String Theory Amplitudes on AdS

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What will this review be about?

A set of tools to compute String Theory amplitudes on AdS

Why scattering amplitudes?

- They allow to test the predictions of our theory.
- They can teach us much about its structures/symmetries.
- There has been great progress regarding amplitudes in flat space, and it's interesting to see how much we can say about AdS.

More specifically:

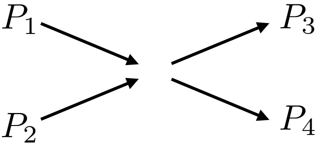
Scattering of four massless strings (gravitons) on $AdS_5 \times S^5$.

First we will review the story in flat space.

Scattering amplitudes

Scattering Amplitudes

Probability that two particles/strings colliding (with momenta p_1, p_2) result into two other particles (with momenta p_3, p_4).



$$= \quad A(g, s, t, u)$$

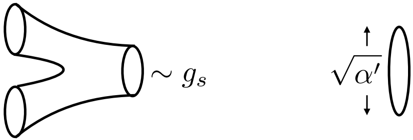
- $A(g, s, t, u)$ depends on many things:
  - The parameters of your theory g.
  - The particles you are scattering (their masses, polarisations, etc)
  - The momenta of the particles being scattered:

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2$$

Four-graviton amplitude - Flat space

4pt graviton amplitude in flat space

- The parameters of the theory are g_s and α'.

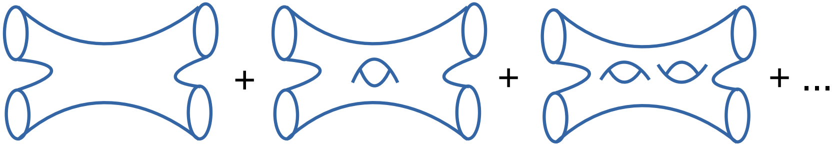


- The amplitude depends on the momenta p_i and polarisations ϵ_i of the (external) gravitons.
- SUSY fixes the dependence on the polarisations:

$$A(g_s, \alpha', p_i, \epsilon_i) = \underbrace{pref(\epsilon_i, p_i)}_{\text{simple prefactor}} \times \underbrace{A(g_s, \alpha', s, t, u)}_{\text{we focus on this}}$$

String theory scattering amplitudes

- The computation organises in a genus expansion



$$A^{(genus\ 0)}(\alpha', s, t, u) + g_s^2 A^{(genus\ 1)}(\alpha', s, t, u) + g_s^4 A^{(genus\ 2)}(\alpha', s, t, u) + \cdots$$

- In flat space we can use the world-sheet theory to compute these amplitudes:

$$A^{(genus\ 0)}(\alpha', s, t, u) \sim \int_{CP^1} |z|^{2\alpha' s - 2} |1 - z|^{2\alpha' t - 2} d^2 z$$

- Note: already at genus-one the expressions are tremendously complicated!

Four-graviton amplitude - Flat space

Leading order in g_s: Virasoro-Shapiro amplitude

$$A_{VS}(\alpha', s, t, u) = \alpha'^3 \frac{\Gamma(-\alpha' s)\Gamma(-\alpha' t)\Gamma(-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}$$

- Crossing symmetric ($s + t + u = 0$)
- Poles due to the exchange of particles (of mass $m = 2\sqrt{n/\alpha'}$ and spin ℓ)

$$A_{VS}(\alpha', s, t, u) \sim \frac{P_\ell(t, u)}{\alpha' s - n}$$

- Regge behaviour

$$A_{VS}(\alpha', s, t, u) \sim t^{-2+\alpha' \frac{s}{2}}, \quad \text{for large } |t|$$

- α' expansion

$$A_{VS}(\alpha', s, t, u) \sim \underbrace{\frac{1}{s\,t\,u}}_{\text{sugra}} + \underbrace{2\zeta(3)\alpha'^3 + 2\zeta(5)\alpha'^5(s^2 + t^2 + u^2) + \cdots}_{\text{stringy corrections}}$$

Four-graviton amplitude - Flat space

A less appreciated property...

- Only odd ζ-values appear in the expansion:

$$A_{VS}(\alpha', s, t, u) = \frac{1}{s\,t\,u} \exp\left(2\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{2n+1} \alpha'^{2n+1}(s^{2n+1}+t^{2n+1}+u^{2n+1})\right)$$

Quite deep from a mathematical point of view!

VS and single-valued periods

- Zeta values (MZV) can be defined in terms of sums

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$$

- Or in terms of polylogarithms evaluated at $z = 1$

$$Li_n(z) = \sum_{k=1} \frac{z^k}{k^n} \to Li_n(1) = \zeta(n)$$

- While these series converge for $|z| < 1$, polylogarithms can be analytically continued to the whole complex plane:

$$Li_1(z) = -\log(1-z), \quad Li_n(z) = \int_0^z Li_{n-1}(t) \frac{dt}{t}$$

- However these functions are not single-valued!

VS and single-valued periods

Single-valued polylogarithms

- Unique map from multi-valued to single-valued polylogarithms

$$Li_n(z) \to \mathcal{L}_n(z,\bar{z})$$

- $\mathcal{L}_n(z,\bar{z})$ is a weight preserving linear combination of $Li_w(z)Li_{w'}(\bar{z})$
- Differential relations are preserved.

$$\begin{aligned} \log z &\to \log z + \log \bar{z} \\ Li_2(z,\bar{z}) &\to \mathcal{L}_2(z) = Li_2(z) - Li_2(\bar{z}) - \log(1-\bar{z}) \log |z|^2 \\ Li_3(z,\bar{z}) &\to \mathcal{L}_3(z) = Li_3(z) + Li_3(\bar{z}) + \cdots \end{aligned}$$

VS and single-valued periods

Single-valued multiple zeta values

- Polylogarithms evaluated at $z = 1 \to$ zeta values.
- Single-valued polylogarithms evaluated at $z = 1$ define what we call single-valued zeta values:

$$\zeta_{sv}(2) = \mathcal{L}_2(1) = 0, \quad \zeta_{sv}(3) = \mathcal{L}_3(1) = 2\zeta(3)$$

- More generally

$$\zeta_{sv}(2n) = 0, \quad \zeta_{sv}(2n+1) = 2\zeta(2n+1)$$

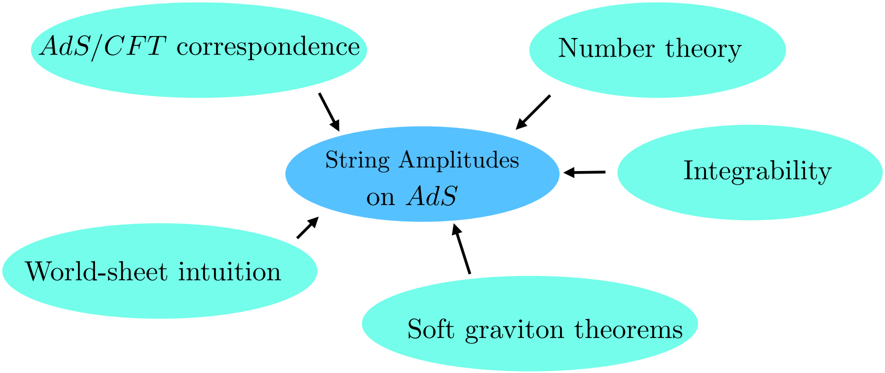
- Odd zeta values are single valued, while $\zeta(2n)$ are not! single valued zetas are a subset of the usual zeta values.

Important message

The α' expansion of the VS amplitude contains only single-valued zetas.

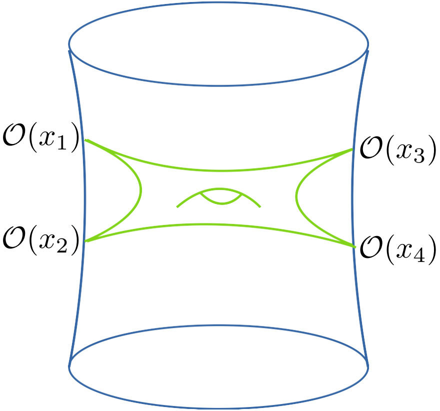
VS in AdS

In curved backgrounds we don't have a world-sheet theory...but for the special case of $AdS_5 \times S^5$ we can make a lot of progress!



AdS/CFT

String amplitudes on *AdS* $\leftrightarrow$ Correlators of local operators in the CFT at the boundary.



$$\mathcal{A}(g_s, \alpha', s, t, u) \leftrightarrow \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle$$

AdS/CFT

String theory on $AdS_5 \times S^5$ $\leftrightarrow$ 4d $\mathcal{N}=4$ SYM

(g_s, R) $\qquad$ (g_{YM}, N)

$$g_s \approx \frac{1}{N}, \qquad \frac{R^2}{\alpha'} = \sqrt{g_{YM}^2 N} \equiv \sqrt{\lambda}$$

String amplitudes on $AdS_5 \times S^5$	Correlators in $\mathcal{N}=4$ SYM
Genus expansion	$1/N$ expansion
Stringy corrections to sugra	$1/\lambda$ corrections
Graviton on AdS	$\mathcal{O}_2$: Scalar operator of dim. 2 in the stress-tensor multiplet

Consider $\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$ in a $1/N$ expansion.

4d $\mathcal{N} = 4$ SYM - Kinematics

The symmetry

- Gauge group $SU(N)$, all fields in the adjoint representation.
- Maximal SUSY + conformal symmetry.

$$PSU(2,2|4) \supset \underbrace{SO(2,4)}_{\text{conformal symmetry}} \oplus \underbrace{SO(6)}_{\text{R-symmetry}}$$

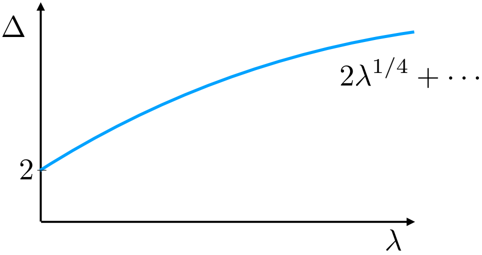
The operators

$$\underbrace{\mathcal{O}^{IJ}(x)}_{\text{sym. traceless of } SO(6)} = Tr\varphi^{(I}\varphi^{J)}$$

Their dimension is always $\Delta = 2$. Simplest protected operator.

Konishi operator

$$\mathcal{K}(x) = Tr\varphi^I\varphi^I$$



4d $\mathcal{N} = 4$ SYM - Kinematics

The observable

$$\langle \mathcal{O}^{I_1 J_1}(x_1)\mathcal{O}^{I_2 J_2}(x_2)\mathcal{O}^{I_3 J_3}(x_3)\mathcal{O}^{I_4 J_4}(x_4)\rangle$$

- Fixed by symmetries up to a function of two cross-ratios!

$$\langle \mathcal{O}^{I_1 J_1}(x_1)\mathcal{O}^{I_2 J_2}(x_2)\mathcal{O}^{I_3 J_3}(x_3)\mathcal{O}^{I_4 J_4}(x_4)\rangle = pref(I_i, J_i, x_i) \times \underbrace{\mathcal{G}(U,V)}_{\text{we focus on this}}$$

$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$

- Similar structure to scattering of gravitons in flat space!

$1/N$ expansion

- $\mathcal{G}(U,V)$ a highly-non trivial function of N and λ.
- Consider the leading non-trivial term in the $1/N$ expansion:

$$\mathcal{G}(U,V) = \underbrace{\mathcal{G}_{disc}(U,V)}_{disconnected} + \frac{1}{N^2}\underbrace{\mathcal{G}_{tree}(U,V)}_{tree-level} + \cdots$$

- Complicated function of λ. Expand further in $1/\lambda$:

$$\mathcal{G}_{tree}(U,V) = \underbrace{\mathcal{G}^{(sugra)}(U,V) + \frac{1}{\lambda^{3/2}}\mathcal{G}^{(1)}(U,V) + \frac{1}{\lambda^{5/2}}\mathcal{G}^{(2)}(U,V) + \cdots}_{\text{Virasoro-Shapiro on AdS}}$$

- $\mathcal{G}^{(sugra)}(U,V)$ can be computed via "Witten-diagrams" (done 22 years ago), but we are interested in the whole tower.

The right language: Mellin space

Mellin space

Formulation where the correlator looks much more like an amplitude.

$\mathcal{G}_{tree}(U,V) \to \mathcal{M}_{tree}(s,t,u)$, with $s+t+u=4$.

$$\boxed{\mathcal{G}_{tree}(U,V) = \int_{-i\infty}^{i\infty} ds dt U^s V^t \underbrace{\Gamma(s,t,u)}_{\text{prefactor}} \quad \underbrace{\mathcal{M}_{tree}(s,t,u)}_{\text{VS amplitude in } AdS_5\times S^5}}$$

$\mathcal{M}_{tree}(s,t,u)$ is a meromorphic function with very nice properties!

AdS VS amplitude

$\underline{\mathcal{M}_{tree}(s,t,u)}$

1. Crossing symmetric.
2. Exchanged operators lead to simple poles:

$$\mathcal{M}_{tree}(s,t) = C^2_{\Delta,\ell} \sum_{m=0}^{\infty} \frac{Q_{\ell,m}(u,t)}{s-(\Delta-\ell)-2m} + \text{regular}$$

3. Regge limit

$$\mathcal{M}_{tree}(s,t) \sim \frac{1}{t^2}, \quad \text{for large } |t| \text{ and } Re(s) < 2$$

4. In the 'flat-space limit' it should reduce to the usual VS amplitude.

Can we use these constraints to fix $\mathcal{M}_{tree}(s,t,u)$?

Extremely powerful when supplemented with another constraint :)

AdS Virasoro-Shapiro around flat space

- Consider $\mathcal{M}_{tree}(s,t)$ in a $1/\lambda$ expansion

$$\mathcal{M}_{tree}(s,t) = \underbrace{\frac{1}{(s-2)(t-2)(u-2)}}_{sugra} +$$

$$\frac{\alpha_{0,0}}{\lambda^{3/2}} + \frac{\alpha_{1,0}(s^2+t^2+u^2)+\gamma_{0,0}}{\lambda^{5/2}} + \frac{\alpha_{0,1}\, s\, t\, u + \cdots}{\lambda^3} + \cdots$$

- Flat-space limit $\rightarrow$ large s,t,u,R, with $s/R^2 \sim$ fixed (recall $\sqrt{\lambda} = R^2/\alpha')$. Rescaling s,t,u by R^2:

$$\mathcal{M}_{tree}(s,t) \rightarrow \frac{1}{s\, t\, u} + \alpha_{0,0}\alpha'^3 + \alpha_{1,0}(s^2+t^2+u^2)\alpha'^5 + \cdots$$

- And this should agree with the flat space answer.

AdS Virasoro-Shapiro around flat space

- $\mathcal{M}_{tree}(s,t,u)$ admits an expansion around flat space

$$\mathcal{M}_{tree}(s,t) = \underbrace{A^{(0)}(s,t)}_{\text{VS in flat space}} + \underbrace{\frac{\alpha'}{R^2}A^{(1)}(s,t) + \frac{\alpha'^2}{R^4}A^{(2)}(s,t) + \cdots}_{\text{curvature corrections}}$$

- Where each bit admits a low energy expansion

$$A^{(0)}(s,t) = \frac{1}{s\,t\,u} + 2\zeta(3)\alpha'^3 + 2\zeta(5)\alpha'^5(s^2+t^2+u^2) + \cdots$$

$$A^{(1)}(s,t) = \underbrace{\frac{s^2+t^2+u^2}{(s\,t\,u)^2}}_{\text{gravity on } AdS} + \underbrace{\alpha_1\alpha'^4 + \alpha_2\alpha'^6(s^2+t^2+u^2) + \cdots}_{\text{unknown coefficients}}$$

Assumption: these unknown coefficients are also single-valued zetas!

AdS VS amplitude

$\underline{\mathcal{M}_{tree}(s,t,u)}$

1. Crossing symmetry.
2. Exchanged operators lead to simple poles:

$$\mathcal{M}_{tree}(s,t) = C^2_{\Delta,\ell} \sum_{m=0}^{\infty} \frac{Q_{\ell,m}(u,t)}{s-(\Delta-\ell)-2m} + \text{regular}$$

3. Regge limit

$$\mathcal{M}_{tree}(s,t,u) \sim \frac{1}{s^2}, \quad \text{for large } |s|$$

4. In the 'flat-space limit' it should reduce to the usual VS amplitude.
5. The low energy expansion of $\mathcal{M}_{tree}(s,t,u)$ contains only single-valued zetas!

AdS Virasoro-Shapiro around flat space

Very powerful when supplemented with the correct structure of poles!

- While $A^{(0)}(s,t)$ has single poles, corrections are more complicated:

$$A^{(1)}(s,t) \sim \frac{r_n^{(0)}(t)}{(\alpha' s - n)^4} + \frac{r_n^{(1)}(t)}{(\alpha' s - n)^3} + \cdots$$

- This follows from the AdS-propagator around flat-space (and also the dispersive sum rules). In general

$$\mathcal{M}_{tree}(s,t) = \underbrace{A^{(0)}(s,t)}_{\text{simple poles}} + \frac{\alpha'}{R^2} \underbrace{A^{(1)}(s,t)}_{\text{quartic poles}} + \frac{\alpha'^2}{R^4} \underbrace{A^{(2)}(s,t)}_{\text{seventh order poles}} + \cdots$$

AdS Virasoro-Shapiro amplitude

Poles + Single-valuedness + World-sheet intuition

$\Downarrow$

Proposal order by order

$$A^{(0)}(s,t) = \int_{CP^1} d^2z |z|^{2\alpha' s-2}|1-z|^{2\alpha' t-2}$$

$$A^{(1)}(s,t) = \int_{CP^1} d^2z |z|^{2\alpha' s-2}|1-z|^{2\alpha' t-2} \underbrace{W_3(z,\bar{z})}_{\text{SV polylogs of weight 3}}$$

$$A^{(2)}(s,t) = \int_{CP^1} d^2z |z|^{2\alpha' s-2}|1-z|^{2\alpha' t-2} \underbrace{W_6(z,\bar{z})}_{\text{SV polylogs of weight 6}}$$

$$\vdots$$

Also consistent with soft graviton theorems.

AdS Virasoro-Shapiro amplitude

$$A^{(1)}(s,t) = \int_{CP^1} d^2z |z|^{2\alpha' s-2}|1-z|^{2\alpha' t-2} \qquad \underbrace{W_3(z,\bar{z})}_{\text{SV polylogs of weight 3}}$$

- Convenient basis $\mathcal{L}_{a,b,c}(z,\bar{z})$, with $a,b,c=0,1$.

$$\frac{\partial}{\partial z}\mathcal{L}_{a,b,c}(z,\bar{z}) = \frac{1}{z-a}\mathcal{L}_{b,c}(z,\bar{z})$$

- Our ansatz:

$$W_3(z,\bar{z}) = P_{0,0,0}(s,t)\mathcal{L}_{0,0,0}(z,\bar{z}) + \cdots + P_{1,1,1}(s,t)\mathcal{L}_{1,1,1}(z,\bar{z}) + P(s,t)\zeta(3)$$

second order homogeneous polynomials

- Structure of poles so constraining, that fixes $W_3(z,\bar{z})$ completely!

AdS Virasoro-Shapiro amplitude

$$A^{AdS}_{VS}(s,t) = \int d^2z \frac{|z|^{2\alpha' s}|1-z|^{2\alpha' t}}{|z|^2|1-z|^2}\left(1+\frac{\alpha'}{R^2}W_3(z,\bar{z})+\frac{\alpha'^2}{R^4}W_6(z,\bar{z})+\cdots\right)$$

$W_3(z,\bar{z})$, $W_6(z,\bar{z})$ fully fixed by our procedure!

- Crossing symmetric ✓
- Single-valued low energy expansion ✓
- The 'structure' of poles is already very constraining! the answer can be fully fixed by *e.g.* localisation results.
- From the answer we can read of a wealth of CFT-data, *e.g.*

$$\Delta_{\mathcal{K}} = 2\lambda^{1/4} - 2 + \frac{2}{\lambda^{1/4}} + \frac{1/2 - 3\zeta(3)}{\lambda^{3/4}} + \cdots$$

In agreement with the results from integrability for planar $\mathcal{N}=4$ SYM!

Final proposal

AdS Virasoro-Shapiro amplitude

$$A_{VS}^{AdS}(s,t) = \int d^2z \frac{|z|^{2\alpha' s}|1-z|^{2\alpha' t}}{|z|^2|1-z|^2}\left(1+\frac{\alpha'}{R^2}W_3(z,\bar{z})+\frac{\alpha'^2}{R^4}W_6(z,\bar{z})+\cdots\right)$$

Can we guess the result to all orders in $1/R$?

Yes! in two particular regimes

- High energy limit $(s,t \gg 1)$
- Regge limit $(t \gg 1$, finite $s)$

The high energy limit

- Scattering amplitudes in the high energy limit $(s, t \gg 1)$ can be computed by saddle point techniques. **[Gross, Mende]**

$$z = \bar{z} = \frac{s}{s+t}$$

- In AdS $1/R$ corrections exponentiate in this limit! **[L.F.A., Hansen, Nocchi]**

$$A_{VS}^{AdS}(s,t)_{HE} = A_{VS}^{flat}(s,t)_{HE} \times \exp\left(\frac{\alpha'}{R^2} W_3(\frac{s}{s+t})\right)$$

- This can be reproduced by a classical scattering problem in AdS.

Regge limit

- The Regge limit is much richer **[L.F.A., Nocchi, Virally, Zhou]**
- Full information on leading twist operators, *i.e. Konishi*

$$A^{AdS}_{VS}(s,t)_{Regge} = \mathcal{R}\left(\partial_S\right) A^{flat}_{VS}(s,t)_{Regge}$$

Curvature of space-time

- The operator $\mathcal{R}$ takes into account the curvature of $AdS_5 \times S^5$.

$$\mathcal{R}(\partial_s) = 1 + \frac{1}{R^2}\left(-\frac{s^2}{6}\partial_s^3 + \cdots\right) + \frac{1}{R^4}\left(\frac{s^4}{36}\partial_s^6 + \cdots\right) + \cdots$$

- Satisfies a differential relation that can be solved to any order!

$$\left(\partial_y + \sum_{n=0}^{\infty} \frac{1}{R^{2n+2}} P_n^{(2)}(s,y)\partial_y^{n+2}\right)\mathcal{R}(y) = \delta(y)$$

Conclusions

Computing the full AdS VS amplitude seems now within reach!

- Single valuedness plays an important role in understanding and constructing scattering amplitudes in flat space. Now also in *AdS*!
- New connections between standard bootstrap techniques, localisation, integrability and number theory.
- The high energy and Regge regimes provide much more manageable limits.
- Similar developments for open strings. [e.g. T. Hansen talk]

In the near future

- All orders/exact in $1/R$?
- Other *AdS* backgrounds?
- Connection to a more direct approach?