Classical Observables from the Exponential Representation of the Gravitational S-Matrix

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Based on work with
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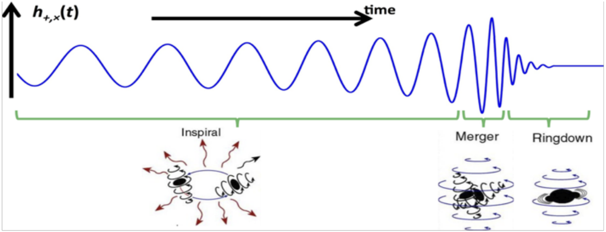
Two-body gravitational interactions



Inspiral-Merger-Ringdown (M. Favata, SXS, K. Thorne)

There are three main analytical approximation methods for describing the two-body dynamics during the inspiral phase

1. the post-Newtonian expansion valid for $(v/c)^2 \simeq G_N(m_1+m_2)/(rc^2) \ll 1$
2. the post-Minkowskian expansion valid for weak field $G_N(m_1+m_2)/(rc^2) \ll 1$ but all values of velocities $\gamma = 1/\sqrt{1-(v/c)^2} \in [1,+\infty[$
3. self-force expansion for small mass ratio $\nu = m/M \ll 1$ for all order in G_N

In this talk we will present two new formalism for the point 2 and point 3

Post-Minkowskian expansion vs Numerical GR

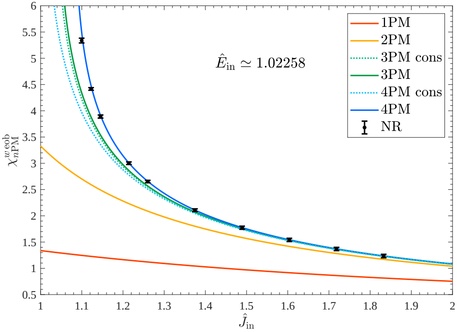


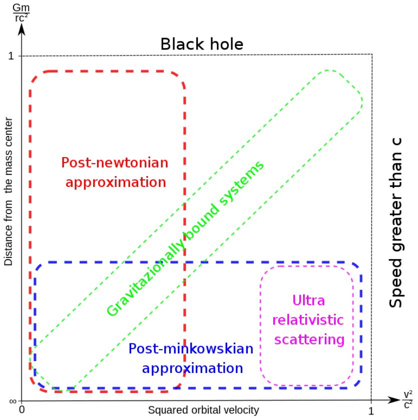
fig 3 of Thibault Damour and Piero Rettegno [arXiv:2211.01399]

Perturbative methods from scattering amplitude can be apply directly leading to new analytic results up to the 4th post-Minkowskian order (3-loop)

[Bern et al.; Bjerrum-Bohr et al.; Damgaard et al.; Plefka et al.; Porto et al.]

with excellent results when compared to numerical general relativity if one includes radiation reaction [Damour, Rettegno]

Two-body gravitational interactions

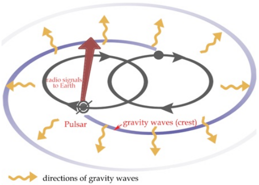


The post-Minkowskian expansion gives analytic expression valid from the static case $\gamma = 1$ to the ultra-relativistic ACV regime $\gamma \to \infty$ [Damour; Heissenberg, di Vecchia, Veneziano; Bjerrum-Bohr, Damgaard, Planté, Vanhove]

The match is obtain only if one includes gravitational radiation

Post-Minkowskian expansion for the binary system

We want to compute gravitational interaction of two massive objects.



A relativistic Hamiltonian for the two body dynamics in centre-of-mass

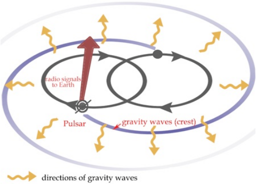
$$\mathcal{H}_{\rm PM}(\gamma,r) = \sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2} + \sum_{L\geq 0} \underbrace{\mathcal{V}_{L+1}(\gamma,r)}_{\propto G_N^{L+1}/r^{L+1}}$$

with a relativistic potential organised in a series of Newton's constant G_N which is the general relativity correction to Newton's potential $L=0$

$$\mathcal{V}_1(\gamma,r) = -\frac{G_N}{E_1E_2}\frac{m_1^2m_2^2}{r}(2\gamma^2-1) \qquad \gamma = \frac{p_1\cdot p_2}{m_1m_2} = \frac{1}{\sqrt{1-\frac{\vec v^2}{c^2}}} \geq 1$$

Post-Minkowskian expansion for the binary system

We want to compute gravitational interaction of two massive objects.



The $L+1$PM potential has polynomial mass dependence [Vines et al.; Damour]

$$\mathcal{V}(\gamma, r) = \sum_{L\geq 0} \frac{G_N^{L+1} m_1^2 m_2^2}{r^{L+1}} \sum_{r_1+r_2=L} v_{r_1,r_2}(\gamma) m_1^{r_1} m_2^{r_2}$$

Consider its Fourier transform to momentum space

$$\mathcal{M}_{L+1}(\gamma, \underline{q}^2) \propto \int e^{-i\underline{q}\cdot\vec{r}} \mathcal{V}_{L+1}(\gamma, r) d^3\vec{r} \propto \frac{G_N^{L+1} m_1^2 m_2^2}{\underline{q}^{2-L}} \sum_{r_1+r_2=L} v_{r_1,r_2}(\gamma) m_1^{r_1} m_2^{r_2}$$

Classical physics from quantum loops

THE GENERATION OF GRAVITATIONAL WAVES.
IV. BREMSSTRAHLUNG*†‡

Sándor J. Kovács, Jr.

and

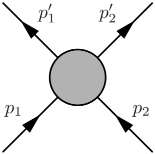
Kip S. Thorne

Received 1977 October 21; accepted 1978 February 28

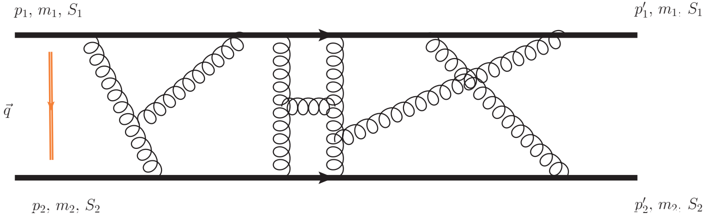
g) *The Feynman-Diagram Approach*

Any classical problem can be solved quantum-mechanically; and sometimes the quantum solution is easier than the classical. There is an extensive literature on the Feynman-diagram, quantum-mechanical treatment of gravitational bremsstrahlung radiation (e.g., Feynman 1961, 1963; Barker, Gupta, and Kaskas 1969; Barker and Gupta

We seek quantum gravity formalism where the classical limit $\hbar \to 0$ gives the general relativity potential

$\lim_{\hbar \to 0}$  $\to \mathcal{M}_L(\gamma, \underline{q}^2) \to \mathcal{V}_{L+1}(p, r)$

Perturbative gravity



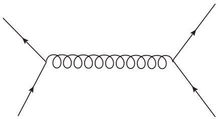
We will be considering the pure gravitational interaction between massive and massless matter of various spin $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G_N} h_{\mu\nu}$

$$\mathcal{S}_{\rm EH} = \int d^4x \sqrt{-g}\left(-\frac{\mathcal{R}}{16\pi G_N} + \frac{1}{2}\sum_{a=1}^{2}\left(g^{\mu\nu}\partial_\mu\phi_a\partial_\nu\phi_a - m_a^2\phi_a^2\right)\right)$$

Evaluating the quantum scattering S-matrix

$$\mathfrak{M}^{\rm GR}(p_1\cdot p_2, \underline{q}, \hbar) = \sum_{L\geq 0} G_N^{L+1}\mathfrak{M}_L(p_1\cdot p_2, \underline{q}, \hbar)$$

One graviton exchange : tree-level (1PM)



$$\mathfrak{M}_0 = -16\pi G_N \hbar \frac{2(p_1\cdot p_2)^2 - m_1^2 m_2^2 - |\hbar\vec{\underline{q}}|^2 (p_1\cdot p_2)]}{|\hbar\vec{\underline{q}}|^2}$$

The $\hbar$ expansion of the tree-level amplitude

$$\mathfrak{M}_0 = \frac{\mathcal{M}_1^{(-1)}(\gamma)}{\hbar|\underline{q}|^2} + \hbar 4\pi G_N p_1\cdot p_2$$

The relativistic classical Newtonian potential is obtained by taking the Fourier transform

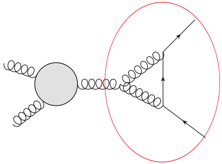
$$V_1(\gamma, r) = \int \frac{d^3\vec{\underline{q}}}{(2\pi)^3}\frac{1}{4E_1E_2}\mathcal{M}_1^{(-1)}(\gamma)\, e^{i\vec{\underline{q}}\cdot\vec{r}} = -\frac{G_N m_1 m_2^2}{E_1E_2}\frac{2\gamma^2-1}{r}$$

The higher order in $\underline{q}^2$ is the quantum contact interaction of order $\hbar$

Classical physics from loops : One-loop (2PM)

The Klein-Gordon equation reads

$$(\Box - \frac{m^2c^2}{\hbar^2})\phi = 0$$



The triangle with a massive leg $p_1^2 = p_2^2 = m^2$ reads

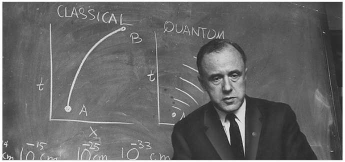
$$\int \frac{G_N^2 \mu^{2\epsilon} d^{4-2\epsilon}\ell}{(\ell+p_1)^2(\ell^2 - \frac{m^2c^2}{\hbar^2})(\ell-p_2)^2}\Bigg|_{\text{finite part}}$$

$$\sim \frac{G_N^2}{m^2}\left(\log\left(\frac{q^2}{\mu^2}\right) + \frac{\pi^2 mc}{\hbar\sqrt{\underline{q}^2}}\right)$$

This one-loop amplitude contains [Iwasaki, Holstein, Donoghue, Bjerrum-Bohr, Vanhove]

- The classical 2nd post-Minkowskian correction G_N^2/r^2 to Newton's potential of order $1/\hbar$
- An infrared quantum correction of order $\hbar^0$

Classical physics from loops : $\hbar$ counting (All PM)



The classical limit $\hbar \to 0$ fixed $\underline{q} \ll m_1, m_2$ of the amplitude [Bjerrum-Bohr, Damgaard, Vanhove, Planté]

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma)}{\hbar^{L+1}|\underline{q}|^{2+\frac{L(4-D)}{2}}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma)}{\hbar|\underline{q}|^{2-L+\frac{L(4-D)}{2}}} + O(\hbar^0)$$

- A classical contribution of order $1/\hbar$ from all loop orders
- The dimensional regularisation scheme gives a control of the IR divergences from radiation [Veneziano et al.; Parra-Martinez et al.; Bjerrum-Bohr et al.]
- The computation is explicitly relativistic invariant

Classical physics from loops : $\hbar$ counting

The connection between quantum scattering and classical gravitational physics has forced to rethink the S matrix for dealing with the $\hbar$ expansion

[Damgaard, Planté, Vanhove]

$$\hat{S} = \mathbb{I} + \frac{i}{\hbar}\hat{T} =: \exp\left(\frac{i\hat{N}}{\hbar}\right)$$

doing the Dyson expansion with the conservative and radiation part

$$\hat{T} = G_N \sum_{L\geq 0} G_N^L \hat{T}_L + G_N^{\frac{1}{2}} \sum_{L\geq 0} G_N^L \hat{T}_L^{\rm rad}$$

$$\hat{N} = G_N \sum_{L\geq 0} G_N^L \hat{N}_L + G_N^{\frac{1}{2}} \sum_{L\geq 0} G_N^L \hat{N}_L^{\rm rad}$$

The higher powers of $1/\hbar$ are more singular than the classical contribution, but **are needed for the consistency of the full quantum amplitude and the correct exponentiation of the amplitude**

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma)}{\hbar^{L+1}|\underline{q}|^{2+\frac{L(4-D)}{2}}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma)}{\hbar|\underline{q}|^{2-L+\frac{L(4-D)}{2}}} + O(\hbar^0)$$

Classical observables

The change in an observable $\hat{O}$ is given by the [Kosower, Maybee, O'Connel] expression $\langle \Delta\hat{O}\rangle := \langle \text{out}|\hat{O}|\text{out}\rangle - \langle \text{in}|\hat{O}|\text{in}\rangle$

$$\langle \Delta\hat{O}\rangle(p_1,p_2,r) = \int \frac{d^D(\hbar \underline{q})}{(2\pi)^{D-2}} \delta(2\hbar p_1\cdot\underline{q} - \hbar^2\underline{q}^2)\delta(2\hbar p_2\cdot\underline{q} + \hbar^2\underline{q}^2) e^{ir\cdot\underline{q}} \langle p_1', p_2'|\hat{O}|p_1,p_2\rangle$$

which can be expanded using the $\hat{N}$-operator

$$\langle \Delta\hat{O}\rangle = \langle \text{in}|\hat{S}^\dagger\hat{O}\hat{S} - \hat{O}|\text{in}\rangle = \sum_{n\geq 1} \frac{(-i)^n}{\hbar^n n!} \langle \text{in}|\underbrace{[\hat{N},[\hat{N},\ldots,[\hat{N},\hat{O}],\ldots,]]}_{n \text{ times}}|\text{in}\rangle$$

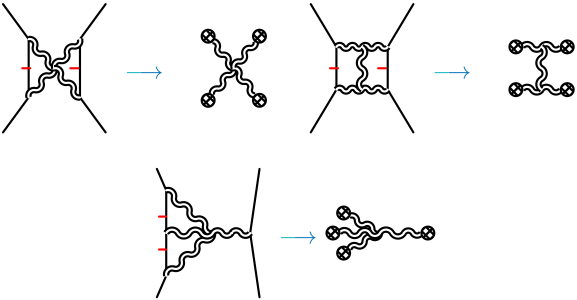
The $\hbar \to 0$ limit gives directly the classical answer [Damgaard, Planté, Vanhove]

$$\lim_{\hbar\to 0} \langle \Delta\hat{O}\rangle = \Delta O^{\text{classical}}(p_1,p_2,r) + O(\hbar)$$

with the exponential representation all superclassical pieces cancel automatically

Relation to the worldline approaches

There is a direct equivalence with the word-line formalism



The completeness relation induces velocity cuts delta-functions on the massive line

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^D k_1}{(2\pi\hbar)^{D-1}} \delta^+(k_1^2 - m_1^2) \frac{d^D k_2}{(2\pi\hbar)^{D-1}} \delta^+(k_2^2 - m_2^2)$$
$$\times \frac{d^{D-1}\ell_1}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_1}} \cdots \frac{d^{D-1}\ell_n}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_n}} \times |k_1, k_2; \ell_1, \ldots, \ell_n\rangle\langle \ell_1, \ldots, \ell_n; k_1, k_2|$$

The $\hat{N}$ operator upto 1PM and 2PM

$$\hat{S} = \exp\left(\frac{i\hat{N}}{\hbar}\right), \qquad N(\gamma, \underline{q}^2) := \langle p_1, p_2 | \hat{N} | p_1', p_2' \rangle$$

$$\tilde{N}(\gamma, J) := \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{N(\gamma, \underline{q}^2)}{4m_1 m_2 \sqrt{\gamma^2 - 1}} e^{i\frac{J\cdot \underline{q}}{p_\infty}}; \qquad p_\infty = \frac{m_1 m_2 \sqrt{\gamma^2 - 1}}{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \gamma}}$$

with the result evaluated at tree-level and one-loop using dimensional regularisation $D = 4 - 2\epsilon$

$$\tilde{N}^{1PM}(\gamma, J) = \frac{G_N m_1 m_2 (2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \Gamma(-\epsilon) J^{2\epsilon}$$

$$\tilde{N}^{2PM}(\gamma, J) = \frac{3\pi G_N^2 m_1^2 m_2^2 (m_1 + m_2)(5\gamma^2 - 1)}{4\sqrt{(p_1 + p_2)^2}} \frac{1}{J}$$

The 1PM (tree-level) and 2PM (one-loop) contributions are the same as for a test mass in the Schwarzschild black hole of mass $M = m_1 + m_2$.

The $\hat{N}$ operator at 3PM

$$\tilde{N}^{3PM}(\gamma,J)=\frac{G_N^3m_1^3m_2^3\sqrt{\gamma^2-1}}{sJ^2}\times\left(\frac{s(64\gamma^6-120\gamma^4+60\gamma^2-5)}{3(\gamma^2-1)^2}\right.$$
$$-\frac{4m_1m_2\gamma(14\gamma^2+25)}{3}+\frac{4m_1m_2(3+12\gamma^2-4\gamma^4)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}}$$
$$\left.+\frac{2m_1m_2(2\gamma^2-1)^2}{\sqrt{\gamma^2-1}}\left(\frac{8-5\gamma^2}{3(\gamma^2-1)}+\frac{\gamma(2\gamma^2-1)\operatorname{arccosh}(\gamma)}{(\gamma^2-1)^{\frac{3}{2}}}\right)\right)$$

At 3PM (two-loop) new phenomena arise

- The conservative part deviates from Schwarzschild as we have contributions which depends (linearly) on the relative mass $\nu=\frac{m_1m_2}{(m_1+m_2)^2}$
- And the important Radiation-reaction terms for the correct high-energy behaviour [Bjerrum-Bohr et al., Para-martinez et al.; Damour; Veneziano et al.]

The radial action

Applying the previous formalism to the momentum kick $\hat{O}^\mu = \hat{P}_1^\mu$ gives in the conservative sector [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

$$\Delta\tilde{P}_1^\mu(\gamma, r)|_{\text{cons}} = -\frac{p_\infty r^\mu}{|r|}\sin\left(-\frac{\partial\tilde{N}(\gamma,J)}{\partial J}\right) + p_\infty^2 L^\mu\left(\cos\left(-\frac{\partial\tilde{N}(\gamma,J)}{\partial J}\right) - 1\right)$$

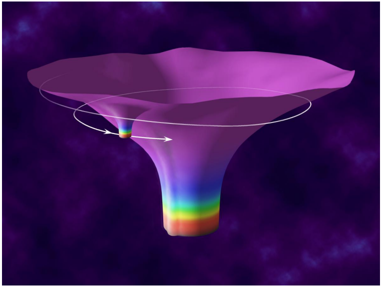
with the angular momentum

$$L^\mu := \frac{m_1 m_2\gamma(p_1^\mu - p_2^\mu) + m_2^2 p_1^\mu - m_1^2 p_2^\mu}{(m_1 m_2)^2(\gamma^2 - 1)}$$

this shows that in the conservative sector the $\tilde{N}(\gamma, J)$ is the radial action used by [Landau, Lifshitz; Damour] for computing the scattering angle in classical GR

$$\chi(\gamma, J) = -\frac{\partial\tilde{N}(\gamma,J)}{\partial J} = \sum_{L=0}^{\infty}\left(\frac{G_N m_1 m_2}{J}\right)^{L+1}\chi_{\text{cons}}^{(L+1)}(\gamma)$$

Strong field regime: Self-force expansion



A principal objective of LISA is to investigate the behaviour of general relativity in a strong gravitational field.

The self-force expansion is a expansion in $\nu \sim m/M \ll 1$ for $m_1 = m \ll m_2 = M$ but to all order in G_N

We reorganise the double summation according to the mass-ratio order

$$N(\gamma, J) = \frac{M^3\nu^2}{|q|^3}\sum_{r\geq 0}\nu^r \sum_{L\geq 2r}(G_N M|q|)^{L+1} c_r^L(\gamma)$$

We have shown in [Mougiakakos, Vanhove] that the cubic formulation of gravity of [Cheung, Remmen] coupled to the worldline allows to construct the **exact spacetime (all G_N order)** from a scattering amplitudes approach

Self-force expansion

We consider a binary system of a heavy body of mass M and a light body of mass m interacting gravitationally

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_l + \mathcal{S}_H,$$

with the *gothic inverse metric* of the Landau-Lifschits formulation

$$\mathfrak{g}^{ab} = \sqrt{-g}g^{ab} = \eta^{ab} - \sqrt{32\pi G_N}h^{ab}$$

and **the cubic formulation** introduced by [Cheung, Remmen]

$$16\pi G_N \mathcal{S}_{EH} = -\int d^D x \left(\left(A^a_{bc} A^b_{ad} - \frac{1}{D-1} A^a_{ac} A^b_{bd} \right) \mathfrak{g}^{cd} + A^a_{bc} \partial_a \mathfrak{g}^{bc} \right),$$

and the wordline actions for the light and heavy body

$$\mathcal{S}_l = -\frac{m}{2} \int d\tau_l \left(\frac{\mathfrak{g}^{\mu\nu} v_\mu v_\nu}{\left(\sqrt{-\mathfrak{g}}\right)^{\frac{2}{D-2}}} + 1 \right), \qquad \mathcal{S}_H = -\frac{M}{2} \int d\tau_H \left(\frac{\mathfrak{g}^{\mu\nu} v_{H\mu} v_{H\nu}}{\left(\sqrt{-\mathfrak{g}}\right)^{\frac{2}{D-2}}} + 1 \right).$$

Notice that the auxiliary field A does not couple to the matter field.

Self-force expansion: effective action

Integrating-out the graviton and the auxiliary field we have the effective action

$$e^{i\mathcal{S}_{\rm eff}[x_l,x_H]} = \int \mathcal{D}h\,\mathcal{D}A\; e^{i\mathcal{S}_{EH}[h,A]+i\mathcal{S}_{GF}[h]+i\mathcal{S}_l[x_l,h]+i\mathcal{S}_H[x_H,h]}.$$

The self-force effective action has an expansion in powers $m/M \ll 1$

$$\mathcal{S}_{\rm eff} = -\frac{M}{2}\int d\tau_H\;\eta^{\mu\nu}v_{H\mu}v_{H\nu} + M\sum_{n=0}^{\infty}\int d\tau_l\;\left(\frac{m}{M}\right)^{n+1}\mathcal{L}_n[x_l(\tau_l),x_H(\tau_H)]$$

where the leading is the worldline action for the heavy body. We parametrize the trajectory of the light body as

$$x_l^\mu(\tau_l) \equiv x^\mu(\tau) = \sum_{n=0}^{\infty}\left(\frac{m}{M}\right)^n \delta x^{(n)\,\mu}(\tau), \qquad x_H^\mu(\tau_H) = u_H^\mu\tau_H + \sum_{n=1}^{\infty}\left(\frac{m}{M}\right)^n \delta x_H^{(n)\,\mu}(\tau_H),$$

Finally, we should note that we haven't specified the kinematics of the system, therefore the formalism is suitable both for the **bound and the scattering problem**

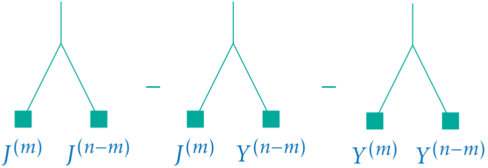
Self-force expansion: exact metric

The off-shell currents for gravition emission from the heavy source M

$$\sqrt{32\pi G_N}\, h^{(n)}_{\mu\nu}(\mathbf{x}) = \int d^{D-1}x\, e^{i\mathbf{k}\cdot\mathbf{x}}\, J^{(n)}_{\mu\nu}(\mathbf{k}),\quad \sqrt{32\pi G_N}\, A^{a\ (n)}_{bc}(\mathbf{x}) = \int d^{D-1}x\, e^{i\mathbf{k}\cdot\mathbf{x}}\, Y^{a\ (n)}_{bc}(\mathbf{k}),$$

is given by sums of cubic graphs only, therefore presenting a recursive nature

$$J^{(n)} = \sum_{m=1}^{n-1} \Bigg(\quad - \quad - \quad \Bigg)$$



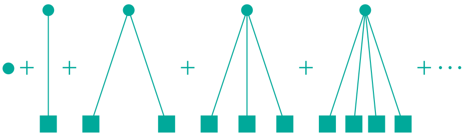
allowing the sum **all loop orders** and lead to a non-perturbative resummation given the **exact** Schwarzschild metric in D dimensions

Self-force expansion: geodesic (0SF)

The geodesic equation for the light mass m is obtained by a **double infinite resummation**:

1. loop for generating the Schwarzschild metric from the heavy mass M
2. the loops for the graviton coupling between the background generated by the heavy mass and the light mass

$$\mathcal{L}_0[x^\mu(\tau), u_H^\mu \tau_H] =$$



This gives a totally consistent self-force formalism without any need to introduce an external metric.

Summary

1. The amplitude computations have led to analytic formulæ for the two body scattering (with no spin) to 4PM. To this order, there is an excellent match with numerical GR when radiation is included.
2. The amplitude approach and the exponential formalism gives a clean and streamlined approach to classical gravitational radiation.
3. This provides a natural organisation of the classical $\hbar \to 0$ limit for the two body system. We have here an example of Bohr's correspondance principle

   The leading idea (Bohr's correspondence principle, 1923) may be stated broadly as follows.[...] In other words, it must be demanded that, for the limiting cases of large masses and of orbits of large dimensions, the new mechanics passes over into classical mechanics. (Max Born Atomic Physics)

4. We have presented a cubic formalism for the self-force formalism which is valid for in the scattering and bound regime and in any dimensions With this formalism the heavy mass background metric is not inserted by hand but derived within the formalism.