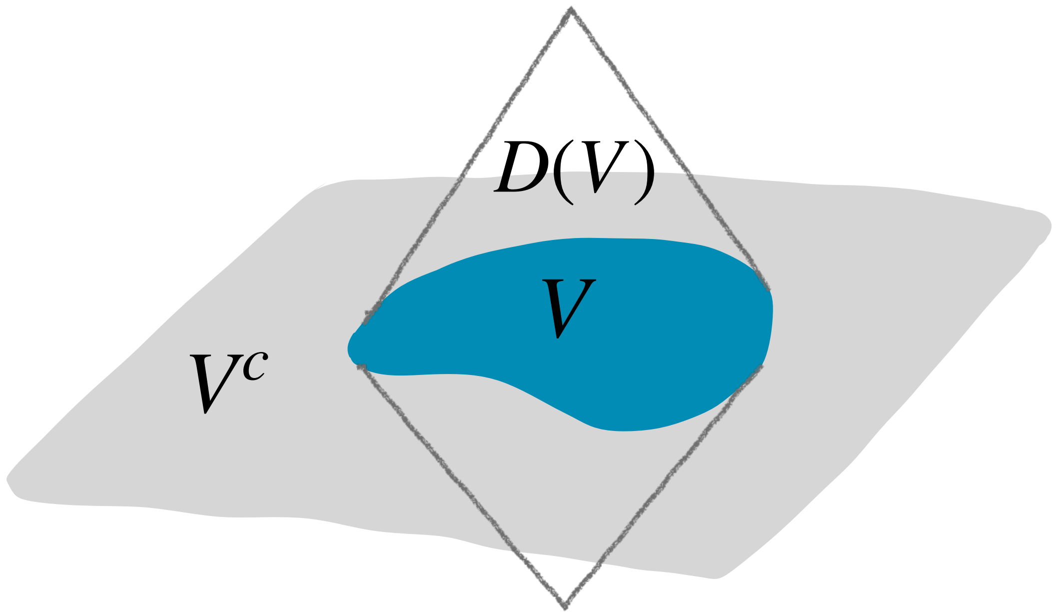
Modular Hamiltonian for free fields

$$H_V \sim \int d\lambda \int ds\,(2\pi s)\,\hat{a}^{\dagger}_{s,\lambda}\hat{a}_{s,\lambda}$$

[Arias, Casini, Huerta, Pontello: 2017]



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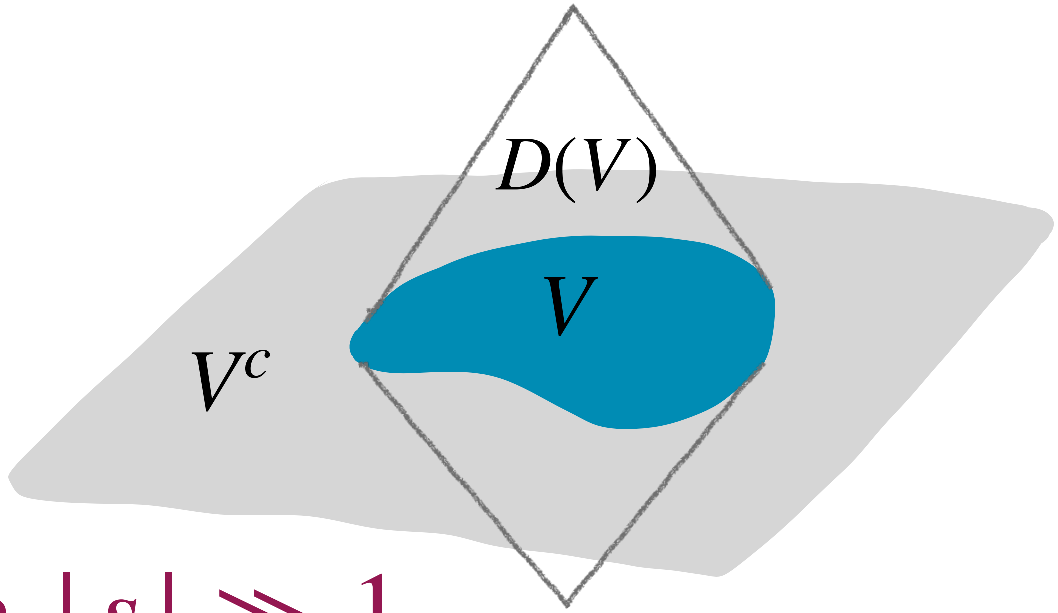

High energy Modular Hamiltonian $|s| \gg 1$

$$H_V^{HE} = \int_V d^{d-1}x \int_{TV} d^{d-1}p\ \beta(\overrightarrow{x}, \hat{p})\ |\overrightarrow{p}|\, a^\dagger_{\overrightarrow{p},\overrightarrow{x}}\, a_{\overrightarrow{p},\overrightarrow{x}}\,.$$

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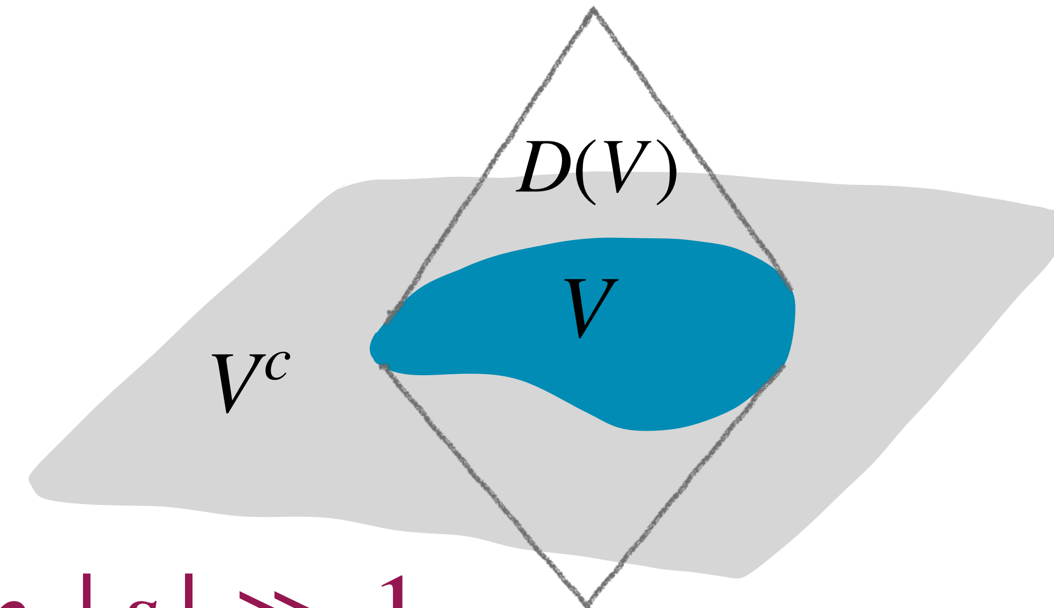

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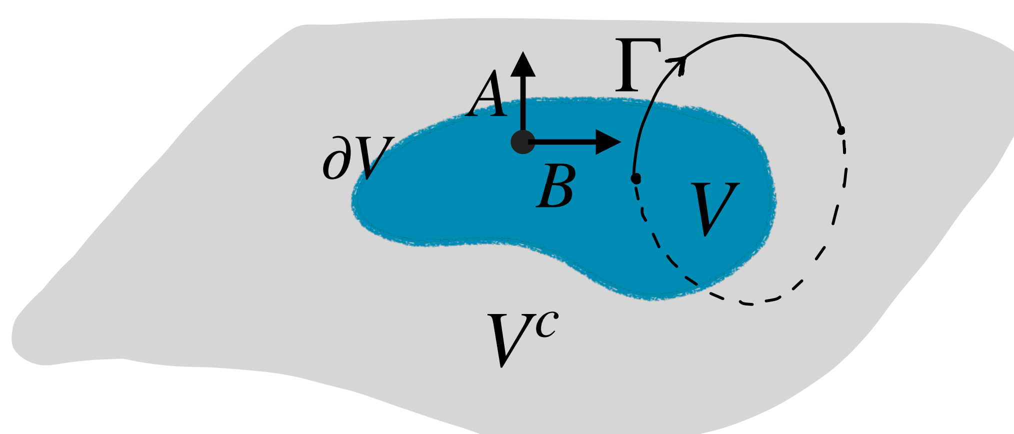
Eikonal equations $|s| \gg 1$

$$A^2 = B^2\,, \quad A \cdot B = 0\,,$$

$$\oint_\Gamma A \cdot dx = 2\pi\,, \qquad \oint_\Gamma B \cdot dx = 0\,,$$



$$\hat{p} = \hat{B}(\overrightarrow{x}) \qquad \beta(\overrightarrow{x}, \hat{p}) = \frac{2\pi}{|B(\overrightarrow{x})|}$$

Example: 2d

$$A^2 = B^2\,, \quad A \cdot B = 0\,, \quad \rightarrow \quad \left(\frac{\partial a}{\partial x}\right)^2 + \left(\frac{\partial a}{\partial y}\right)^2 = \left(\frac{\partial b}{\partial x}\right)^2 + \left(\frac{\partial b}{\partial y}\right)^2\,, \qquad \frac{\partial a}{\partial x}\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y}\frac{\partial b}{\partial y} = 0\,,$$

$$A = \nabla a\,, \quad B = \nabla b\,.$$

Implies $\alpha = a + ib$ is analytic function

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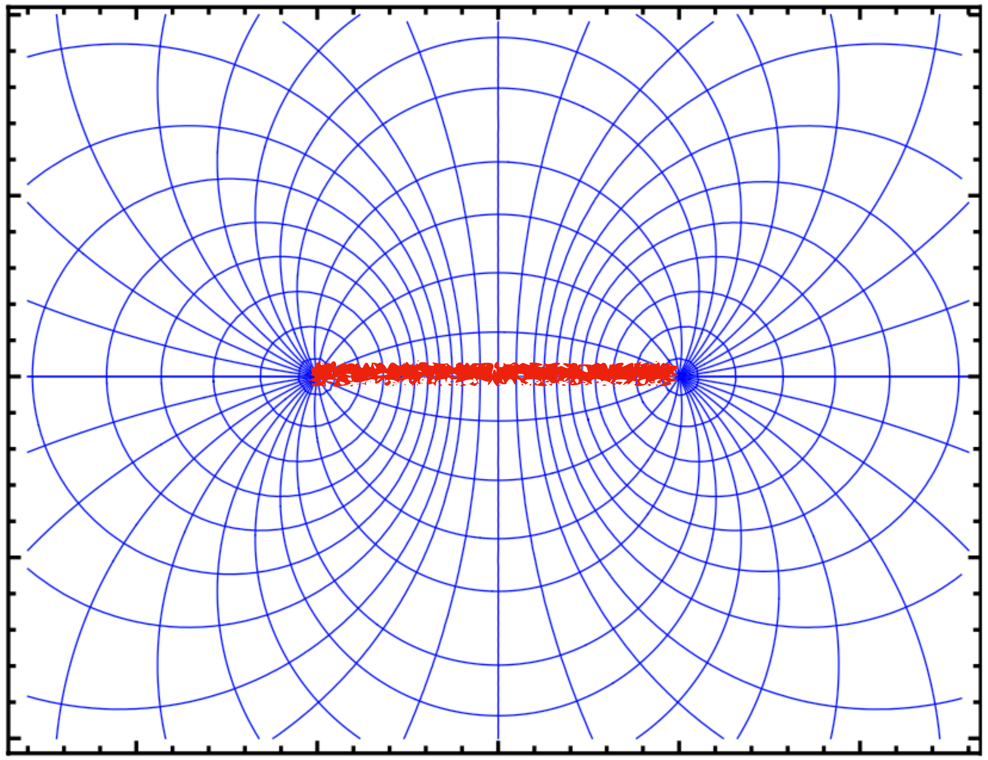
Implies $\alpha = a + ib$ is analytic function

$$\oint_\Gamma A \cdot dx = 2\pi \,, \qquad \oint_\Gamma B \cdot dx = 0 \,, \qquad \rightarrow \qquad \alpha(z) = i \log\left(\prod_{n=1}^{N} \frac{z - l_i}{r_i - z}\right)$$

One interval

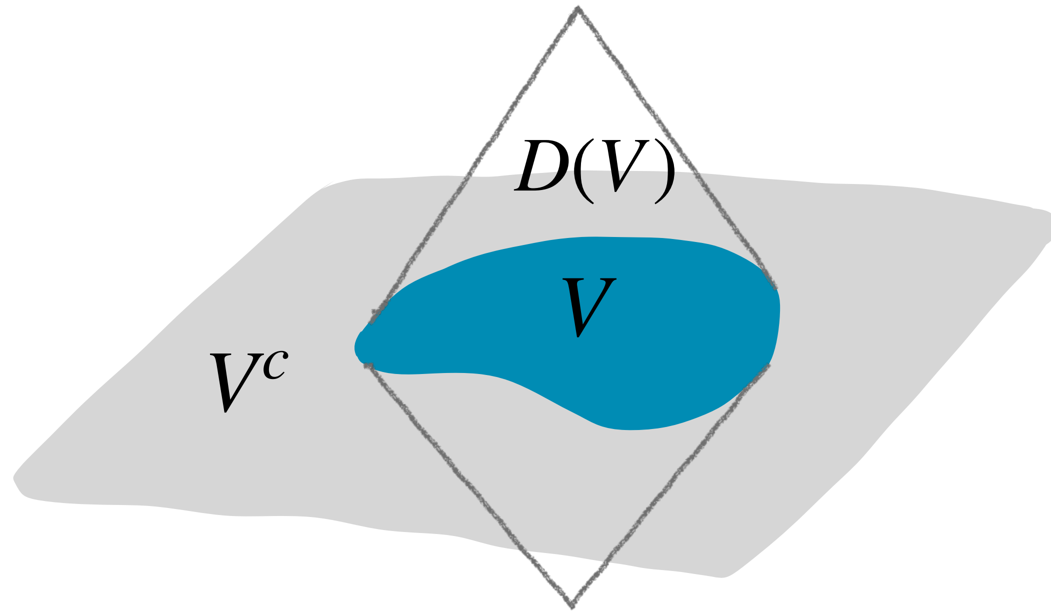


Entanglement temperatures

$$\beta(x) = \frac{2\pi}{|B(x)|}$$

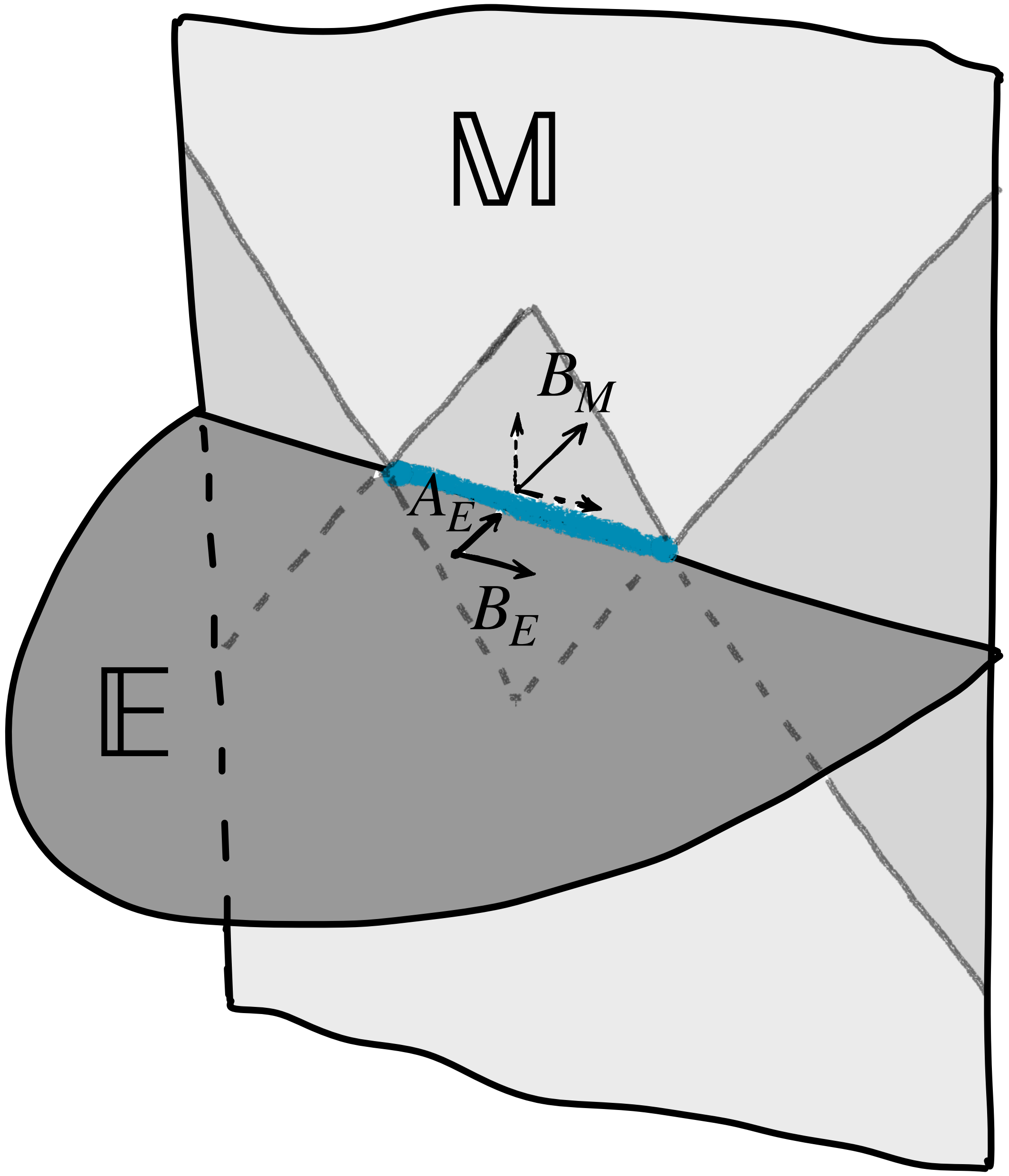
$$\beta(x) = 2\pi \left(\sum_{i=1}^{N} \left[\frac{1}{x - l_i} + \frac{1}{r_i - x}\right]\right)^{-1}$$

Lorentzian Eikonal equations



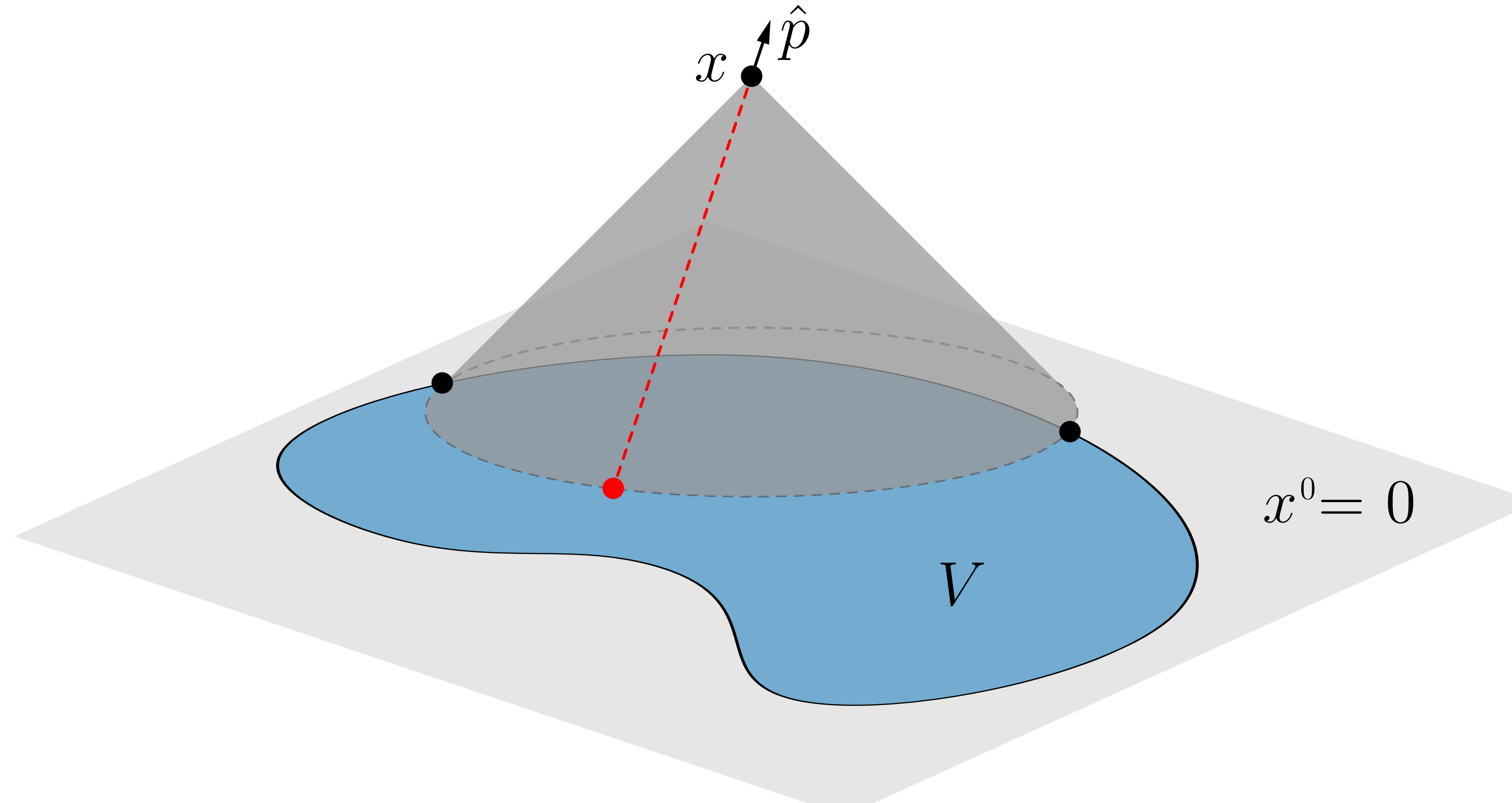
$$H_V^{HE} = \int_V d^{d-1}x \int_{TV} d^{d-1}p \; \beta(\overrightarrow{x}, \hat{p}) \mid \overrightarrow{p} \mid a^\dagger_{\overrightarrow{p}, \overrightarrow{x}} \, a_{\overrightarrow{p}, \overrightarrow{x}} \, .$$

Lorentzian Eikonal equations



The boundary conditions imply

$$B_M^2 = 0 \qquad B_M^\mu \nabla_\mu B_M^\nu = 0$$



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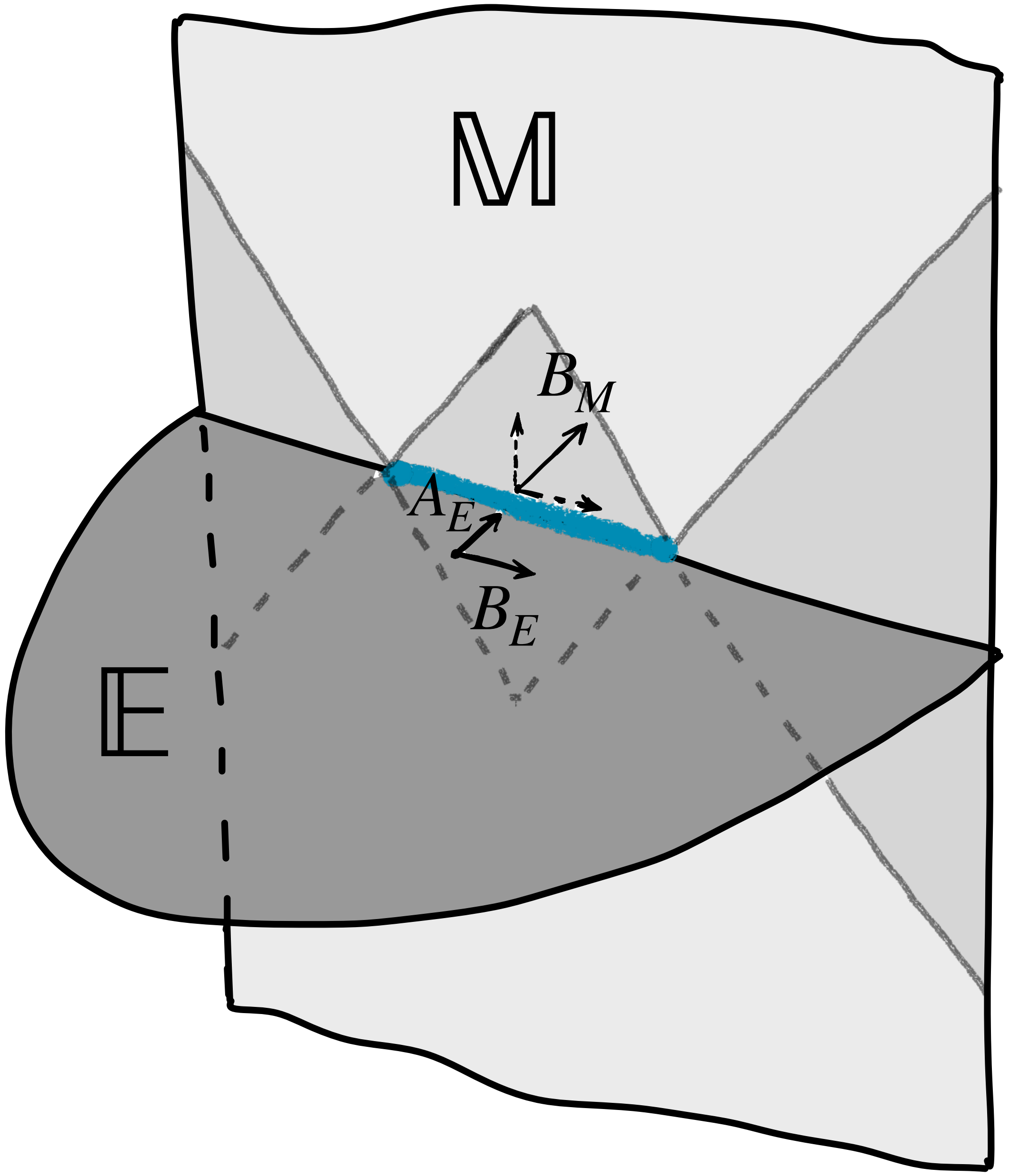

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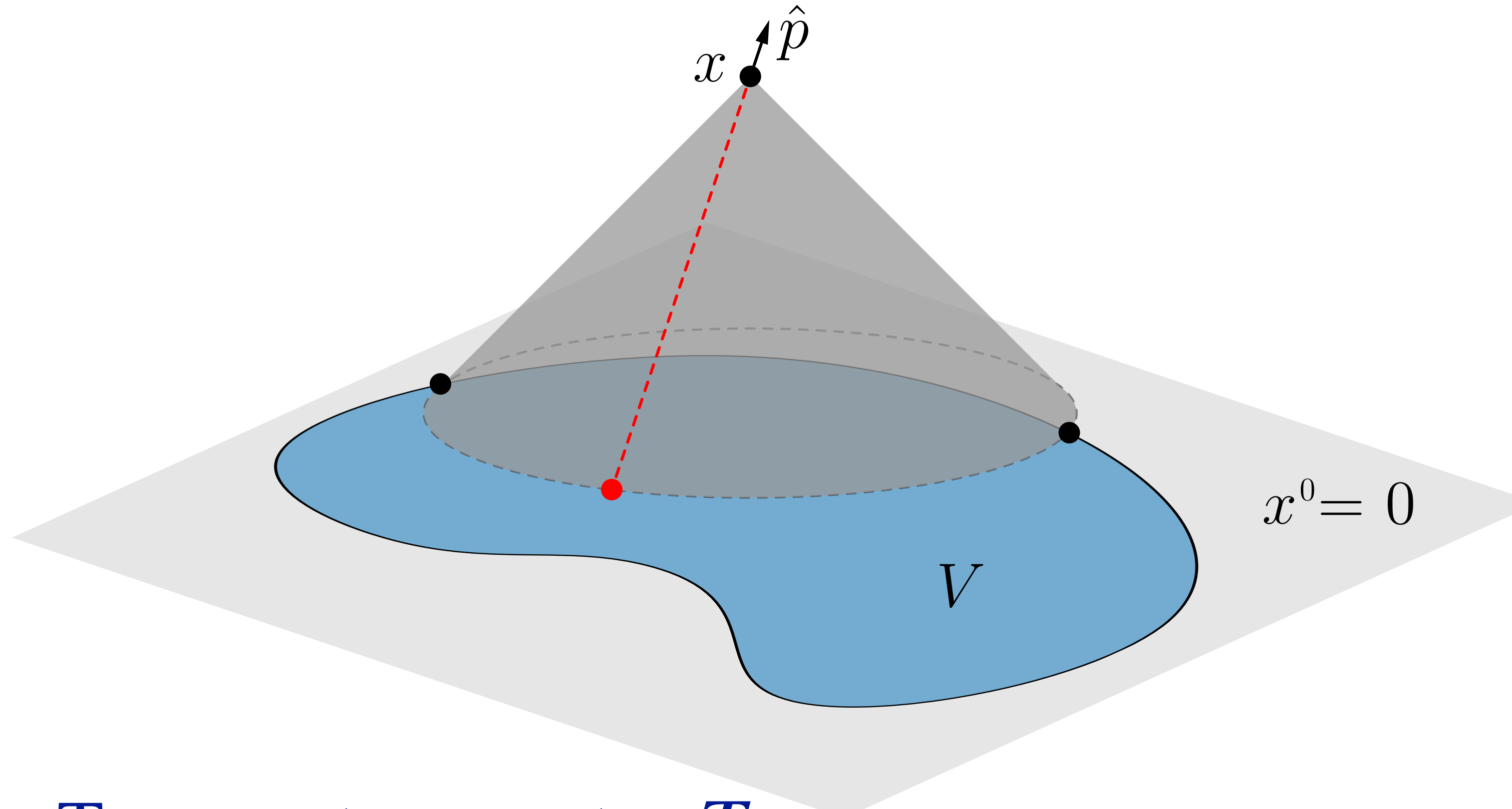
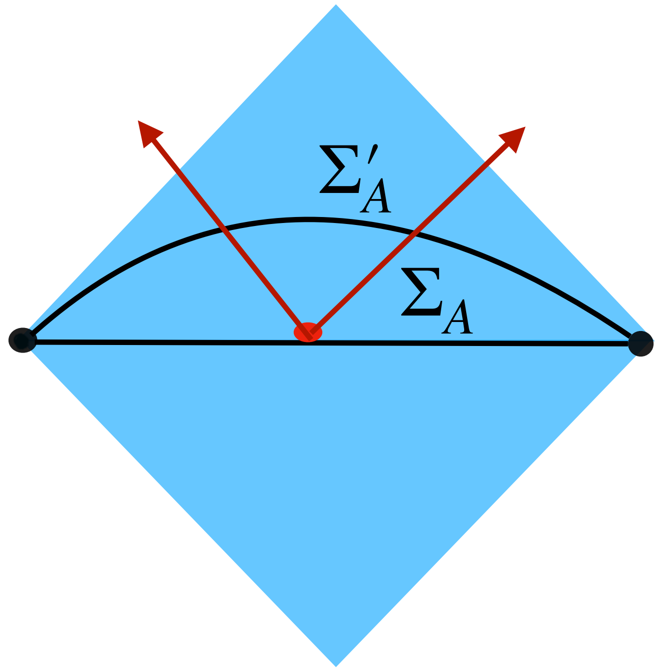


Temperature vector T_μ

$$T^\mu = \frac{B_M^\mu}{2\pi} \qquad T^\mu \nabla_\mu T_\nu = 0$$

$$S_{\rm rel}(\rho_A \,|\, \sigma_A) \approx \beta_A(x, \hat{p})\, E \quad \longrightarrow \quad T^\mu(x, \hat{p}) = \frac{1}{S_{\rm rel}} p^\mu$$

Lorentzian modular Hamiltonian



$$H_A^{\mathrm{HE}} = \int_{\Sigma_A} d\sigma\, \hat{n}_\mu \int_{T\Sigma_A} \frac{d^{d-1}p}{(\hat{n}\cdot p)}\; p^\mu S_{\mathrm{rel}}(x,p)\; a^\dagger_{\vec{p},\vec{x}}\, a_{\vec{p},\vec{x}}\,,$$

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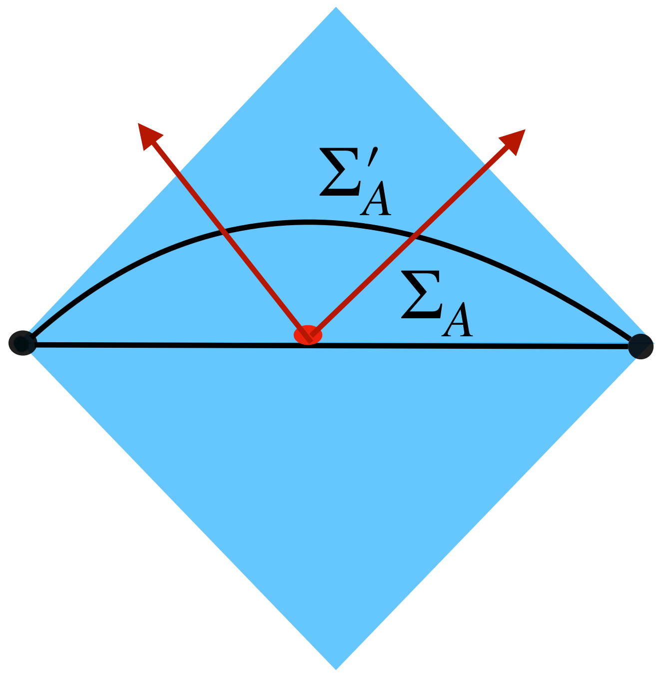

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Conservation ↔ Boltzmann equation

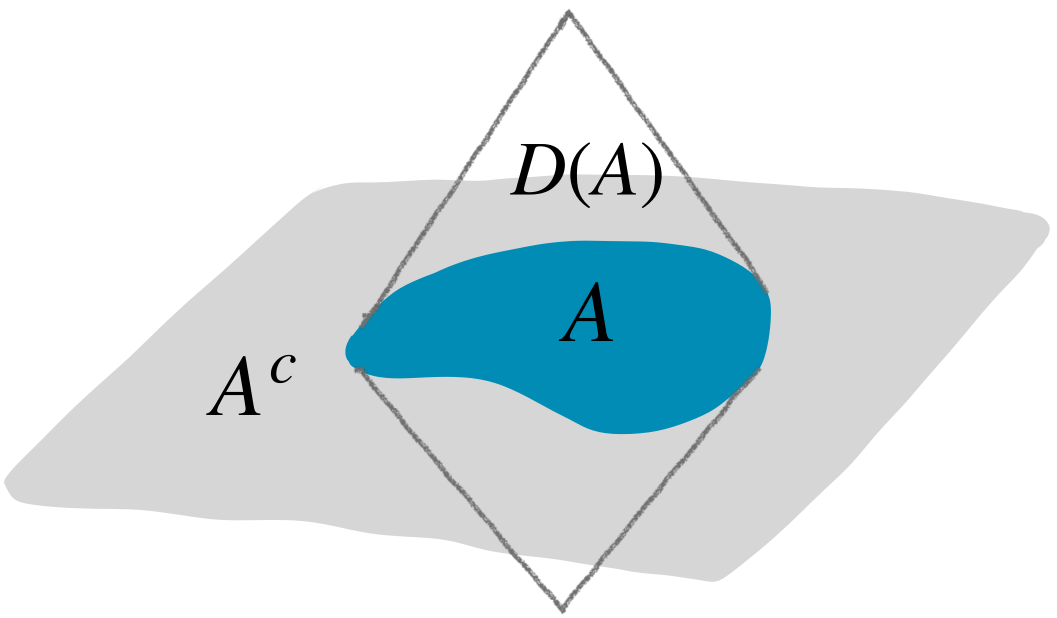
$$\frac{dS_{\mathrm{rel}}(x(\lambda),p(\lambda))}{d\lambda} = 0 \iff T^\mu \nabla_\mu T^\nu = 0$$

Any physical observable that depends on H_A^{HE}

Small n (Hot) Rényi entropies

$$S_n(A) = \frac{1}{1-n} \log \left(\mathrm{tr}\, \hat{\rho}_A^n \right)$$

$$\hat{\rho}_A = \frac{e^{-H_A}}{Z(A)}, \qquad Z_n(A) = \mathrm{tr}\, e^{-n H_A}$$



$$S_n(A) = \frac{1}{1-n} \left[\log Z_n(A) - n \log Z(A) \right]$$

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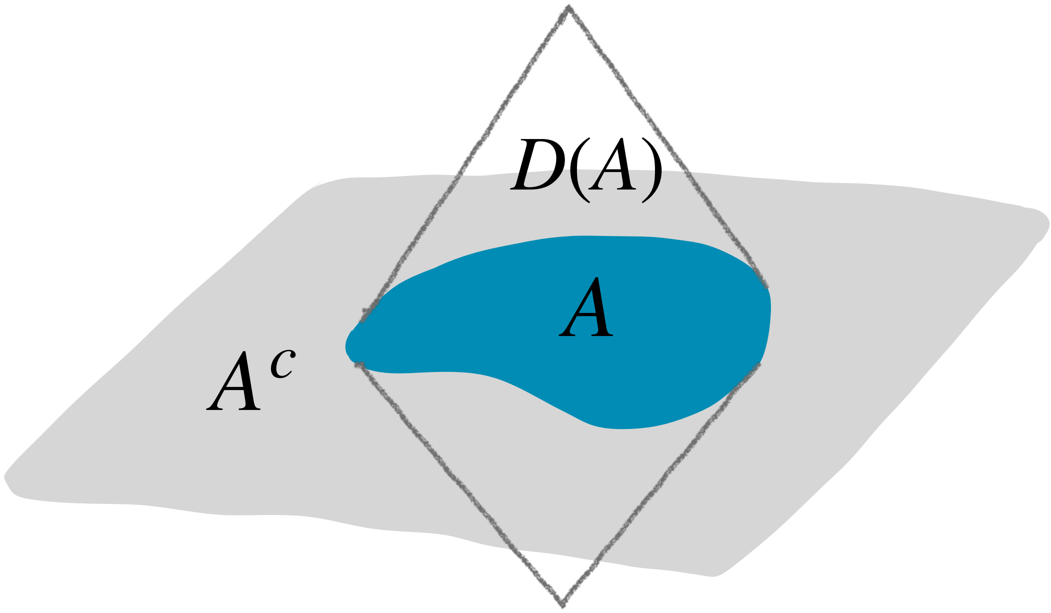

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$S_n(A)$ with $n \to 0$ or $T = 1/n \to \infty$

$$S_n(A) \approx \log Z_n(A) \approx -nF_n$$

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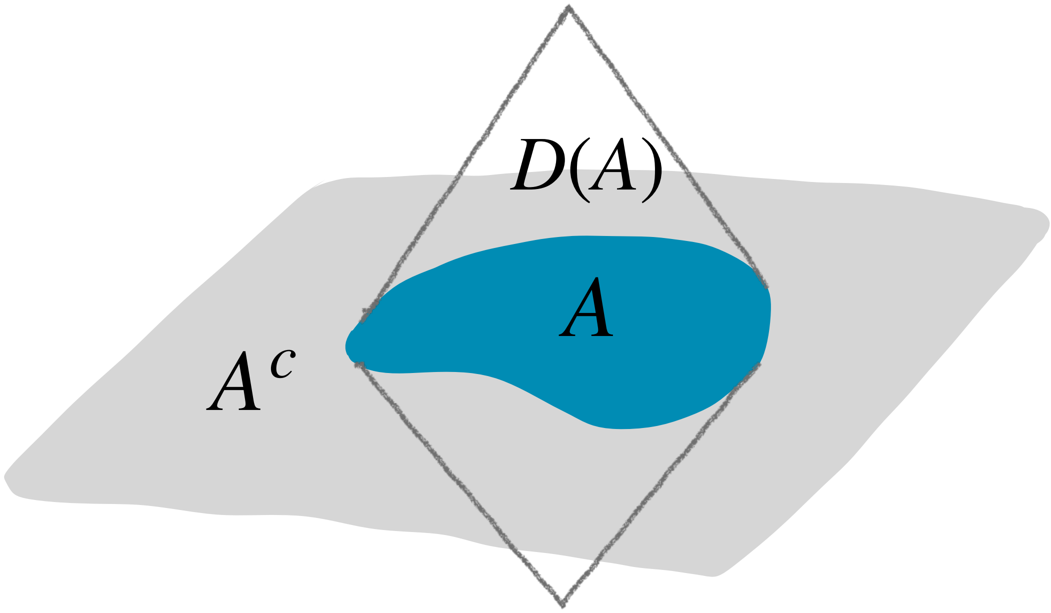

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For a thermal state in a CFT

$$-\beta F = \sigma \frac{V}{\beta^{d-1}}$$

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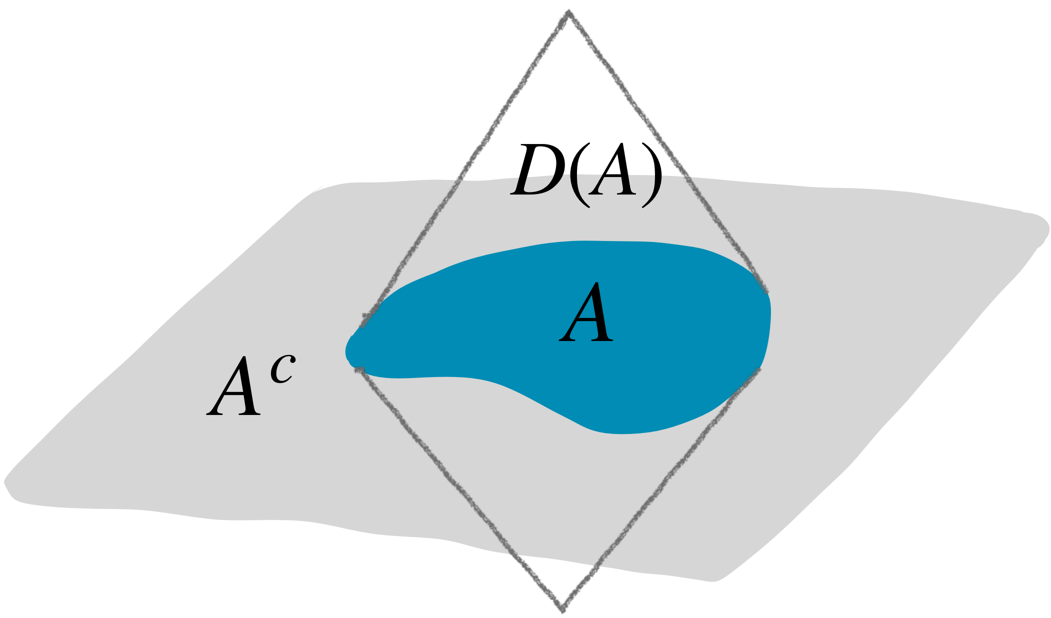

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Conjeture: when $n \to 0$

$$S_n(A) \sim -\, n\, F_n \sim \sigma \frac{g\,(A)}{n^{d-1}}$$

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Checks

Spheres ✔

Multiple intervals ✔

Strip ✔

$$S_n(A) \sim \sigma \frac{g(A)}{n^{d-1}}$$

$$g(A) = \frac{1}{\mathrm{vol}(\mathbb{S}^{d-2})} \int_A d^{d-1}x \int \frac{d^{d-2}\Omega}{\beta^{d-1}(x, \hat{\Omega})}$$

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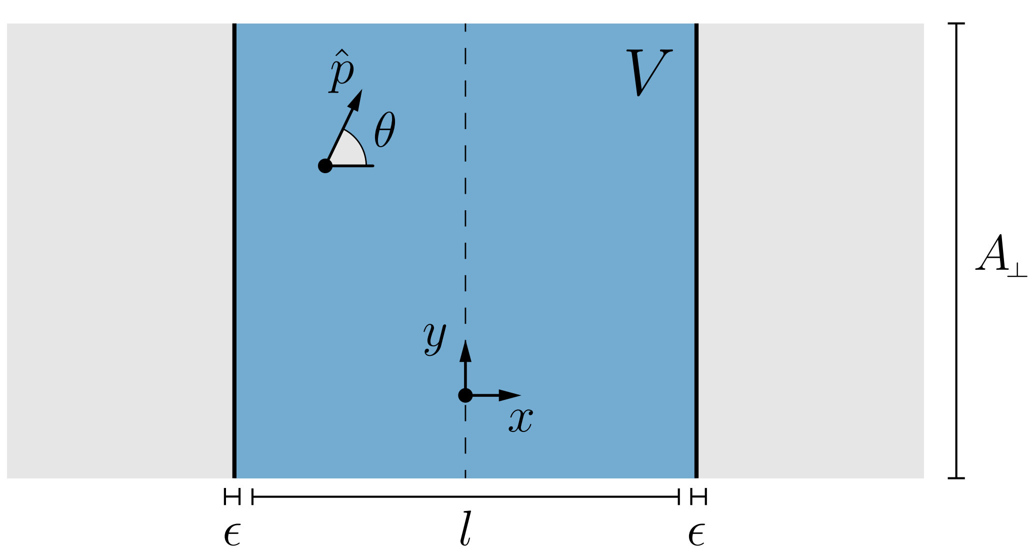

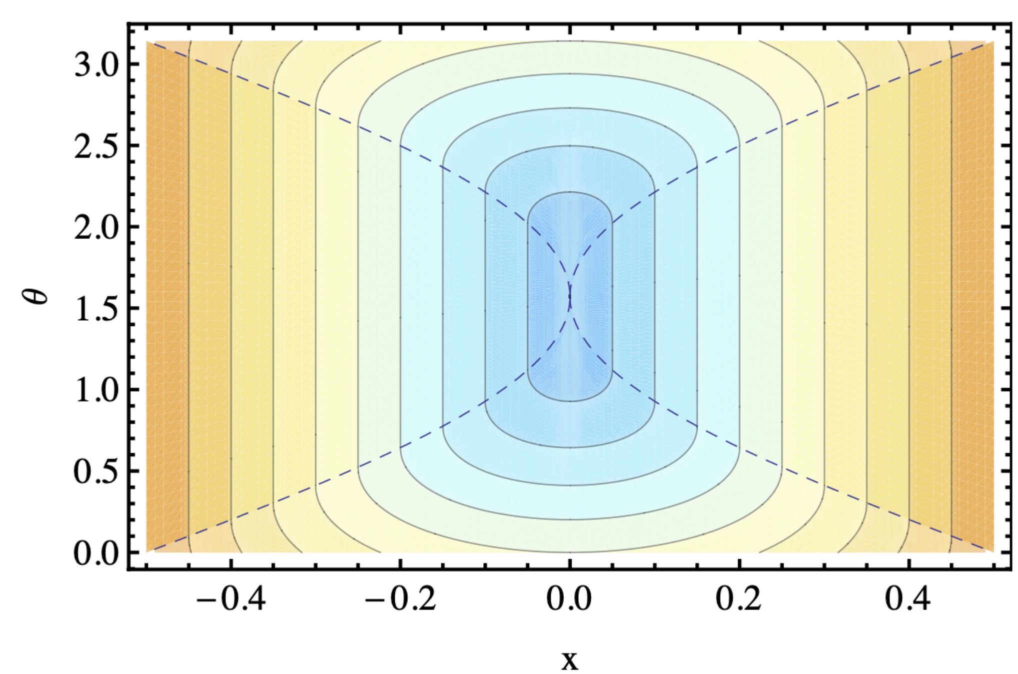

For free theories

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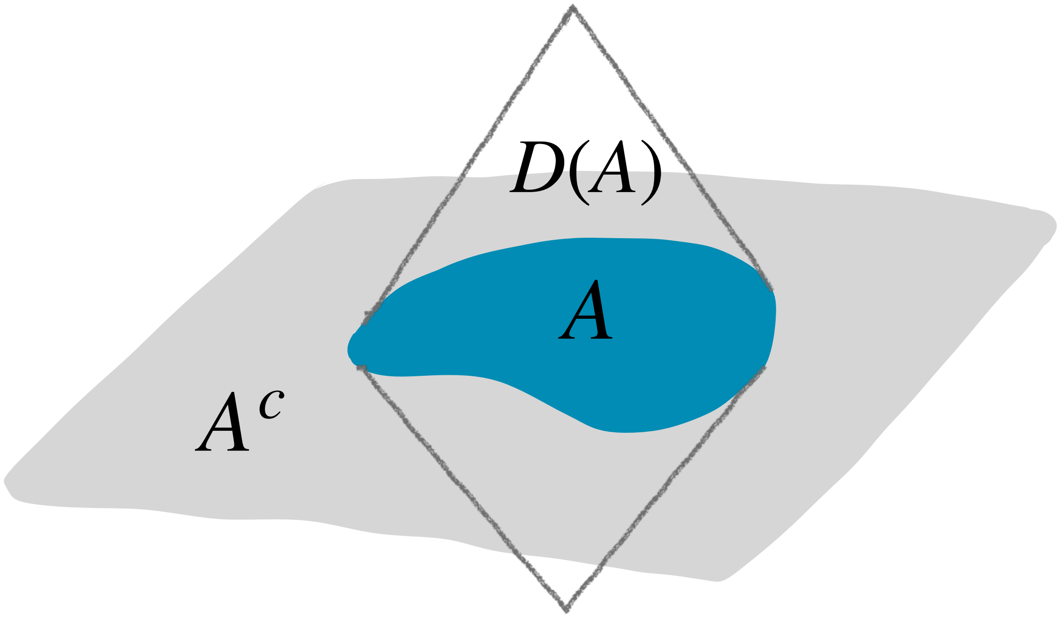

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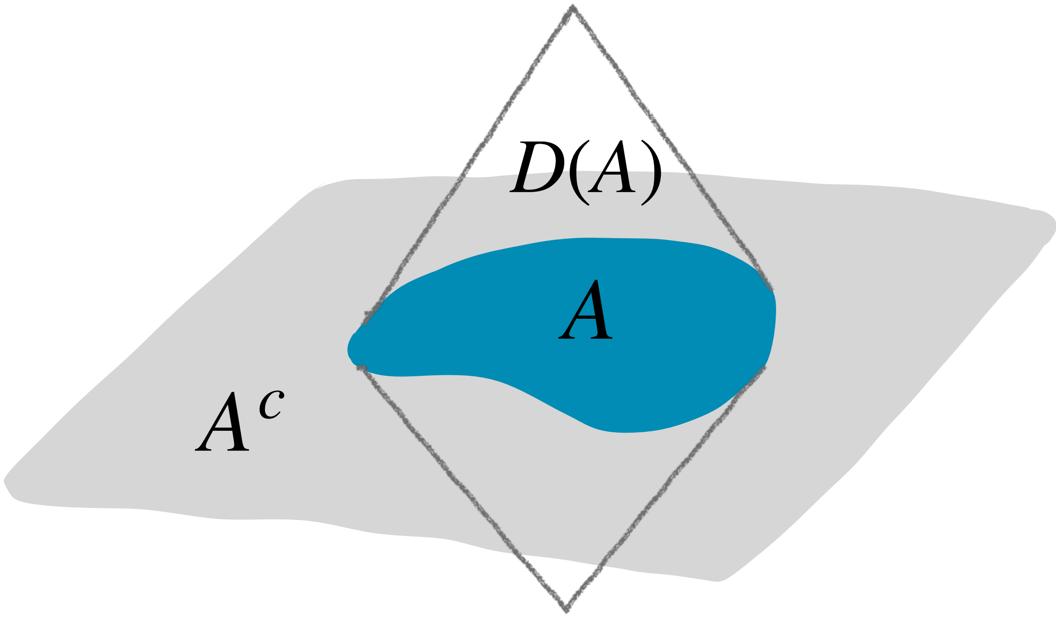

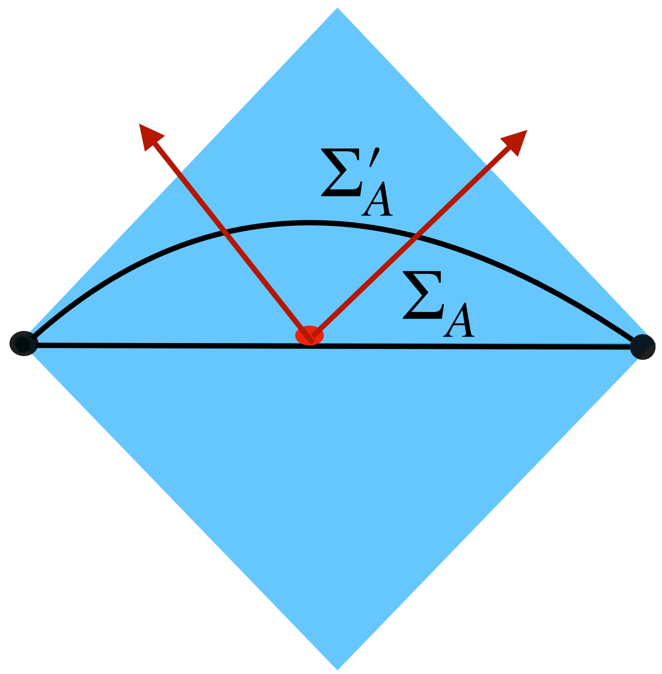

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$$\psi(x,p) = \frac{(\pm)\log\left(1 \pm e^{-nS_{rel}(x,p)}\right)}{(2\pi)^{d-1}}$$

Current and energy momentum

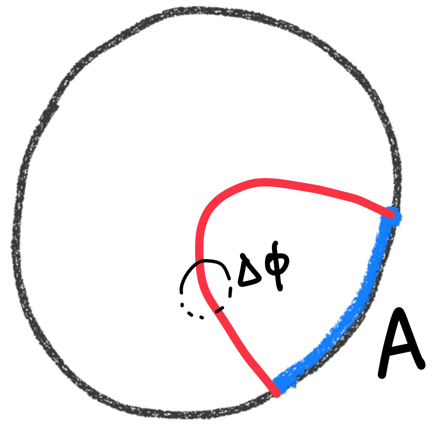
$$J^\mu = \frac{\sigma}{n^{d-1}} \frac{1}{\mathrm{vol}(\mathbb{S}^{d-2})} \int d\tilde{\Omega}_{d-2} \frac{\hat{p}^\mu}{\beta(x,\hat{p})^{d-1}}\,, \qquad T^{\mu\nu} = \frac{\sigma}{n^{d-1}} \frac{(d-1)}{\mathrm{vol}(\mathbb{S}^{d-2})} \int d\tilde{\Omega}_{d-2} \frac{\hat{p}^\mu \hat{p}^\nu}{\beta(x,\hat{p})^d}$$

Small n Rényi entropies in holography

$$\rho_A^{(n)} = \frac{e^{-nH_A}}{\mathrm{tr}[e^{-nH_A}]}, \qquad \langle T_{\mu\nu} \rangle_n = \mathrm{tr}[\hat{T}_{\mu\nu} \rho_A^{(n)}]$$

Dual geometry [Dong, 2016] cosmic brane with tension $T_n = \dfrac{n-1}{4nG_N}$

$$\tilde{S}_n(A) = \min \frac{\mathrm{Area(Cosmic\ Brane_n)}}{4G_N}$$



where $\tilde{S}_n(A) = n^2 \partial_n \left(\dfrac{n-1}{n} S_n(A) \right)$,

and $\tilde{S}_n(A)$ is vN entropy of $\rho_A^{(n)}$

$$\Delta\varphi = 2\pi \frac{n-1}{n}$$

Small n Rényi entropies in holography

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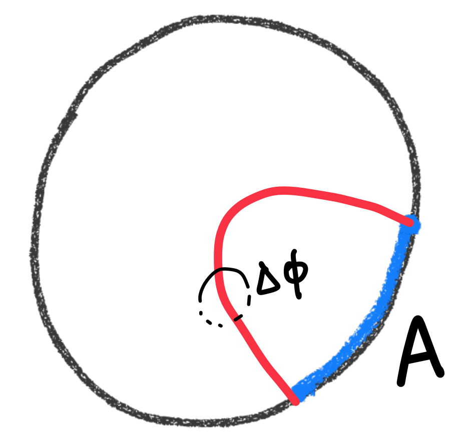

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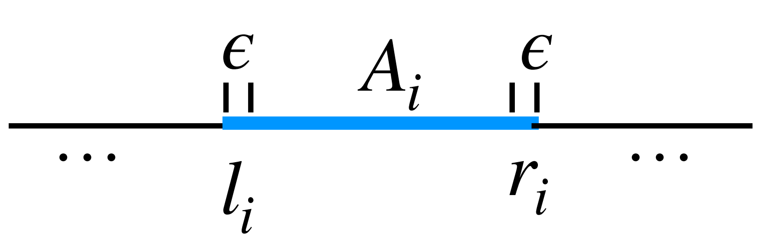
In 2d local temperatures are universal



$$T_{zz}(z) = K_n\left(\sum_{i=1}^{N}\left[\frac{1}{z-l_i}+\frac{1}{r_i-z}\right]\right)^2\,,$$

In 2d the general solution is known: [Bañados, 1999]

$$L(z) = -\frac{1}{4G_N} T_{zz}(z)\ , \qquad \bar{L}(\bar{z}) = -\frac{1}{4G_N} T_{\bar{z}\bar{z}}(\bar{z})$$

Exact Fefferman-Graham form

$$ds^2 = \frac{d\zeta^2}{\zeta^2} + \frac{1}{\zeta^2}\left(dz + \zeta^2 \bar{L}(\bar{z})\, d\bar{z}\right)\left(d\bar{z} + \zeta^2 L(z)\, dz\right)$$

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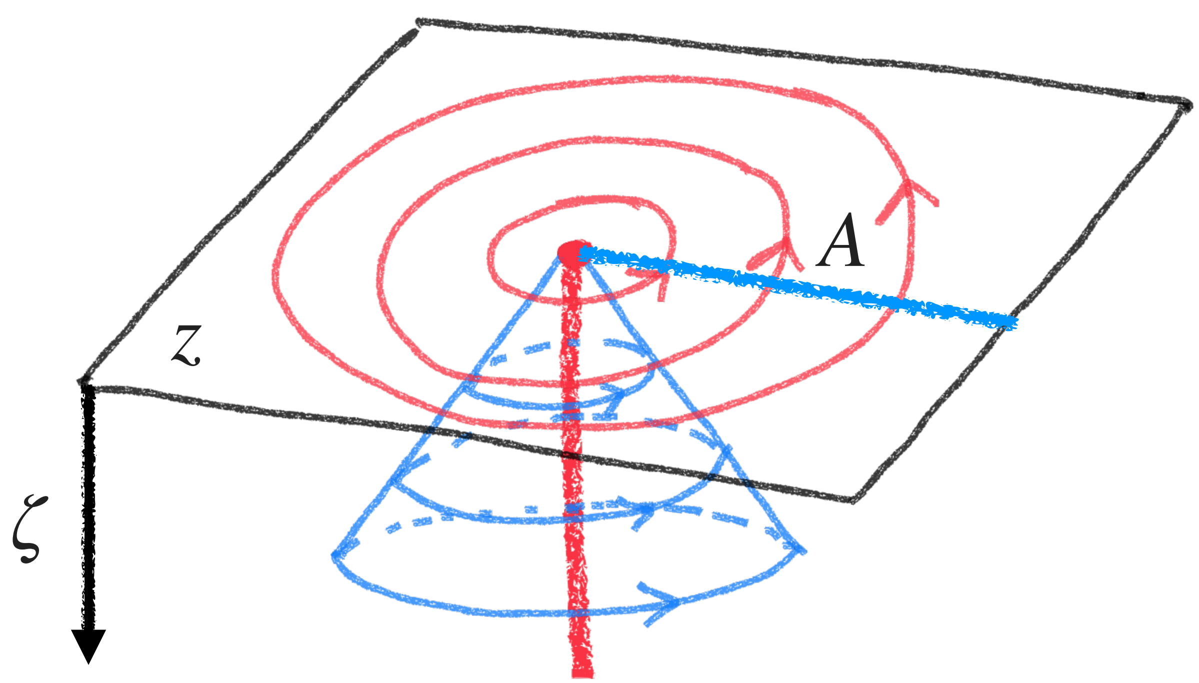
Exact Fefferman-Graham form

$$ds^2 = \frac{d\zeta^2}{\zeta^2} + \frac{1}{\zeta^2}\left(dz + \zeta^2 \bar{L}(\bar{z})\,d\bar{z}\right)\left(d\bar{z} + \zeta^2 L(z)\,dz\right)$$

Rindler: $$L(z) = \frac{|K_n|^2}{z^2}\,, \qquad z = r\,e^{i\theta}\,,$$

$$ds^2 = \frac{d\zeta^2}{\zeta^2} + |K_n|^2\left(1+\frac{\zeta^2}{r^2}\right)^2\frac{dr^2}{\zeta^2} + |K_n|^2\left(1-\frac{\zeta^2}{r^2}\right)^2\frac{r^2}{\zeta^2}d\theta^2\,, \qquad \zeta \in [0, r]$$



$$\Delta\theta = 2\pi\frac{n-1}{n} \quad \leftrightarrow \quad |K_n|^2 = \frac{1-n^2}{4n^2}$$

In 2d the general solution is known: [Bañados, 1999]

$$L(z) = -\frac{1}{4G_N}T_{zz}(z)\,, \qquad \bar{L}(\bar{z}) = -\frac{1}{4G_N}T_{\bar{z}\bar{z}}(\bar{z})$$

Exact Fefferman-Graham form

$$ds^2 = \frac{d\zeta^2}{\zeta^2} + \frac{1}{\zeta^2}\left(dz + \zeta^2\bar{L}(\bar{z})\,d\bar{z}\right)\left(d\bar{z} + \zeta^2 L(z)\,dz\right)$$

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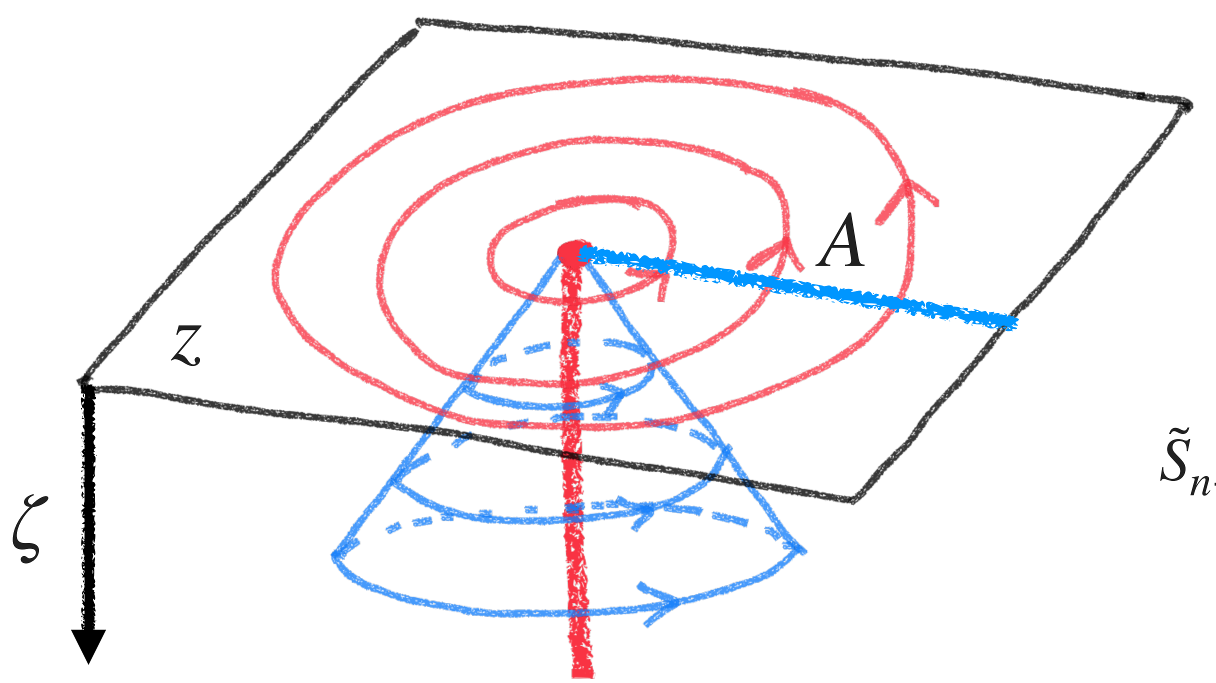

$$\Delta\theta = 2\pi\frac{n-1}{n} \quad \leftrightarrow \quad |K_n|^2 = \frac{1-n^2}{4n^2}$$

$$ds^2_{\text{ind}} = 4|K_n|^2\frac{dr^2}{r^2}$$

$$\tilde{S}_{n\to 0} = \frac{\text{Area(B}_\text{n})}{4G_N} \approx \frac{1}{4G_N\,n}\int\frac{dr}{r} = \frac{1}{4G_N\,n}\log\left(\frac{L_{IR}}{\epsilon}\right)$$

$$\tilde{S}_n \approx 2S_n$$

For arbitrary number of intervals:

$$w(z) = \prod_{i=1}^{N} \left(\frac{z - l_i}{r_i - z} \right) \quad \text{and} \quad L(z) = \frac{|K_n|^2}{w^2(z)} \left(\frac{\partial w(z)}{\partial z} \right)^2$$

$$|K_n|^2 = \frac{1 - n^2}{4n^2}$$

Write Bañados solution in $w = re^{i\theta}$

$$ds^2 = \frac{d\zeta^2}{\zeta^2} + |K_n|^2 \frac{(1+\chi^2)^2}{\chi^2} \frac{dr^2}{r^2} + |K_n|^2 \frac{(1-\chi^2)^2}{\chi^2} d\theta^2 , \quad \text{where} \quad \chi = \frac{r}{\zeta} \left| \frac{\partial w}{\partial z} \right|$$

Surface of zero determinant $\chi = 1 \rightarrow \zeta = r \left| \frac{\partial w}{\partial z} \right|$

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Surface of zero determinant $\chi = 1 \rightarrow \zeta = r \left| \frac{\partial w}{\partial z} \right|$

$$ds^2_{\text{ind}} = \frac{(\partial_r \zeta \, dr + \partial_\theta \zeta \, d\theta)^2}{\zeta^2} + 4|K_n|^2 \frac{dr^2}{r^2} \underset{n \to 0}{\approx} \frac{1}{n^2} \frac{dr^2}{r^2}$$

For arbitrary number of intervals:

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$$\tilde{S}_{n \to 0} = \frac{\text{Area}(\text{B}_\text{n})}{4G_N} \approx \frac{L_{AdS}}{4G_N} \frac{1}{n} \left(\sum_{i,j} \log |r_j - l_i| - \sum_{i<j} \log |r_j - r_i| - \sum_{i<j} \log |l_j - l_i| - N \log \epsilon \right)$$

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Surface of zero determinant $\chi = 1 \to \zeta = r \left| \frac{\partial w}{\partial z} \right|$

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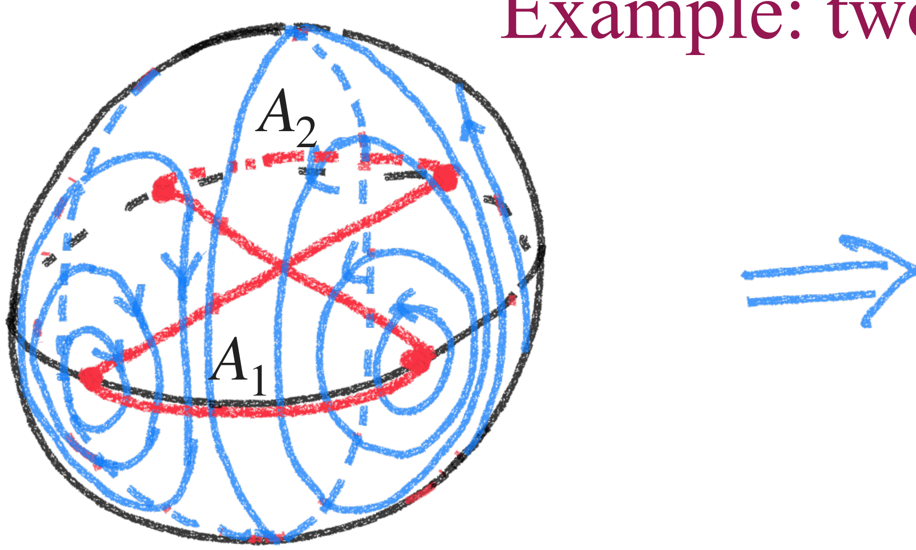
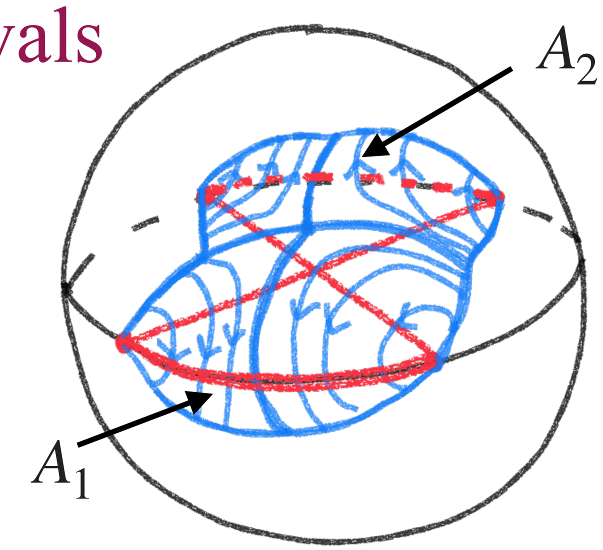
Example: two intervals





Summary

- ET in QFT, computed in Euclidean and propagated in Lorentzian
- Effective description of highly excited states in terms of ET. Allows to calculate Rényi entropies at large modular temperature
- Discussed holographic dual to small n Rényi entropies in $d = 2$

Future work

- Can we find average ET from a fluid problem
- Understand the extent of universality in the ETs, **Holography**?
- Study the Euclidean conserved currents. Is there a **Euclidean Boltzmann equation**?

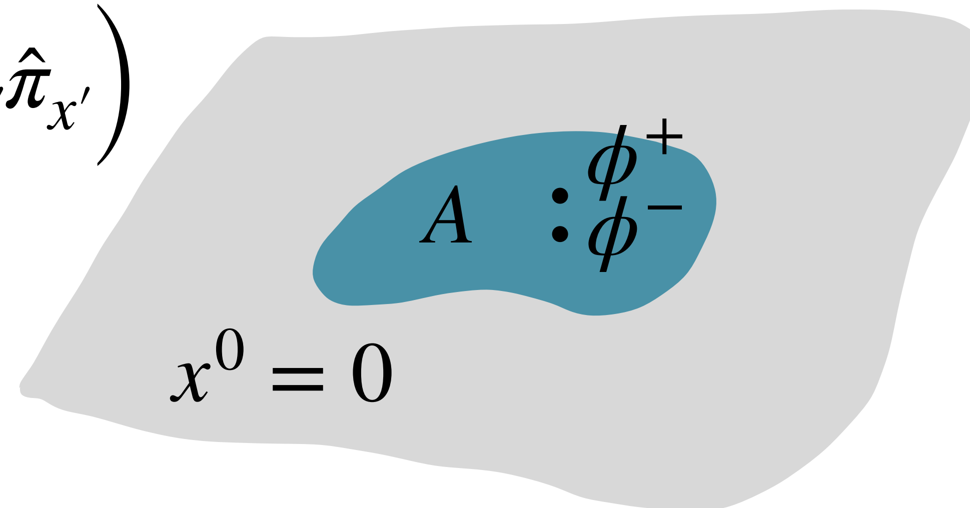
Thanks for listening!

Additional slides

Modular Hamiltonian for free fields

$$H_A \sim \int_A d^{d-1}x \int_A d^{d-1}x' \left(\hat{\phi}_x M_{xx'} \hat{\phi}_{x'} + \hat{\pi}_x N_{xx'} \hat{\pi}_{x'}\right)$$

$$H_A \sim \int d\lambda \int ds\, (2\pi s)\, \hat{a}^\dagger_{s,\lambda} \hat{a}_{s,\lambda}$$



[Arias, Casini, Huerta, Pontello: 2017]

$$\Box\, \phi_{s,\lambda} = 0\,, \qquad \phi^+_{s,\lambda}(\vec{x}) = e^{-2\pi s}\, \phi^-_{s,\lambda}(\vec{x})\,, \qquad x^0 = 0\,, \qquad \vec{x} \in A\,.$$

$$a^\dagger_{s,\lambda,0} = \int_A d^{d-1}x\, \partial_0 \phi^+_{s,\lambda}\, \hat{\phi}_x\,, \qquad a^\dagger_{s,\lambda,1} = \int_A d^{d-1}x\, \phi^+_{s,\lambda}\, \hat{\pi}_x\,,$$

Eikonal limit $|s| \gg 1$ [Arias, Blanco, Casini, Huerta: 2017]

$$\phi(x) = f(x)\, e^{s\,\alpha(x)}$$

Derivation of the conjeture

For QFTs with a free UV fixed point

The free energy for free Bosons (−) o Fermions (+)

$$-\beta F = \log Z(\beta) = \log \mathrm{tr} e^{-\beta H}\,, \qquad H = \int d^{d-1}p\, E_p\, a_p^\dagger a_p$$

$$-\beta F = V \int \frac{d^{d-1}p}{(2\pi)^{d-1}} (\pm) \log\left(1 \pm e^{-\beta E_p}\right)\,,$$

The "free energy" $n\, F_n(A)$, $n \to 0$

$$H_A \approx \int_A d^{d-1}x \int d^{d-1}p\, \beta(x,\hat{p})\, |\vec{p}\,|\, a_p^\dagger(x) a_p(x)\,. \qquad E_p \approx |\vec{p}\,|$$

$$-nF_n(A) = \log \mathrm{tr} e^{-nH_A} = \int_A d^{d-1}x \int \frac{d^{d-1}p}{(2\pi)^{d-1}} (\pm) \log\left(1 \pm e^{-n\beta(x,\hat{p})|\vec{p}\,|}\right)\,,$$

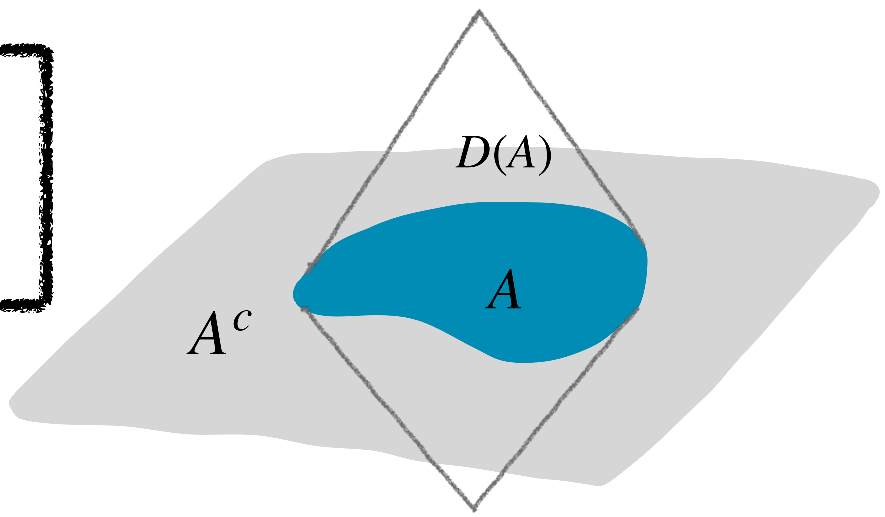
$$S_n(A) \sim -\, n\, F_n \sim \sigma \frac{g(A)}{n^{d-1}}$$

$$g(A) = \frac{1}{\mathrm{vol}(\mathbb{S}^{d-2})} \int_A d^{d-1}x \int \frac{d^{d-2}\Omega}{\beta^{d-1}(x,\hat{\Omega})}$$

$$\sigma = \frac{\mathrm{vol}(\mathbb{S}^{d-2})}{(2\pi)^{d-1}} \int_0^\infty d\chi\, \chi^{d-2} (\pm) \log(1 \pm e^{-\chi})$$

Rényi currents

$$S_n(A) = \int_{\Sigma_A} d\sigma \, \hat{n}_\mu \, J^\mu(x)$$



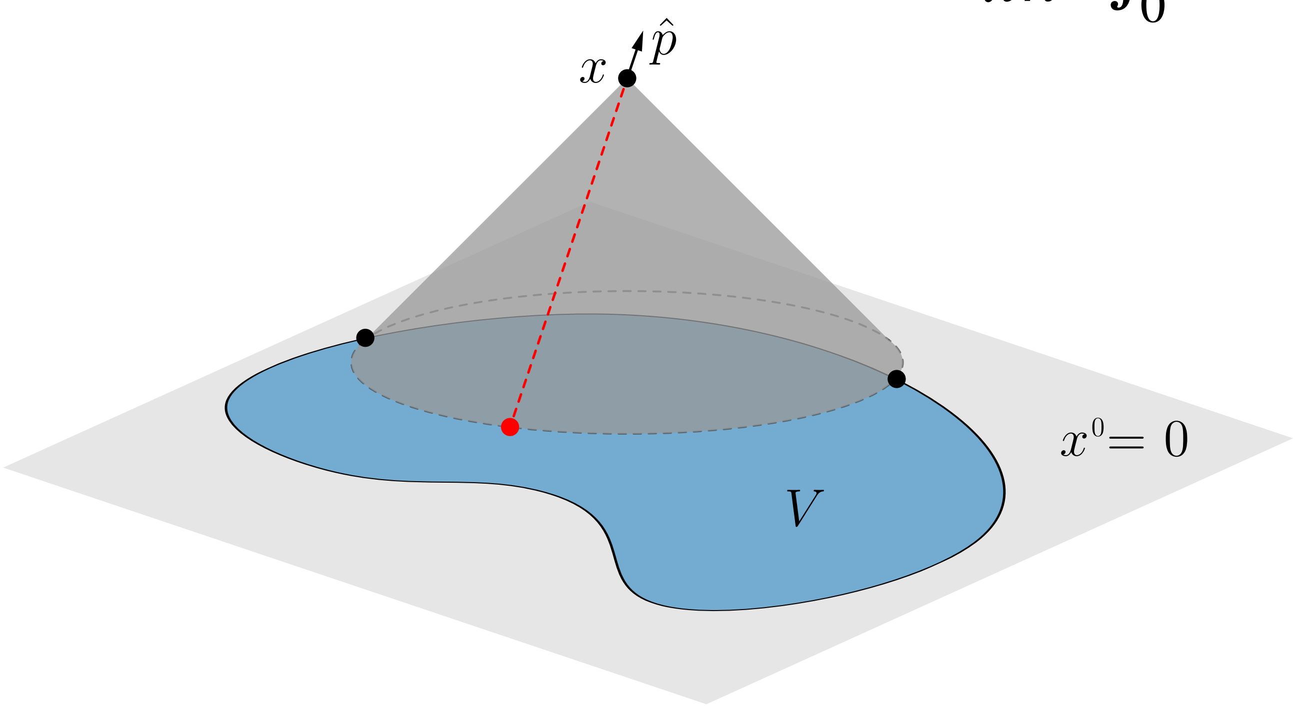
Sphere and multi-intervals have Killing symmetry

Perfect fluid

$$J^\mu = \frac{\sigma \; u^\mu}{n^{d-1}\beta_{\text{th}}^{d-1}(x)}\,, \qquad T^{\mu\nu} = \frac{\sigma}{n^d}\frac{\left(du^\mu u^\nu + g^{\mu\nu}\right)}{\beta_{\text{th}}(x)^d} \qquad \beta_{\text{th}}(x) = (-\xi^2)^{1/2}$$

Wall?

$$T^{\mu\nu}(x) = \frac{\sigma}{\pi n^3}\int_0^{2\pi} d\theta \frac{\hat{p}^\mu \hat{p}^\nu}{\beta(x,\hat{p})^3} = \text{Diag}\{\rho, P_x, P_\perp\}\,, \qquad t = 0$$

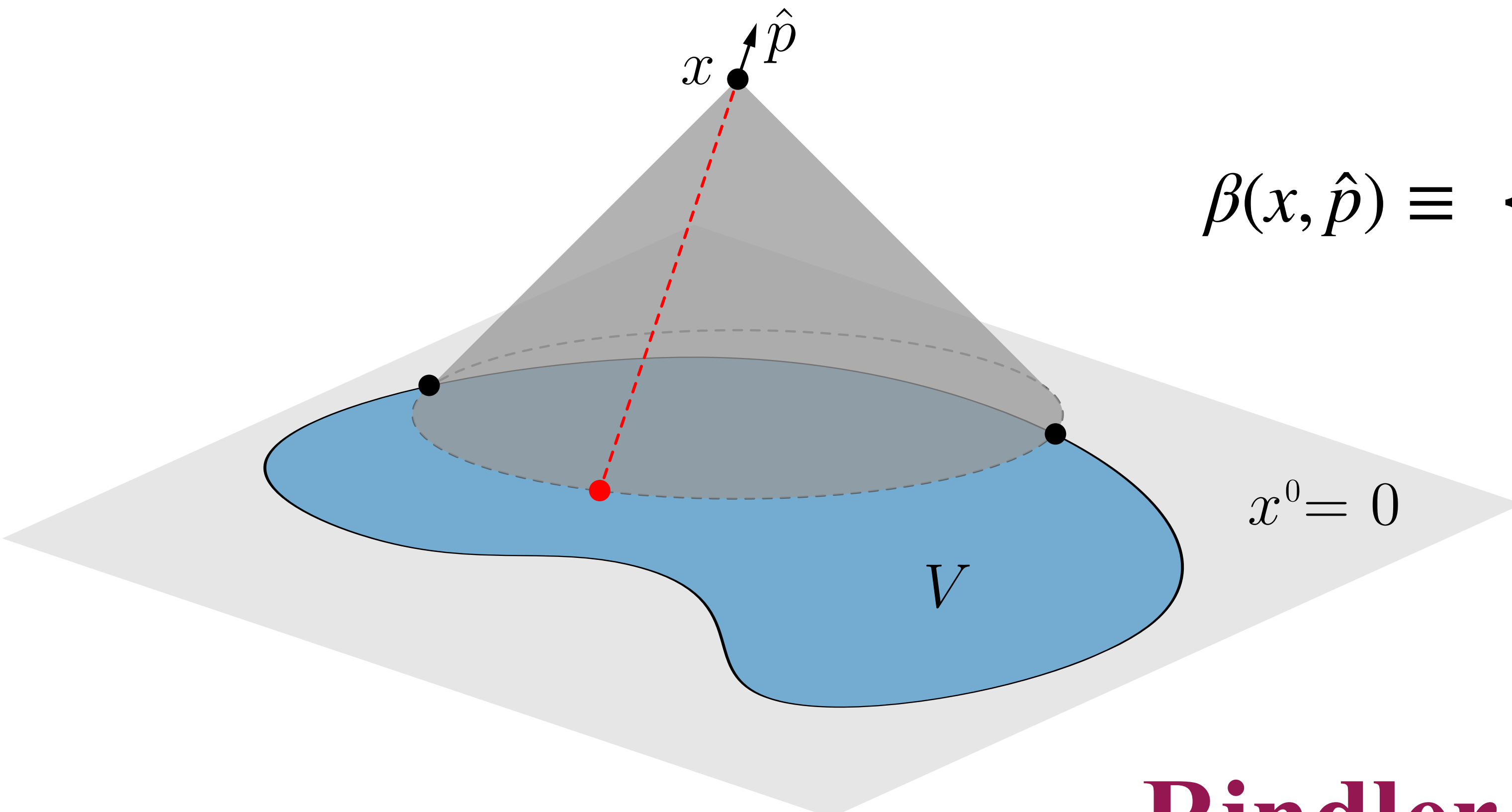


$$J^\mu = \frac{\sigma}{n^{d-1}}\frac{1}{\text{vol}(\mathbb{S}^{d-2})}\int d\tilde{\Omega}_{d-2}\frac{\hat{p}^\mu}{\beta(x,\hat{p})^{d-1}}$$

$$\downarrow$$

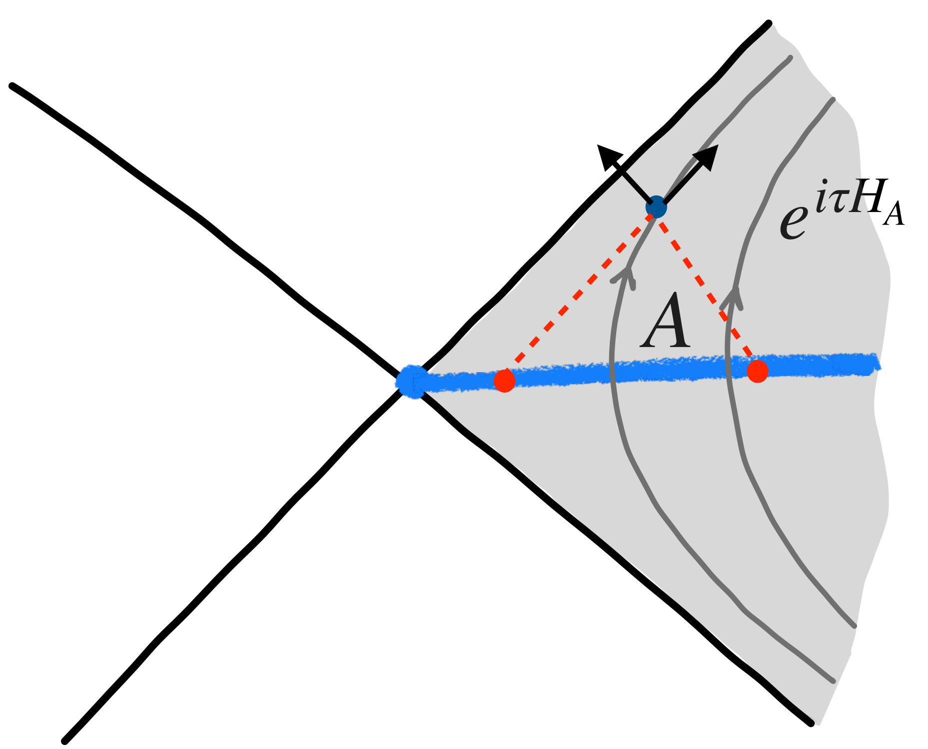
$$J^\mu = \frac{\sigma \; u^\mu}{n^{d-1}\beta_{\text{th}}^{d-1}(x)}\,,$$

Wall



$$\beta(x,\hat{p}) \equiv \begin{cases} \pi(1-2|x|)\,, & \sin\theta > 1-2|x| \\ \dfrac{\pi}{2}\dfrac{\cos^2\theta - 4x^2}{1-|\sin\theta|}\,, & \sin\theta < 1-2|x| \end{cases}$$

Rindler

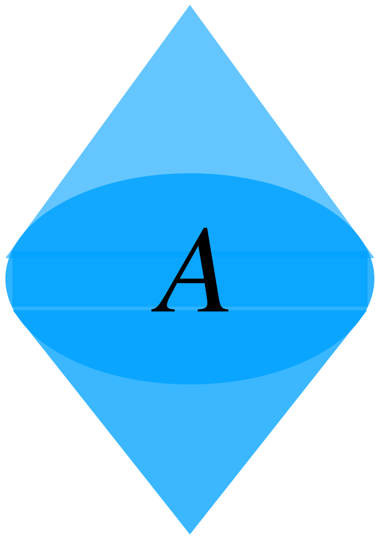


$$\beta_{\mathrm{Unruh}}(x) = 2\pi\, x_1 = 2\pi\sqrt{x_1^2 - t^2}$$

$$J_n^\mu = \frac{\sigma\ u^\mu}{n^{d-1}\beta_{\mathrm{th}}^{d-1}(x)}\,, \qquad \beta_{\mathrm{th}}(x) = 2\pi\sqrt{x_1^2 - t^2}$$

$$u^\mu = \left(\hat{x}\frac{t}{\sqrt{x^2-t^2}} + \hat{t}\frac{x}{\sqrt{x^2-t^2}}\right)^\mu$$

Examples/checks of the conjeture



$$S_n(A) \sim \sigma \frac{g(A)}{n^{d-1}}$$

$$g(A) = \frac{1}{\mathrm{vol}(\mathbb{S}^{d-2})} \int_A d^{d-1}x \int \frac{d^{d-2}\Omega}{\beta^{d-1}(x, \hat{\Omega})}$$

Spheres

$$\beta(r) = 2\pi \frac{(R^2 - r^2)}{2R}$$

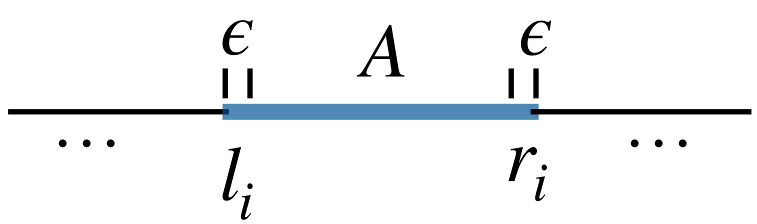
$$g(A) = \int_A \frac{d^{d-1}x}{\beta^{d-1}(x)} = \frac{\mathrm{vol}_{\mathrm{reg}}(\mathbb{H}^{\mathrm{d}-1})}{(2\pi)^{d-1}},$$

$$\lim_{n\to 0} S_n = \frac{\sigma}{n^{d-1}} \frac{\mathrm{vol}_{\mathrm{reg}}(\mathbb{H}^{\mathrm{d}-1})}{(2\pi)^{d-1}}, \qquad \lim_{n\to 0} S_n = \sigma \frac{V}{\beta^{d-1}},$$

Universal CFT result that follows from mapping to Hyperbolic space

Examples/checks of the conjeture

$$S_n(A) \sim \sigma \frac{g(A)}{n}$$



$$g(A) = \frac{1}{2}\int_A dx \left[\frac{1}{\beta(x,+)} + \frac{1}{\beta(x,-)}\right]$$

[Arias, Blanco, Casini, Huerta: 2017]
[Arias, Casini, Huerta, Pontello: 2017]

Multiple intervals

$$\beta(x) = 2\pi \left(\sum_{i=1}^{N} \left[\frac{1}{x-l_i} + \frac{1}{r_i - x} \right] \right)^{-1} \quad \rightarrow \quad g(A) = \frac{1}{2\pi} \sum_{i=1}^{N} \int_A \left(\frac{dx}{x-l_i} + \frac{dx}{r_i - x} \right)$$

$$S_n(A) \approx \frac{\sigma}{n\,\pi} \left(\sum_{ij} \log|r_j - l_i| - \sum_{i<j} \log|r_j - r_i| - \sum_{i<j} \log|l_j - l_i| - N \log \epsilon \right) .$$

For free fermions $\sigma_f = \pi/6$ [Casini, Huerta: 2009]

Two intervals

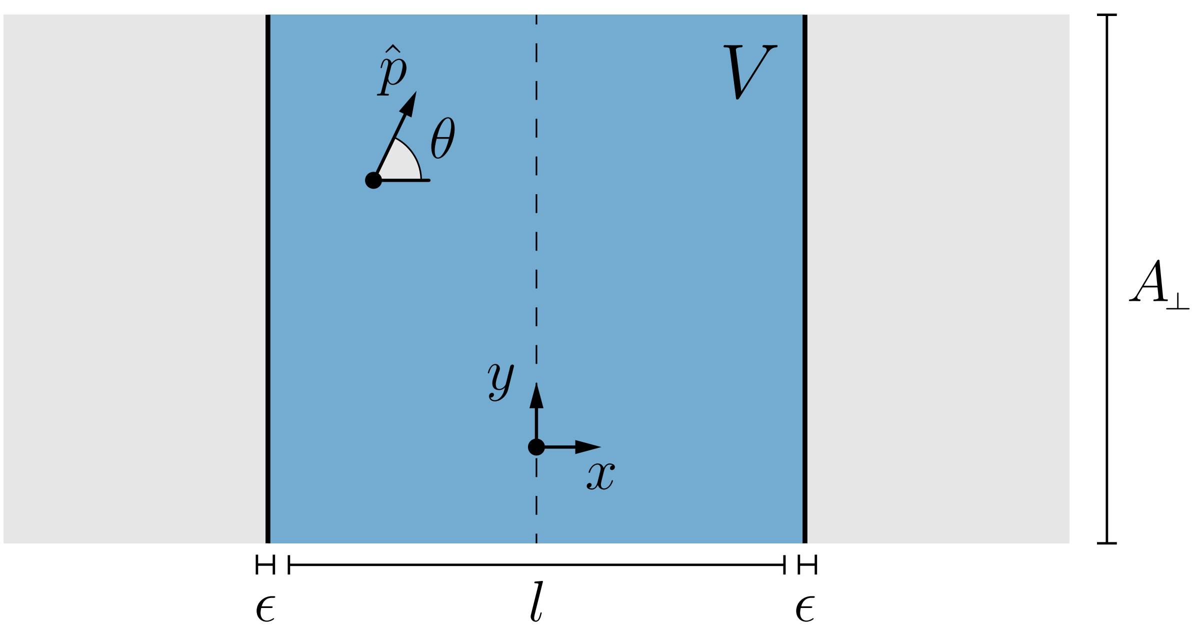
For a free scalar [Arias, Casini, Huerta, Pontello: 2018]

Examples/checks of the conjeture

$$S_n(A) \sim \sigma \frac{g(A)}{n^{d-1}}$$

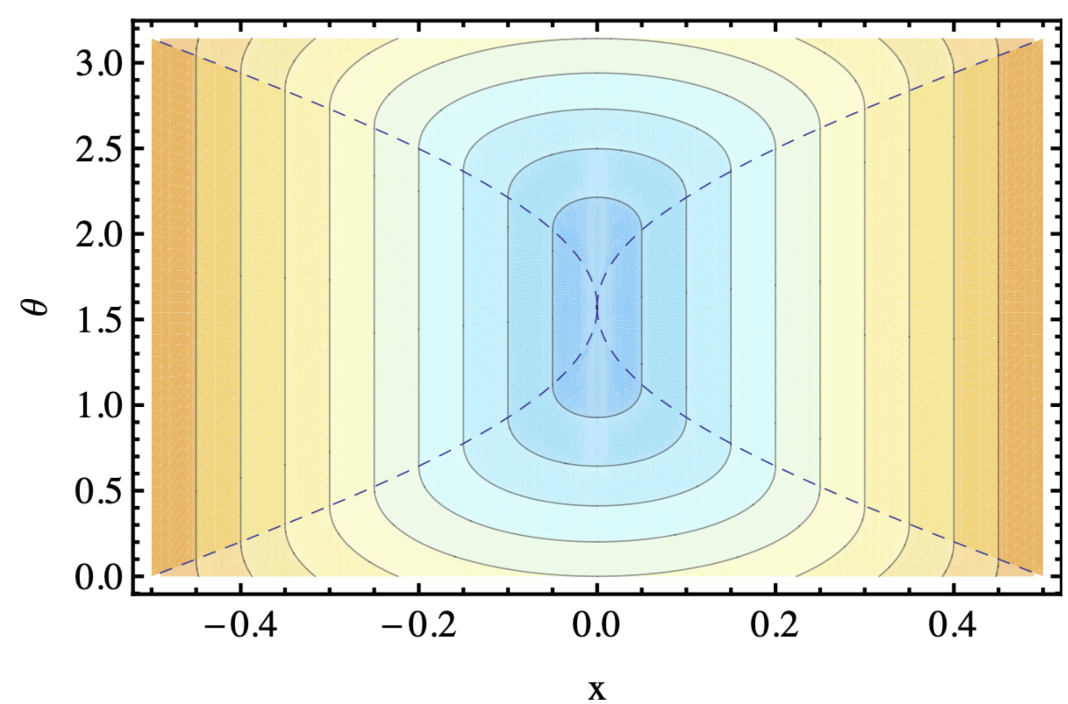
$$\beta(x,\hat{p}) \equiv \begin{cases} \pi(1-2|x|)\,, & \sin\theta > 1-2|x| \\ \frac{\pi}{2}\frac{\cos^2\theta - 4x^2}{1-|\sin\theta|}\,, & \sin\theta < 1-2|x| \end{cases}$$

Wall



[Casini, Huerta: 2005]

$$S_n(A) \sim \alpha_n \frac{A_\perp}{\epsilon^{d-2}} - \kappa_n \frac{A_\perp}{l^{d-2}}$$



[Arias, Casini, Huerta, Pontello: 2017]