A new index for a new black hole

Dario Martelli

UNIVERSITÀ
DI TORINO

[based mainly on 2012.08530 & 2303.14199 + 2106.05571 & 2404.07173]

EUROSTRINGS 2024
MEETS
FUNDAMENTAL PHYSICS UK

Southampton, 2 - 6 September, 2024

University of Southampton

Outline

1. Topological twist vs no twist (intro)
2. The supersymmetric and accelerating black holes
3. Rigid supersymmetry on the spindle
4. The spindle index from localization
5. Outlook

The set up

- In this talk I will focus on four-dimensional, asymptotically AdS_4, supersymmetric black holes with three-dimensional SCFT duals
- These can also be thought of as arising from wrapping M2-branes on two-dimensional surfaces Σ, in a way that preserves supersymmetry
- The 3d $\mathcal{N} = 2$ theory living on the M2-branes (e.g. ABJM) is compactified to a 1d SCQM
- The black hole can be viewed as a flow from the 3d theory at the conformal boundary to the 1d theory at the extremal horizon
- Much of what I will discuss have higher dimensional analogs, but I will focus on AdS_4/CFT_3 today

Supersymmetry with the topological twist

- Couple the theory to a background R-symmetry gauge field A_μ

$$\nabla_\mu \epsilon = (\partial_\mu + \omega_\mu)\epsilon = 0 \quad \rightarrow \quad (\partial_\mu + \omega_\mu - A_\mu)\epsilon = 0$$

  where $\omega_\mu \equiv {\omega_\mu}^{12}$ is the spin connection on $\Sigma = \Sigma_g$

- Taking $A_\mu = \omega_\mu$ supersymmetry is preserved by $\partial_\mu \epsilon = 0$
- ϵ becomes effectively a scalar $\rightarrow$ topologically twisted theory
- Geometrically: A is the connection on a line bundle L, that gets identified with the tangent bundle of Σ_g:

$$\int_{\Sigma_g} c_1(L) = \int_{\Sigma_g} \frac{dA}{2\pi} = \int_{\Sigma_g} \frac{d\omega}{2\pi} = \int_{\Sigma_g} c_1(T\Sigma_g) = 2(1-g)$$

Supersymmetry with no twist

- For genus $g = 0$, namely S^2, supersymmetry can also be preserved differently, by the standard Killing spinors that exist on all spheres

$$\nabla_\mu \epsilon = \gamma_\mu \epsilon$$

- In this case there is no background R-symmetry gauge field A
- More generally, we can couple the theories to a number of "flavour" background fields A_i, with the Killing spinors charged under a linear combination, e.g. the diagonal $A_R = \sum_i A_i$
- The supersymmetry constraints on the fluxes $n_i \equiv \int_{\Sigma_g} \frac{dA_i}{2\pi}$ become

topological twist $\quad \sum_i n_i = 2(1 - \mathrm{g})$

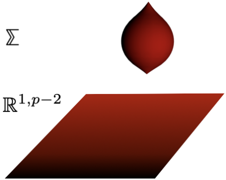
no twist $\quad \sum_i n_i = 0$

Topologically twisted vs untwisted BPS black holes

- These two mechanisms to preserve supersymmetry are realized by
  1. Topologically twisted (static, magnetic) black holes
  2. Kerr-Newman (rotating, dyonic) black holes
- The details of these two classes of black holes differ, but in all cases their entropy is a function of the magnetic fluxes $S = S(n_i)$ (as well as of the electric charges for the Kerr-Newman BHs)
- The challenge is to reproduce this entropy from a microscopic computation, using holography. The key ingredients are:
  1. Identify the dual $d = 3$ SCFT from the internal geometry
  2. Identify the $d = 3$ background from the conformal boundary
  3. Compute the exact partition function through localization
  4. Compute and extremize the free energy at large N

Orbifold holography

- [Ferrero,Gauntlett,DM,Perez-Ipiña,Sparks] proposed that $\boldsymbol{p}$-branes can be wrapped on a "spindle" Σ, extending the AdS/CFT correspondence to the realm of orbifolds
- This is a nickname for $\mathbb{WP}^{\mathbf{1}}_{[\boldsymbol{n_+},\boldsymbol{n_-}]}$, that is a two-sphere with two orbifold singularities $\mathbb{C}/\mathbb{Z}_{\boldsymbol{n_+}}$, $\mathbb{C}/\mathbb{Z}_{\boldsymbol{n_-}}$ at its poles



- The focus of this talk is M2-branes, namely $\boldsymbol{p} = \mathbf{2}$, where the $(\mathbf{1}+\mathbf{0})-$dimensional SCQM has a dual $\mathrm{AdS}_{\mathbf{2}} \times \Sigma$ solution

The spindle's new features

Besides the obvious difference with respect to smooth Σ due to the conical singularities, the spindle has additional features

1. it does not admit metrics with constant curvature
2. it is a "bad" orbifold, i.e. not a global quotient of a manifold
3. it has a $U(1)_\Sigma$ symmetry
4. supersymmetry preserved in two ways [Ferrero,Gauntlett,Sparks]

$$\sum_I n_I = \frac{\sigma_1}{n_+} + \frac{\sigma_2}{n_-} \qquad \sigma_1\sigma_2 \equiv \sigma = \pm 1 : \quad \text{twist/anti-twist}$$

$\sigma = +1$ is topologically a twist: $\pm \int_\Sigma \frac{dA_R}{2\pi} = \int_\Sigma c_1(T\Sigma) = \frac{1}{n_+} + \frac{1}{n_-}$

$\sigma = -1$: for $n_+ = n_- = 1$ reduces to S^2 with no twist

Supersymmetric accelerating black holes

- Asymtotically AdS_4 black holes constructed in minimal gauged supergravity in $D = 4$ (partial generalizations in STU model)
- Can be thought of as solutions deforming the (supersymmetric or otherwise) Kerr-Newman-AdS black hole: rotating (J), electrically (Q_e) and magnetically (Q_m) charged and accelerating
- The horizon S^2 is replaced by a spindle $\mathbb{\Sigma}$. The asymptotic boundary is (topologically) $\mathbb{\Sigma} \times S^1$, with anti-twist for A_R through $\mathbb{\Sigma}$
- There is a magnetic flux

$$Q_m = \frac{1}{4\pi} \int_{\mathbb{\Sigma}} dA_R \left(= \sum_i n_i \right) = \frac{1}{n_-} + \sigma \frac{1}{n_+}$$

- In the minimal case only the anti-twist ($\sigma = -1$) is realized, while in STU one can have $\sigma = \pm 1$ (although the solutions are known only in special cases [Ferrero,Inglese,DM,Sparks][Ferrero,Gauntlett,Sparks])

On-shell action = entropy function

- Deforming the BPS black hole to a "supersymmetric but not extremal" and complex solution, leads to the on-shell action [Gauntlett,Cassani,DM,Sparks]

$$\mathcal{E}(\varphi,\epsilon;Q_m)=\frac{1}{2i}\left(Q_m^2\epsilon+\frac{\varphi^2}{\epsilon}\right)\,,\quad \varphi-\frac{\chi}{4}\epsilon=\pi i\,,\quad \chi\equiv\frac{n_++n_-}{n_+n_-}$$

  and can be written in terms of gravitational blocks $\mathcal{F}_3(\Delta)\propto N^{3/2}\Delta^2$:

$$\mathcal{E}(\varphi,\epsilon;Q_m)=\frac{1}{\epsilon}\big(\mathcal{F}_3(\varphi+Q_m\epsilon)+\mathcal{F}_3(\varphi-Q_m\epsilon)\big)$$

- From the Legendre transform of this, namely extremizing

$$\mathcal{S}(\varphi,\epsilon;Q_m,Q_e,J)=\mathcal{E}(\varphi,\epsilon;Q_m)-(\epsilon J+\varphi Q_e)$$

  and further imposing that this is real, one obtains the BH entropy

$$S(Q_e,Q_m)=\frac{\pi}{4}\left(-\chi+\sqrt{\chi^2+16\left(Q_e^2+Q_m^2\right)}\right)$$

Rigid background from the accelerating black hole

- We want to reproduce this from a microscopic computation of the partition function of the dual $\mathcal{N} = 2$ SCFTs, using as background the boundary geometry on $\Sigma \times S^1$
- The gauge field with anti-twist through Σ reads

$$A = -\frac{\cos\theta}{1 + \alpha^2 a^2 \cos^4\theta}\Big[\frac{\alpha}{\kappa}\left(e - g\alpha a \cos^2\theta\right)dt + \left(g + g\alpha^2 a^2 \cos^2\theta - e\alpha a \sin^2\theta\right)d\phi\Big]$$

- a, α, g, e, m are related by supersymmetry, leaving two parameters on which the background depends. I won't write the 3d metric...
- This background has novel features
  1. It has conical orbifold singularities at the poles of the spindle
  2. It has a complex metric (inherited by the bulk analysis)
  3. The gauge field obeys the anti-twist

General rigid backgrounds

- Before attempting to implement localization on this background we need to tame these features in rigid new minimal supergravity
- Our complexified rigid geometry does not fit in the analysis of [Closset,Dumitrescu,Festuccia,Komargodski], that assumed
  - The metric is real
  - The Killing vector bilinear $K^\mu = \zeta_+ \gamma^\mu \zeta_-$ is real

  both of which are false for our background $\longrightarrow$ start from scratch
- [Inglese,DM,Pittelli] studied the most general Euclidean-complex background admitting two Killing spinors ζ_+, ζ_- satisfying

$$(\nabla_\mu \mp \mathrm{i} A_\mu)\zeta_\pm = -\frac{H}{2}\gamma_\mu \zeta_\pm \mp \mathrm{i} V_\mu \zeta_\pm \mp \epsilon_{\mu\nu\rho}\frac{V^\nu}{2}\gamma^\rho \zeta_\pm$$

- Conditions imposed by these on the complex spinor bilinears

$$v = \zeta_+\zeta_- \qquad K^\mu = \zeta_+\gamma^\mu\zeta_- \qquad P^\mu_\pm = \zeta_\pm\gamma^\mu\zeta_\pm / v$$

General rigid backgrounds

- K^μ is a complex Killing vector $\rightarrow$ $K = \partial_\psi + \omega\partial_\varphi$, where ω is a complex constant $\rightarrow$ two real Killing vectors $\partial_\psi, \partial_\varphi$

- The most general metric can be written as

$$\mathrm{d}s^2 = f^2\mathrm{d}x^2 + h_{ij}\mathrm{d}\psi_i\mathrm{d}\psi_j \quad \text{with} \quad \psi_1 = \psi\,, \psi_2 = \varphi$$

  where $f(x)$, $h_{ij}(x)$ are complex functions of x

- Denoting $h = \det(h_{ij})$ the bilinears read

$$\begin{aligned} K &= (h_{11} + \omega h_{12})\mathrm{d}\psi + (h_{12} + \omega h_{22})\mathrm{d}\varphi \\ P_\pm &= \mathrm{e}^{\pm\mathrm{i}(\alpha_1\psi + \alpha_2\varphi)}\big(\pm f\mathrm{d}x + \mathrm{i}(\sqrt{h}/v)(-\omega\mathrm{d}\psi + \mathrm{d}\varphi)\big) \\ v^2 &= h_{11} + 2\omega h_{12} + \omega^2 h_{22} \end{aligned}$$

  and the background fields read $V = \frac{1}{v}\Big[\mathrm{i}HK - \star\mathrm{d}K\Big]$ and

$$A^C \equiv A - \tfrac{3}{2}V = \frac{v^3}{4f\sqrt{h}}\left[\frac{1}{\omega}\left(\frac{h_{11}}{v^2}\right)'\mathrm{d}\psi - \left(\frac{h_{22}}{v^2}\right)'\mathrm{d}\varphi\right] + \mathrm{d}\theta$$

Twist and anti-twist

- Specialize to $S^1 \times \Sigma$: $\psi \sim \psi + 2\pi$ parameterizes the S^1 and $x \in [-1, 1], \varphi \sim \varphi + 2\pi$ are coordinates on Σ
- Boundary conditions at the N$\equiv \{x = 1\}$ and S$\equiv \{x = -1\}$ poles: setting $f = 1$ and denoting by $\varrho_\pm$ the coordinates near to N/S poles

$$h_{11} \sim h_{11}^{\pm} \qquad h_{22} \sim \frac{1}{n_\pm^2}\varrho_\pm^2 \qquad h_{12} \sim h_{12}^{\pm}\varrho_\pm^2$$

at the north and south poles, where $h_{12}^{\pm}, h_{11}^{\pm}$ are constants

- The type of twist for a generic background is fixed by the function v

$$\frac{1}{2\pi}\int_\Sigma \mathrm{d}A = -\frac{1}{2}\left(\frac{s_+}{n_+} + \frac{s_-}{n_-}\right)$$

where $s_\pm$ denote the signs of $v/\sqrt{h_{11}}$ at the N and S poles of Σ

  - $-s_+ = s_-(= -1) \rightarrow$ anti-twist
  - $s_+ = s_-(= -1) \rightarrow$ twist

Simple backgrounds

- The boundary of the accelerating black hole of course fits into this general framework and we can now start the localization procedure
- However, since the final result will depend only on the type of twist and the Killing vector parameter ω we can consider "simple" background metrics such as, for example:

$$\mathrm{d}s^2 = f^2(x)\mathrm{d}x^2 + \left(1 - x^2\right)(\mathrm{d}\varphi - \Omega \mathrm{d}\psi)^2 + \beta^2 \mathrm{d}\psi^2$$

with Ω, β constants and $f(x)$ a function subject to the previous boundary conditions:

  - anti-twist: $\omega = \Omega - \mathrm{i}\beta \quad \leftrightarrow \quad v/\beta = x$
  - twist: $\omega = \Omega \quad \leftrightarrow \quad v/\beta = -1$

- The background fields A, V and the Killing spinors are obtained from the general expressions

Two new indices at the price of one

- For $n_+ = n_- = 1$ the backgrounds reduce to 1) $S^1 \ltimes S^2$ with no twist and 2) $S^1 \ltimes S^2$ with the standard topological twist
- As a check on our calculations, in these limits we must recover 1) the superconformal index and 2) the topologically twisted index
- The partition functions on these backgrounds defines two new indices
- Anti-twisted spindle index, that reduces to the superconformal index for $n_+ = n_- = 1$
- Twisted spindle index, that reduces to the (refined) topologically twisted index for $n_+ = n_- = 1$
- In fact, in [Inglese,DM,Pittelli] we obtained more than what we hoped: a single formula for a new index that we called the spindle index, unifying and generalising the above two indices!

Localization

- To implement localization we follow the well-established strategy [Pestun], with suitable modifications, where required
- The intermediate calculations are done separately for the twist and anti-twist case, but the end results can then be written in terms of universal expressions, typically depending on $\sigma = \pm 1$
- Strategy: derive the BPS locus, compute the classical (Chern-Simons) contribtions and the 1-loop determinants
- As usual, to compute the 1-loop determinants we write these as

$$Z_{\text{1-L}} = \frac{\det_{\mathrm{Ker} L_{P_+}} \left(L_K + \mathcal{G}_{\Phi_G} \right)}{\det_{\mathrm{Ker} L_{P_-}} \left(L_K + \mathcal{G}_{\Phi_G} \right)}$$

where $\delta^2_{\mathrm{susy}} = L_K + \mathcal{G}_{\Phi_G}$

1-loop determinants

- $L_K, L_{P_\pm}$ are explicit linear differential operators given by

$$L_K = \mathcal{L}_K - \mathrm{i}\, q_R^\phi\, \Phi_R$$
$$L_{P_\pm} = \mathcal{L}_{P_\pm} - \mathrm{i}\, q_R^\phi\, \iota_{P_\pm}\left(A^C + V\right) - \mathrm{i}\, \iota_{P_\pm}\mathcal{A}$$

with $\Phi_R \equiv \iota_K\left(A^C + V\right) - \mathrm{i} vH = \frac{1}{2}(\alpha_1 + \omega\alpha_2)$

- The gauge field $\mathcal{A}$ has fluxes through the spindle

$$\frac{1}{2\pi}\int_\Sigma d\mathcal{A} = \frac{m_-}{n_-} - \frac{m_+}{n_+} \qquad m_+, m_- \in \mathbb{Z}$$

- Unpaired eigenmodes: the modes annihilated by $L_{P_\pm}$ take the form

$$e^{im_\varphi^\pm\varphi + im_\psi^\pm\psi} B^\pm_{m_\varphi^\pm, m_\psi^\pm}(x) \qquad m_\varphi^\pm, m_\psi^\pm \in \mathbb{Z}$$

1-loop determinants

- Normalizability at the poles of Σ gives, for example in the twist case:

$$-\left\lfloor -\frac{p_-}{n_-} \right\rfloor \leq m_\varphi^- \leq \left\lfloor \frac{p_+}{n_+} \right\rfloor$$

where $p_+ \equiv m_+ - \sigma r/2$ and $p_- \equiv m_- + r/2$

- The "floor functions" or "integer parts" are a new distinctive feature associated to orbifolds and are the main technical complication

- In this way one obtains infinite products depending on ω and various combinations of these integer parts of the fluxes; after regularization, lead to q-Pochammers

- The same results can be obtained extracting the eigenvalues from the equivariant orbifold index theorem [Vergne], which picks up contributions at the N and S poles of the spindle!

Orbifold index theorem

- The equivariant orbifold index theorem gives the index as a sum over 1) the two fixed points of the spindle and 2) at each fixed point there are $n_\pm$ "images", each weighted by a root of unity $\omega_\pm = e^{2\pi i/n_\pm}$

$$ind(L_{P_-}; e^{-i\omega\delta^2}) = \frac{1}{n_+}\sum_{j=0}^{n_+-1}\frac{\omega_+^{-jp_+}q_+^{-p_+}}{1-\omega_+^{j\sigma}q_+^{\sigma}} + \frac{1}{n_-}\sum_{j=0}^{n_--1}\frac{\omega_-^{-jp_-}q_-^{-p_-}}{1-\omega_-^{-j}q_-^{-1}}$$

where $q = e^{i\omega}$, $q_\pm = q^{1/n_\pm}$

- Remarkably (using $r \in 2\,\mathbb{Z}$) this can be resummed into

$$ind(L_{P_-}; e^{-i\omega\delta^2}) = \frac{q^{-\sigma\lfloor\sigma\, p_+/n_+\rfloor}}{1-q^{\sigma}} + \frac{q^{\lfloor -p_-/n_-\rfloor}}{1-q^{-1}}$$

1. Extracting the eigenvalues from this reproduces exactly those obtained with the unpaired modes method
2. Setting $n_+ = n_- = 1$, for $\sigma = \pm 1$ reduces to the index theorem used for the topologically twisted and superconformal indeces, respectively

Chern-Simons terms contributions

- For each gauge flux $\frac{1}{2\pi}\int_{\Sigma}\mathcal{F}=\frac{m}{n_+n_-}$, $m\in\mathbb{Z}$ the contribution of a CS term with level k is

$$e^{S_{\text{CS}}}=e^{2\pi i k\frac{m}{n_+n_-}u}$$

- Under a large gauge transformation of the gauge holonomy u, $u\to u+\ell$, $\ell\in\mathbb{Z}$ this is not invariant (unless $n_+=n_-=1$)
- To restore gauge invariance we proposed that the naive CS term is modified to an effective CS term (as in [Beem,Dimofte,Pasquetti])

$$e^{S_{\text{CS}}|_{\text{eff}}}=e^{2\pi i k b u}$$

where $b=1+\sigma\lfloor\sigma p_+/n_+\rfloor+\lfloor -p_-/n_-\rfloor$

- Checks: partition function makes sense, dualities, large N

Entropy (function) from the large N limit

- In [Colombo,Hosseini,DM,Pittelli,Zaffaroni] we studied the large N limit of the spindle index (a more detailed paper will appear soon..)
- The main technical difficulty that was solved is how to deal with the fractional parts of the magnetic fluxes, that appear in the matrix model representation of the spindle index
- Write out the matrix model resulting from assembling the pieces for a generic ("non-chiral") $\mathcal{N} = 2$ Chern-Simons quiver theory, with gauge group $\mathcal{G}$, $|\mathcal{R}|$ chiral multiplets transforming in bi-fundamental or adjoint representations and a global symmetry $U(1)^{\delta}$
- For example, for ABJM: $\mathcal{G} = U(N) \times U(N)$, $|\mathcal{R}| = 4$ (bi-fundamentals), the CS levels are $(k_1, k_2) = (k, -k)$, $\delta = 4$

Black holes microstates from the large N limit

- The upshot is that at leading order in the large N limit, the logarithm of the spindle index, $\log Z_{\mathbb{\Sigma}\times S^1}$, takes the form of gravitational blocks

$$\log Z_{\mathbb{\Sigma}\times S^1} = \frac{\mathcal{F}_3(\Delta_I^-)}{\omega} - \sigma\frac{\mathcal{F}_3(\Delta_I^+)}{\omega} + \mathcal{O}(N)$$

for either choice of twist $\sigma = \pm 1$, and with generic flavour charges satisfying (for the anti-twist)

$$\sum_{I\in W} \Delta_I^\pm = 2\pi\left(1 - \frac{\omega}{n_\pm}\right)$$

For example, for ABJM: $\mathcal{F}_3(\Delta_I) \propto N^{3/2}\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$

- If you are interested in hearing about the details of this computation, that presents several novel features with respect to previous large N limit computations, I refer you to the talk by E. Colombo on Thursday

Outlook

- ... $\rightarrow$ BPS accelerating black holes in AdS_4 $\rightarrow$ localization on $\Sigma \times S^1$ $\rightarrow$ *spindle index* $\rightarrow$ microstates from the large N limit $\rightarrow$ ...
- First example of "orbifold holography" program has been completed!
- Extensions to other sugra solutions$_4$/SQFT$_3$ will appear soon [Colombo,Hosseini,DM,Pittelli,Zaffaroni], [Crisafio,Fontanarossa†,DM]
- Corrections sub-leading in N (extending work by [Bobev] and others..)
- Numerous other constructions involving branes wrapped on spindles (or "half spindles")…. [many authors]
- Supergravity solutions containing higher (than two) dimensional orbifolds, e.g. $\mathrm{AdS}_2 \times \mathbb{M}_4$, 6d black holes, ...
- Supersymmetric field theories on orbifolds promise to lead to several interesting new features and surprises!

†See the talk by A. Fontanarossa on Thursday

Thank you!