Bootstrapping relativistic transport from causality

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Based on [arXiv:2212.07434] and [arXiv:2305.07703] with Michal Heller, Michał Spaliński, and Ben Withers

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This talk in three slides

- Relativistic hydrodynamics: EFT for IR behavior of conserved currents.

➥ Instrumental in nuclear physics, astrophysics and cosmology.

- When stochastic fluctuations are negligible, relativistic hydrodynamics is a classical EFT built in a gradient expansion: operators + transport coefficients

$$T_{\mu\nu} = T_{\mu\nu}^{ideal} - \eta[T]\nabla_{\langle\mu}U_{\nu\rangle} - \xi[T]\Delta_{\mu\nu}\nabla_{\alpha}U^{\alpha} + O(\nabla^2)$$

Does an arbitrary sequence of transport coefficients define a valid theory of relativistic hydrodynamics?

- No!

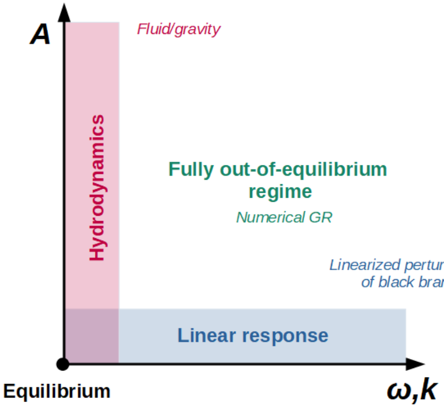
- Relativistic causality implies that all consistent theories of relativistic hydrodynamics are contained inside a universal convex geometry in the infinite-dimensional space of transport coefficients: the **hydrohedron**.

The actual talk

1. Basics
2. The natural UV cutoff of relativistic hydrodynamics
3. Two-sided bounds on transport and the hydrohedron

Basics

QFT near thermal equilibrium



In this talk: relativistic hydrodynamics $\bigcap$ linear response theory

Linear response basics

- Basic object: retarded two-point function.

$$G_R(x,y) = -i\theta(x^0 - y^0)\langle[\mathcal{O}(x),\mathcal{O}(y)]\rangle \longrightarrow G_R(\omega,\mathbf{k}), \quad p^\mu = (\omega,\mathbf{k}), \quad \mathbf{k} = (k,\mathbf{0}).$$

- Basic notion: mode. A mode is a singularity of $G_R(\omega,k)$ characterized by a dispersion relation

$$\omega = \omega(k).$$

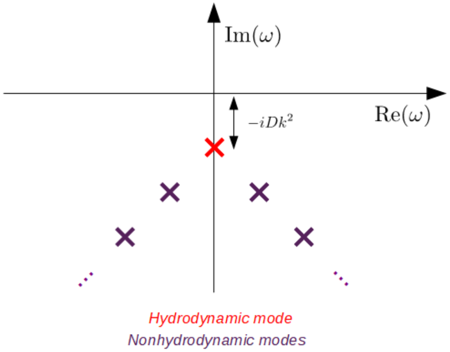
➤ Modes organize the spatiotemporal response of the equilibrium state to external perturbations,

$$\langle\mathcal{O}(t,\mathbf{x})\rangle = \int_k \rho(t,\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \rho(t,\mathbf{k}) = \sum_n e^{-i\omega_n(k)t}\nu_n(\mathbf{k}) + \ldots$$

- In holography, modes = single poles of $G_R(\omega,k)$ = black brane quasinormal modes.

Two fundamental kinds of modes: hydrodynamic vs nonhydrodynamic

- Hydrodynamic, $\omega_H(k) \to 0$ for $k \to 0$: long-lived and slowly-varying excitation.
- Nonhydrodynamic, $\omega_{NH}(0)$ finite: transient excitation.



Dispersion relations and the hydrodynamic gradient expansion

- The crucial fact for us is that relativistic hydrodynamics predicts the small-k expansion of the dispersion relations,

$$\omega(k) = \sum_{n=1}^{\infty} c_n k^n, \quad c_n = \alpha_n + i\beta_n,$$

- c_n: transport coefficients.

- **Example**: hydrodynamic shear and sound modes of a neutral relativistic fluid,

$$\omega_{\mathrm{shear}}(k) = -iDk^2 + \ldots, \quad \omega^{\pm}_{\mathrm{sound}}(k) = \pm c_s k - i\frac{\Gamma_s}{2}k^2 + \ldots$$

Relativistic causality

Relativistic causality imposes a positivity constraint on $\omega(k)$

- Causality: $G_R(x, 0) = 0$ outside the future lightcone.

➥ If $G_R(x, 0)$ is a tempered distribution, then $G_R(p)$ is analytic in the open future lightcone of $\text{Im}(p)$,

$$p^\mu = (\omega, \mathbf{k}), \quad -\text{Im}(\omega)^2 + \text{Im}(\mathbf{k})^2 < 0, \quad \text{Im}(\omega) > 0.$$

[Streater & Wightman, '89] [Haag, '92]

➥ All dispersion relations must obey the fundamental causality condition

$$|\text{Im}(k)| - \text{Im}(\omega(k)) \geq 0$$

Analyticity

- We will assume that the dispersion relations in the microscopic QFT are **analytic at $k = 0$**.

➥ This means that their small-k expansion can be matched exactly to the prediction of relativistic hydrodynamics.

➤ Physically, the absence of nonanalytic terms implies that stochastic fluctuations are negligible.

➤ This assumption restricts us to large-N QFTs.

Relativistic causality & analyticity consequences

In the rest of the talk, we will explore the consequences
of the causality inequality and analyticity

- Sanity check: at low orders, $|c_s| \leq 1$, $D \geq 0$.
- This is just the tip of the iceberg: relativistic causality & analyticity impose powerful nontrivial constraints on relativistic transport.
- These constraints take the form of a **novel infinite set of two-sided bounds on all the transport coefficients.**
- **These two-sided bounds are optimal when expressed in units of R, the convergence radius of the small-k expansion.**

A word on bounds on transport

- Bounds on transport are not a new topic. Most celebrated example: Kovtun-Son-Starinets conjecture,

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}.$$

- Our methods cannot prove such Planckian lower bounds on diffusivities (see 2310.16948 by Delacrétaz for advances on this front).

- They do provide the sharp counterpart of previous qualitative upper bounds on the diffusivity obtained from causality considerations (see 1706.00019 by Hartman, Hartnoll & Mahajan).

The natural UV cutoff of relativistic hydrodynamics

Empirical properties of R

- R can be measured or computed explicitly provided the underlying microscopic QFT is known.

- R has been under intense scrutiny in holography in recent years.

  [Withers, '18] [Grozdanov, Kovtun, Starinets & Tadic, '19] [Abbasi & Tahery, '20] [Jansen & Pantelidou, '20] [Areán, Davison, Goutéraux & Suzuki, '20] [Baggioli, Gran & Tornso, '21] [Wu, Baggioli & Li, '21] [Asadi, Soltanpanahi & Taghinavaz, '21] [Grozdanov, Starinets & Tadic, '21] [Jeong, Kim & Sun, '21] [Huh, Jeong, Kim & Sun, '21] [Liu & Wu, '21] [Cartwright, Amano, Kaminski, Noronha & Speranza, '21]...

- In every example known to date, R is finite and set by a branch-point singularity of $\omega(k)$.

First-principle constraints on R

The empirical properties of R follow from the causality condition and analyticity [arXiv:2212.07434].

1. If $\omega(k)$ is entire, it is a polynomial of at most degree one,

$$\omega(k) = c_0 + c_1 k.$$

➥ Since dissipative transport coefficients appear at $O(k^2)$ or higher, R is finite in any dissipative relativistic theory of transport.

➥ R is set by the singularity of $\omega(k)$ closest to the origin in the complex k-plane.

2. If $\omega(k)$ has a pole of any finite order, it violates the causality condition sufficiently close to it.

➥ R is set by a branch-point singularity.

Nonhydrodynamic modes are inevitable

Example: $\langle T_{\mu\nu} \rangle$.

(a) Relativistic hydrodynamics predicts a single $\omega_H^{(shear)}$ in the shear channel.

(b) Our analysis predicts that $\omega_H^{(shear)}$ is endowed with a branch-point singularity.

(c) Rotational invariance implies that shear and sound modes degenerate at $k = 0$.

(a+b) Going through the branch cut of $\omega_H^{(shear)}$ to the secondary sheet, we must find a nonhydrodynamic shear mode at $k = 0$.

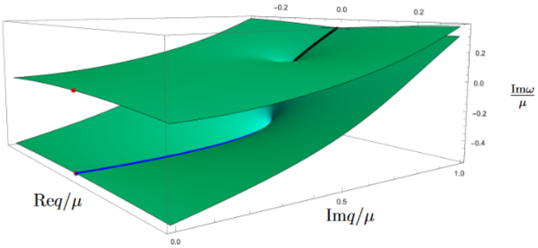
(a+b+c) There has to be a nonhydrodynamic mode in the sound channel as well.

R as a UV cutoff

- *R* marks the momentum scale at which nonhydrodynamic degrees of freedom become relevant and cannot be neglected anymore.

➥ **Natural UV cutoff of relativistic hydrodynamics in the linear response regime.**

- This general interpretation of *R* agrees with all the previous explicit examples in holography: mode collisions.



From arXiv:1803.08058 by Withers

Two-sided bounds on transport and the hydrohedron

The moment problem

We carve out the space of transport coefficients by transforming the causality condition into a **moment problem**.

This strategy is borrowed from the S-matrix bootstrap program.

- The causality condition implies that the unit-normalized density ($k = re^{i\theta}$)

$$\mu(\theta) = \frac{r|\sin(\theta)| - \text{Im}(\omega)(r,\theta)}{4r},$$

is positive semidefinite,

$$\int_{-\pi}^{\pi} d\theta \mu(\theta) P(\theta) \geq 0, \quad \text{for} \quad P(\theta) \geq 0.$$

- Let $P_N(\theta)$ be the most general positive semidefinite trigonometric polynomial of order N, and focus on the moments of $\mu(\theta)$,

$$\gamma_n = \int_{-\pi}^{\pi} d\theta e^{-in\theta} \mu(\theta), \quad n = 0, \pm 1, \pm 2, \ldots$$

- From $\int_{-\pi}^{\pi} d\theta \mu(\theta) P_N(\theta) \geq 0$, it follows that

$$(T_N)_{ij} = \gamma_{j-i}, \quad i,j = 0, 1, \ldots, N$$

is a positive semidefinite Hermitean matrix.

- The moments γ_n are related to the transport coefficients,

$$\omega(k) = \sum_{n=0}^{\infty} \alpha_{2n+1} k^{2n+1} + \sum_{n=1}^{\infty} i\beta_{2n} k^{2n} \qquad \Longrightarrow \qquad \gamma_0 = 1,\ \gamma_{2n+1} = i\frac{\pi}{4} r^{2n} \alpha_{2n+1},$$

$$\alpha_{2n+1}, \beta_{2n} \in \mathbb{R} \qquad \gamma_{2n} = -\frac{1}{(4n^2-1)} - \frac{\pi}{4} r^{2n-1} \beta_{2n}.$$

➥ Imposing that $T_N \succeq 0$ for $N = 1, 2, 3, \ldots$ gives rise to an infinite set of hierarchical two-sided bounds on all transport coefficients.

The hydrohedron

- This infinite set of bounds leads to a convex geometry in the infinite-dimensional space spanned by the transport coefficients: **hydrohedron**.

- Every theory lying outside the hydrohedron is incompatible with relativistic causality: **the landscape of all consistent theories of relativistic transport lies inside or at the edges of the hydrohedron.**

- To make manifest this universal structure, **it is essential to measure the transport coefficients in units of R.**

  R absorbs dependence on: spacetime dimension, coupling strength, temperature and thermodynamic potentials...

$$R = R(T, \lambda_i, \mu_i, \ldots)$$

- This makes learning new general lessons about R extremely valuable.

Diffusion cross-section

- Dispersion relation,

$$\omega(k) = \sum_{n=1}^{\infty} i\beta_{2n}k^{2n}, \quad \beta_{2n} \in \mathbb{R}, \quad \beta_2 = -D$$

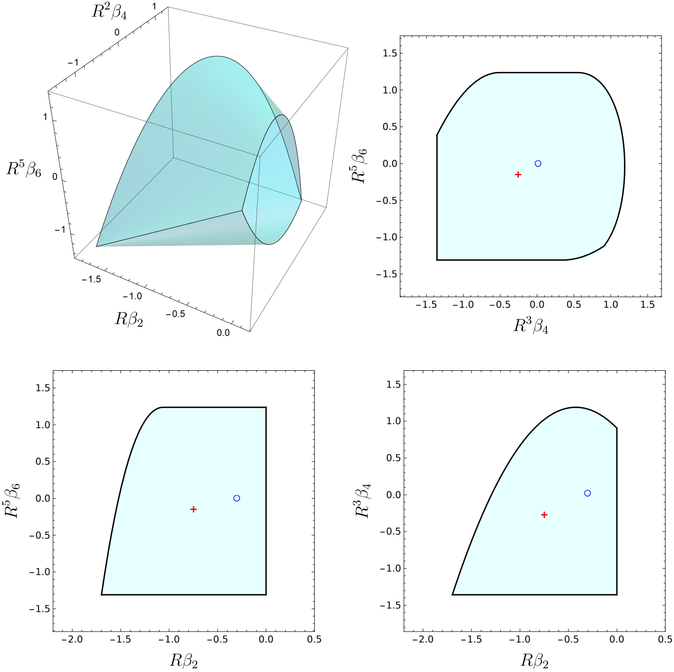
- Hierarchical bounds,

$$-\frac{16}{3\pi} \leq R\beta_2 \leq 0,$$

$$-\frac{64}{15\pi} \leq R^3\beta_4 \leq \frac{256 - 15\pi R\beta_2(8 + 3\pi R\beta_2)}{90\pi},$$

$$\frac{-32768 + 1575\pi^2 \left(R\beta_2 - R^3\beta_4\right)^2 - 240\pi \left(13R\beta_2 + 14R^3\beta_4\right)}{525\pi \left(16 + 3\pi R\beta_2\right)} \leq R^5\beta_6 \leq \frac{4096 - 525\pi^2(R\beta_2 + R^3\beta_4)^2 - 120\pi(31R\beta_2 + 14R^3\beta_4)}{175\pi \left(8 - 3\pi R\beta_2\right)},$$

$$\cdots$$


$R^2\beta_4$
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$R\beta_2$

Sound cross-section

- Dispersion relation,

$$\omega(k) = \sum_{n=0}^{\infty} \alpha_{2n+1} k^{2n+1} + \sum_{n=1}^{\infty} i\beta_{2n} k^{2n}, \quad \alpha_{2n+1}, \beta_{2n} \in \mathbb{R}, \quad \alpha_1 = c_s, \beta_2 = -\Gamma_s/2$$

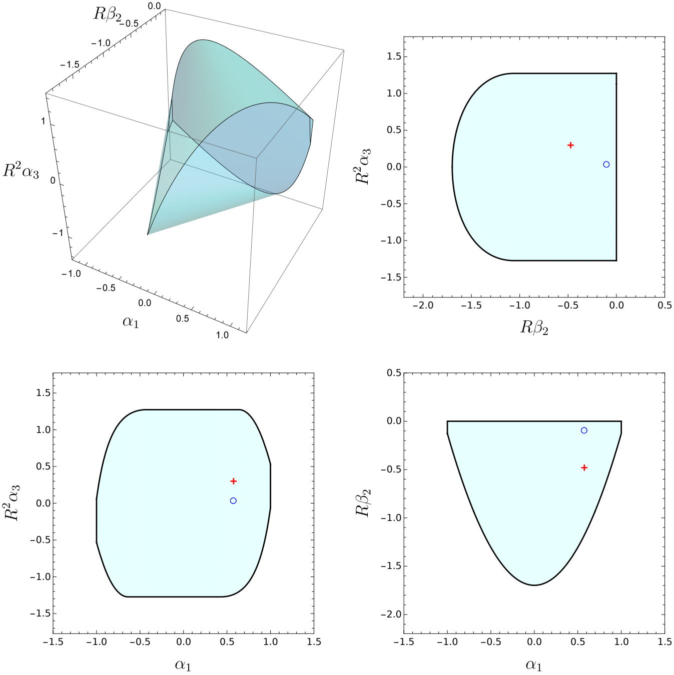
- Hierarchical bounds,

$$|\alpha_1| \leq 1,$$

$$-\frac{16}{3\pi} + \frac{\pi}{2}\alpha_1^2 \leq R\beta_2 \leq 0,$$

$$\frac{128 - 9\pi^2(\alpha_1 - R\beta_2)^2 - 12\pi(\alpha_1 + 2R\beta_2)}{9\pi(-4 + \pi\alpha_1)} \leq R^2\alpha_3 \leq \frac{128 - 9\pi^2(\alpha_1 + R\beta_2)^2 + 12\pi(\alpha_1 - 2R\beta_2)}{9\pi(4 + \pi\alpha_1)},$$

$$\cdots$$



$R\beta_2$
$R^2\alpha_3$
α_1
$R\beta_2$
$R^2\alpha_3$
α_1
$R\beta_2$
α_1

The edge of the hydrohedron

Is there any special physical theory lying at the boundary of the hydrohedron?

- The hydrohedron boundaries set by the moment problem are open: they correspond to sequences of transport coefficients that can be obtained in closed form, resummed, and always feature poles.

- Other boundaries such as $|c_s| = 1,\ D = 0$ are not necessarily open.

- **There are physical theories living at special points at these other boundaries**.

- One can demonstrate that when $|c_s| = 1$, all higher-order transport coefficients vanish and the dispersion relation is

$$\omega(k) = \pm k.$$

In hydrodynamics: stiff perfect fluid, fluid boosted to the speed of light.
Outside hydrodynamics: free massless fields, 2D CFTs.

Some open questions

Regarding R:

- Is there a Planckian bound on R, $R \geq cT$?
- Is R related to the nonhydrodynamic gap?

Regarding the hydrohedron:

- Is there a generalization of the hydrohedron to nonlinear transport?
- Can one incorporate stochastic effects?

Take-home messages

- Bootstrap approach to relativistic transport based on microscopic causality.

➥ Nontrivial consequences in the linear response regime!

➤ **Nonhydrodynamic d.o.f. are unavoidable in dissipative relativistic systems.**

➤ **Hydrohedron**: universal convex geometry in the space of transport coefficients containing the landscape of consistent theories.

➤ **Crucial role of** R, the natural cutoff of the hydrodynamic gradient expansion.

Many thanks for your time!