Higher-form symmetries and the A-twist

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Supersymmetric QFT often gives us an analytical handle on strongly-coupled physics.

[Seiberg, Witten, 1994; ...]

Exact, **non-perturbative** computations are possible. In the path-integral formulation, this is generally explored through supersymmetric localisation:

$$\langle \mathcal{O} \rangle = \int [D\phi DA]\, \mathcal{O} e^{-S[\phi, A, \cdots]} = \sum_{\text{top. sector } k} \int_{\mathfrak{M}_k} d\varphi_k dA_k\, \mathcal{O}|_{\mathcal{M}_k} e^{-S[\varphi_k, A_k, \cdots]}$$

To make this precise, we need to replace (Euclidean) space-time by a more general d-dimensional manifold (more generally, a geometric background [Festuccia, Seiberg, 2011]):

$$\mathbb{R}^d \quad \rightsquigarrow \quad M_d$$

Supersymmetric partition functions

There are two (related) approaches:

(1) $M_d \cong \mathbb{R}^d + \Omega$ deformation: modifies the UV couplings at the origin. [Nekrasov, 2002; ...]

(2) M_d compact and smooth. UV physics preserved. [Pestun, 2007; Kapustin, Willett, Yaakov, 2009; ...]

(One can obtain (2) from gluing (1) patches, and one can also extend (2) to some some singular backgrounds [e.g. spindles; see Dario Martelli's talk].)

In the following, 'supersymmetric partition function' with $M_d =$ (2).

We are here interested in gauge theories with four supercharges and a $U(1)_R$ symmetry:

$$4\text{d}\ \mathcal{N}=1 \quad \to \quad \boxed{3\text{d}\ \mathcal{N}=2} \quad \to \quad 2\text{d}\ \mathcal{N}=(2,2)$$

They often flow to interesting superconformal field theories in the IR.

Generalised symmetries and global structures

[Gaiotto, Kapustin, Seiberg, Willett, 2014]

Over the past 10 years, the **symmetries of QFTs** have been much revisited. In particular, we better understand the higher-form symmetries acting on higher-dimensional operators:

$$[\mathcal{U}^\gamma, \mathscr{L}_p] = \mathfrak{R}(\gamma)\mathscr{L}_p \ , \qquad \gamma \in \Gamma^{(p)}$$

Here $\mathscr{L}_p$ is p-dimensional. In the Euclidean path integral, the symmetry operator is a topological operator defined on a submanifold $\mathcal{C}_{d-p-1}$ linking $\mathscr{L}_p$:

$$\mathcal{U}^\gamma(\mathcal{C}_{d-p-1}) \ .$$

We will focus on $\Gamma^{(p)}$ a discrete abelian group ('invertible symmetries').

On compact manifolds, the insertion of $\mathcal{U}^\gamma$ only depends on the homology class:

$$[\mathcal{C}_{d-p-1}] \in H_{d-p-1}(M_d, \Gamma^{(p)})$$

Dually, we introduce the 'flat' background gauge fields:

$$B_{p+1} \in H^{p+1}(M_d, \Gamma^{(p)})$$

Generalised symmetries and global structures

Common examples are the **one-form 'center' symmetries** of gauge theories with (real, compact) gauge group G with non trivial center:

$$\Gamma^{(1)} \cong \Gamma \subseteq Z(G) \ .$$

- $\mathcal{U}^{\gamma}(\mathcal{C}_{d-2})$ acts on the Wilson loops $\mathscr{L}_1 = W$ by a phase.
- We can compactify space-time to $\mathbb{R}^{d-1} \times S^1$ and wrap W over S^1. This Polyakov loop is a local operator in $\mathbb{R}^{d-1}$ and there is an ordinary (0-form) symmetry acting on it.
- Color confinement iff $\Gamma^{(1)}$ preserved by the vacuum ($\langle W \rangle \sim e^{-A} = 0$).

Given a gauge theory with gauge group G and symmetry $\Gamma^{(1)}$, we can construct the gauge theory with the smaller gauge group:

$$G' = G/\Gamma' \ , \qquad \Gamma' \subseteq \Gamma$$

by gauging a non-anomalous subgroup $\Gamma' \subseteq \Gamma^{(1)}$ of the one-form symmetry.

The choice of G for a fixed Lie algebra $\mathfrak{g}$ is often called the 'global structure' of the gauge theory. In 4d, it does not affect the spectrum of local operators, only the lines.

[Aharony, Seiberg, Tachikawa, 2013]

Higher-form symmetries and supersymmetry

The symmetry operators $\mathcal{U}^\gamma$ are topological, hence supersymmetric. Thus, in principle, we can use supersymmetric methods to compute exactly any expectation value:

$$\langle \mathcal{U}^\gamma \rangle_{M_d} = Z_{M_d}(B_\gamma)$$

on a supersymmetric M_d. This would allow us to gauge higher-form symmetries explicitly:

$$Z_{M_d}[\mathcal{T}] \quad \rightsquigarrow \quad Z_{M_d}[\mathcal{T}/\Gamma^{(p)}] = \sum_{B \in H^{p+1}(M_d, \Gamma^{(p)})} Z_{M_d}[\mathcal{T}](B)$$

Perhaps surprisingly, there has been little work exploring higher-form symmetries with supersymmetry, so far.
(Early work on global structures on Lens spaces: [Razamat, Willett, 2013].)

In the following, we will focus on 3d $\mathcal{N} = 2$ supersymmetric gauge theories. This already gives rise to an intricate structure.

The 3d A-twist and 1-form symmetries

The 3d A-twist and the twisted index

2d $\mathcal{N} = (2,2)$ theories with a $U(1)_R$ 'vector' R-symmetry can be defined on any closed Riemann surface Σ through the topological A-twist: [Witten, 1988]

$$L_R \cong \sqrt{\mathcal{K}_\Sigma} \ .$$

The nilpotent scalar supercharges $\mathcal{Q}$, $\bar{\mathcal{Q}}$ can be used to define a **2d TQFT:**

A-model = 2d Cohomological TQFT $\mathbf{Z} \cong H_{\rm SUSY}($2d $\mathcal{N} = (2,2)$ QFT$)$

The 3d $\mathcal{N} = 2$ half-BPS backgrounds are Seifert 3-manifolds – circle-fibered over the (orbifold) surface $\hat{\Sigma}$: [CC, Dumitrescu, Festuccia, Komargodski, 2013; CC, Kim, Willett, 2017-2018]

$$S^1 \longrightarrow M_3 \xrightarrow{\pi} \hat{\Sigma}$$

The 3d A-twist is a pull-back of the A-twist on $\hat{\Sigma}$ through π. Simplest case:

$$M_3 = S^1 \times \Sigma_g \ .$$

The supersymmetric partition then computes the **topologically twisted index:**

$$Z_{\Sigma_g \times S^1}[\mathcal{T}] = {\rm Tr}_{\mathcal{H}_{\Sigma_g}} \left((-1)^{\rm F} y^{Q_F} \right)$$

[Nekrasov, Shatashvili, 2014; Benini, Zaffaroni, 2015-2016; CC, Kim, 2016]

The 3d A-twist and the twisted index

To compute this index, it is best to exploit the 2d TQFT structure:

3d $\mathcal{N}=2$ theory on $S^1 \times \mathbb{R}^2 \cong$ effective 2d $\mathcal{N}=(2,2)$ Kaluza-Klein (KK) description.

The effective A-model has a finite number of states on $S^1 \subset \Sigma$:

$$\mathcal{H}_{S^1} \cong \mathrm{Span}_{\mathbb{C}}\{|\hat{u}\rangle\}$$

$\hat{u}$ are the **Bethe vacua**, due to the Bethe/gauge correspondence. [Nekrasov, Shatashvili, 2009]
They are the solutions to the effective twisted superpotential of the 2d KK theory:

$$\exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial u}\right) = 1 \;, \qquad\qquad u \in \mathfrak{g}_{\mathbb{C}} \cong \mathrm{Lie}(G)_{\mathbb{C}}$$

Then:

$$Z_{\Sigma_g \times S^1}[\mathcal{T}] = \mathrm{Tr}_{\mathcal{H}_{S^1}}\left(\mathcal{H}^{g-1}\right) = \sum_{\hat{u} \in \mathcal{S}_{\mathrm{BE}}} \mathcal{H}(\hat{u})^{g-1}$$

Here $\mathcal{H}$ is the handle-gluing operator, given by: [Vafa, 1991; Nekrasov, Shatashvili, 2009]

$$\mathcal{H}(u) = \mathrm{Hess}(\mathcal{W}(u))\, \exp\left(2\pi i \Omega(u)\right)$$

where Ω is the effective dilaton coupling on Σ.

The 3d A-twist and the twisted index

This Bethe-vacua formalism only works for 3d gauge theories with gauge group $G = \tilde{G}$ simply-connected and/or unitary – that is, with:

$$\pi_1(\tilde{G}) \cong \mathbb{Z}^{n_T}$$

For instance, it works for $G = U(N)$ or $SU(N)$ but not for $PSU(N) = SU(N)/\mathbb{Z}_N$:

$$\pi_1(SU(N)) = 0 \;, \qquad \pi_1(PSU(N)) = \mathbb{Z}_N$$

In particular, it works for $SU(2)$ but not $SO(3) = PSU(2)$.

To extend the A-twist formalism to any G, we need to: [CC, Furrer, Khlaif, 2024]

1. Study the one-form (center) symmetry $\Gamma^{(1)}_{\rm 3d}$ of the $\tilde{G}$ theory as it acts on $\mathcal{H}_{S^1}$.
2. Explicitly gauge $\Gamma^{(1)}_{\rm 3d}$ in the A-model formalism.

Previous works along those lines: [Willett, 2019 (unpublished)]

[Eckhard, Kim, Schafer-Nameki, Willett, 2019; Gukov, Pei, Reid, Shehper, 2021]

Higher-form symmetries in the 2d description

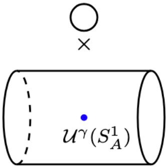
The one-form symmetry operators $\mathcal{U}^{\gamma}(\mathcal{C})$ are one-dimensional. They can themselves be charged under $\Gamma^{(1)}$ ('t Hooft anomaly).

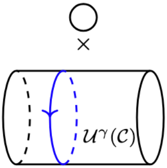
Upon compactification to 2d, we have:

$$\Gamma^{(1)}_{3\mathrm{d}} = \Gamma \qquad \longrightarrow \qquad \Gamma^{(1)} \cong \Gamma \ , \qquad \Gamma^{(0)} \cong \Gamma \ .$$

with the topological point and line operators:

$$\longrightarrow \qquad \Pi^{\gamma} \cong \mathcal{U}^{\gamma}(S^1_A) \ , \qquad \mathcal{U}^{\gamma} \ .$$





One-form symmetry $\Gamma^{(1)}$ and decomposition

The two-dimensional $\Gamma^{(1)}$ is necessarily preserved by the 2d vacua. ($\Gamma^{(p)}$ cannot be spontaneously broken in dimension $d < p+2$. [Gaiotto, Kapustin, Seiberg, Willett, 2014])

In fact, Π^γ for $\gamma \in \Gamma^{(1)}$ acts as:

$$\Pi^\gamma |\hat{u}\rangle = \chi_{\hat{u}}(\gamma)|\hat{u}\rangle \ , \qquad \chi_{\hat{u}} \in \hat{\Gamma}^{(1)} = \mathrm{Hom}(\Gamma^{(1)}, U(1))$$

Inserting the topological line Π^γ is equivalent to inserting a background $G = \tilde{G}/\Gamma$ bundle over Σ which is not a $\tilde{G}$ bundle:

$$\Lambda_{\mathrm{mw}}^{\tilde{G}/\Gamma} \supset \Lambda_{\mathrm{mw}}^{\tilde{G}} \ , \qquad \Gamma^{(1)} \cong \Lambda_{\mathrm{mw}}^{\tilde{G}/\Gamma}/\Lambda_{\mathrm{mw}}^{\tilde{G}} \ .$$

This topological point operator is then a flux operator, which is known to be governed by the effective twisted superpotential: [CC, Kim, Willett, 2017]

$$\Pi(u)^\gamma \equiv \exp\left(2\pi i \gamma \frac{\partial \mathcal{W}}{\partial u}\right)$$

Note that the Bethe vacua satisfy:

$$\Pi(\hat{u})^{\mathfrak{m}} = 1 \ , \qquad \forall \mathfrak{m} \in \Lambda_{\mathrm{mw}}^{\tilde{G}}$$

One-form symmetry $\Gamma^{(1)}$ and decomposition

[Hellerman *et al*, 2009; Sharpe *et al*, 2022]

The existence of a one-form symmetry in 2d implies decomposition in 'universes':

$$\mathcal{H}_{S^1} = \bigoplus_{\chi \in \hat{\Gamma}^{(1)}} \mathcal{H}^{\chi}_{S^1} \ , \qquad \mathcal{H}^{\chi}_{S^1} \equiv \mathrm{Span}_{\mathbb{C}} \left\{ |\hat{u}\rangle \ \middle| \ \hat{u} \in \mathcal{S}^{\chi}_{\mathsf{BE}} \right\} \ ,$$

$\Gamma^{(1)}$ is always non-anomalous. Gauging it is exceedingly simple, and projects us onto a specific 'universe':

$$Z^{[\mathcal{T}/\Gamma^{(1)}]}_{\Sigma_g \times S^1}(\vartheta) = \sum_{\hat{u} \in \mathcal{S}^{\chi=\vartheta}_{\mathsf{BE}}} \mathcal{H}(\hat{u})^{g-1} = \mathrm{Tr}_{\mathcal{H}^{\vartheta}_{S^1}} (\mathcal{H}^{g-1}) \ .$$

Here ϑ is a background gauge field for the dual (-1)-form symmetry.

Zero-form symmetry $\Gamma^{(0)}$ and twisted sectors

Note that, off-shell, $u \sim u + \mathfrak{m}$ and

$$u \in \mathfrak{g}_{\mathbb{C}} \ , \qquad \gamma + \mathfrak{m} \in \Lambda_{\rm mw}^{\tilde{G}/\Gamma} \subset \mathfrak{g}_{\mathbb{C}}$$

The action of $\Gamma^{(0)}$ on Bethe vacua is more interesting. It acts by permutations:

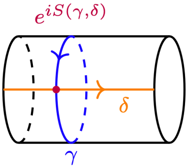
$$\mathcal{U}^{\gamma}(\mathcal{C})|\hat{u}\rangle = |\hat{u} + \gamma\rangle$$

Hence, in each vacuum $\hat{u}$, the symmetry $\Gamma^{(0)}$ is (generically) spontaneously broken to a subgroup:

$$\Gamma^{(0)} \quad \longrightarrow \quad {\rm Stab}(\hat{u}) \subseteq \Gamma^{(0)}$$

Ahead of gauging (a.k.a. ‘orbifolding’), we should also consider the **twisted sectors:**

$$\mathcal{H}_{S^1}^{(\delta)} \cong {\rm Span}_{\mathbb{C}}\Big\{ |\hat{u};\delta\rangle \ \Big| \ \hat{u} \in \mathcal{S}_{\rm BE} \, , \hat{u}+\delta \sim \hat{u} \Big\} \, .$$



Zero-form symmetry $\Gamma^{(0)}$ and twisted sectors

The twisted sectors exist for Bethe vacua with non-zero stabiliser:

$$\mathcal{S}_{\mathsf{BE}}^{(\gamma)} \equiv \left\{ \hat{u} \in \mathcal{S}_{\mathsf{BE}} \;\middle|\; \hat{u} + \gamma \sim \hat{u} \right\} , \qquad \gamma \in \Gamma^{(0)} ,$$

Given an orbit $\hat{\omega} = \mathrm{Orb}(\hat{u})$ under $\Gamma^{(0)}$, the number of twisted sector is:

$$|\mathrm{Stab}(\hat{u})| = \frac{\left|\Gamma^{(0)}\right|}{|\mathrm{Orb}(\hat{u})|} ,$$

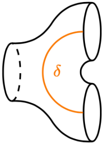
In the KK theory under consideration, $\Gamma^{(0)}$ is non-anomalous. We can then gauge it and get the gauged Hilbert space:

$$\mathcal{H}_{S^1}^{[\mathcal{T}/\Gamma^{(0)}]} \cong \mathrm{Span}_{\mathbb{C}} \left\{ |\hat{\omega}; s_{\hat{\omega}}\rangle \;\middle|\; \hat{\omega} \in \mathcal{S}_{\mathsf{BE}}/\Gamma^{(0)} \ , \ s_{\hat{\omega}} = 1, \cdots, |\mathrm{Stab}(\hat{\omega})| \right\} ,$$

Zero-form symmetry $\Gamma^{(0)}$ and twisted sectors

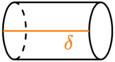
On Σ_g, we can do the gauging explicitly, summing over topological line insertions, using the TQFT structure:

$= \sum_{\hat{u}} \langle \hat{u} | \mathcal{H}^{-\frac{1}{2}}$

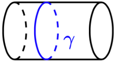


$= \sum_{\hat{u}} \mathcal{H}^{\frac{1}{2}} |\hat{u}\rangle \langle \hat{u}; \delta | \langle \hat{u}; \delta |$

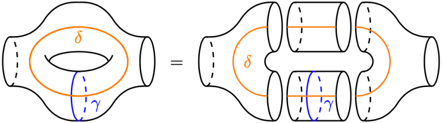
$= \sum_{\hat{u}} \mathcal{H}^{-\frac{1}{2}} |\hat{u}\rangle$



$= \sum_{\hat{u}} |\hat{u}; \delta\rangle \langle \hat{u}; \delta |$



$= \sum_{\hat{u}} |\hat{u}\rangle \langle \hat{u} | \mathcal{U}^{\gamma}$



't Hooft anomalies and orbit structure

In 3d, there can be a non-trivial anomaly for $\Gamma^{(1)}_{3\mathrm{d}}$, which becomes a mixed anomaly in the 2d A-model:

$$S^{\mathrm{anom}}_{4\mathrm{d}}[B] = 2\pi\mathfrak{a}\int_{\mathfrak{M}_4} B\cup B \qquad \rightsquigarrow \qquad S^{\mathrm{anom}}_{3\mathrm{d}}[B] = 2\pi\mathfrak{a}\int_{\mathfrak{M}_3} B\cup C$$

This anomaly implies that the two types of topological operators don't commute:

$$\mathcal{A}\,:\,\Gamma^{(0)}\times\Gamma^{(1)}\to\mathbb{R}/\mathbb{Z}\;, \qquad \Pi^{\gamma_{(1)}}\mathcal{U}^{\gamma_{(0)}} = e^{2\pi i\mathcal{A}(\gamma_{(0)},\gamma_{(1)})}\,\mathcal{U}^{\gamma_{(0)}}\Pi^{\gamma_{(1)}}$$

The anomaly coefficients can be extracted from $\mathcal{W}$, and only depend on the Chern-Simons levels of the 3d gauge theory.

One interesting implication of $\mathcal{A}$ is that it constrains the orbit structure under $\Gamma^{(0)}$. For instance, there can exist $\hat{u}$ fixed by the full $\Gamma^{(0)}$ iff the anomaly vanishes. ('Larger' anomaly means larger orbits.)

Example: $\Gamma=\mathbb{Z}_N$**, anomaly** $\mathfrak{a}\in\mathbb{Z}_N$**:** All orbits are of size:

$$|\hat{\omega}| = n\frac{N}{\gcd(\mathfrak{a},N)}\leq N\;, \qquad n\in\mathbb{Z}_{>0}$$

Topologically twisted index for general G

For any non-anomalous $\Gamma^{(1)}_{3\text{d}}$, we can now compute the topologically twisted index for:

$$G = \tilde{G}/\Gamma$$

namely:

$$Z^{\mathcal{T}/\Gamma^{(1)}_{3\text{d}}}(\theta, C^D) = \frac{1}{|\Gamma|^{2g}} \sum_{\delta \in \Gamma^{(1)}} \sum_{[\gamma] \in H_1(\Sigma_g, \Gamma^{(0)})} e^{i(\theta, B_\delta)} e^{2\pi i (C^D, C_\gamma)} \left\langle \Pi^\delta \, \mathcal{U}^\gamma \right\rangle_{\Sigma_g} \, .$$

Turning off background gauge fields for the dual symmetries, we obtain:

$$Z^{\mathcal{T}/\Gamma^{(1)}_{3\text{d}}}_{\Sigma_g \times S^1} = \text{Tr}_{\mathcal{H}^{[\tilde{G}/\Gamma]}_{S^1}} \left(\mathcal{H}_G^{g-1} \right)$$

with the trace over the Hilbert space

$$\mathcal{H}^{[\tilde{G}/\Gamma]}_{S^1} \cong \text{Span}_{\mathbb{C}} \left\{ |\hat{\omega}; s_{\hat{\omega}}\rangle \;\middle|\; \hat{\omega} \in \mathcal{S}^{\vartheta=1}_{\text{BE}}/\Gamma^{(0)} \ , \ s_{\hat{\omega}} = 1, \cdots, |\text{Stab}(\hat{\omega})| \right\} .$$

and the action:

$$\mathcal{H}_G |\hat{\omega}; s_{\hat{\omega}}\rangle = \frac{\mathcal{H}(\hat{\omega})}{|\hat{\omega}|^2} |\hat{\omega}; s_{\hat{\omega}}\rangle \ , \qquad \mathcal{H}(\hat{\omega}) \equiv \mathcal{H}(\hat{u}) \ , \forall \hat{u} \in \hat{\omega}$$

3d $\mathcal{N} = 2$ $SU(N)_K$ Chern–Simons (revisited)

Chern-Simons theories with gauge algebra $\mathfrak{su}(N)$ – $\tilde{G} = SU(N)$

Consider the $SU(N)_K$ $\mathcal{N}=2$ Chern–Simons theory. For $K \geq N$, this is equivalent to the pure $SU(N)_k$ Chern–Simons theory at level $k \equiv K - N$.

The twisted superpotential is simply:

$$\mathcal{W} = \frac{K}{2}(u,u)$$

It is well-known that the (ordinary) Witten index is: [Witten, 1999; Ohta, 1999]

$$\mathrm{I}_W = Z_{T^2\times S^1}[SU(N)_K] = \binom{K-1}{N-1}$$

More generally, the index on Σ_g is given by the Verlinde formula: [Verlinde, 1988]

$$Z_{\Sigma_g\times S^1}[SU(N)_K] = N^{g-1}\left(\frac{K}{2^N}\right)^{(g-1)(N-1)} \sum_{\underline{l}\in\mathcal{J}_{N,K}} \prod_{1\leq a<b\leq N} \left(\sin\frac{\pi(l_a-l_b)}{K}\right)^{2-2g}$$

with the indexing set for the Bethe vacua:

$$\mathcal{J}_{N,K} \equiv \left\{ (l_1,\cdots,l_N;\ell)\in\mathbb{Z}_K^N\oplus\mathbb{Z}_N, \middle| \, 0\leq l_1<\cdots<l_N\leq K\,,\, \sum_{a=1}^N l_a - \ell \in K\mathbb{Z} \right\}\,,$$

Chern-Simons theories with gauge algebra $\mathfrak{su}(N)$ – $\tilde{G} = SU(N)$

The $SU(N)_K$ theory has one-form symmetry and anomaly:

$$\Gamma^{(1)}_{3\mathrm{d}} = \mathbb{Z}_N \ , \qquad \mathfrak{a} = -K \ (\mathrm{mod}\ N)$$

In particular, we can gauge any non-anomalous subgroup $\mathbb{Z}_r$ for $r|N$ and obtain the CS theories:

$$(SU(N)/\mathbb{Z}_r)_K \qquad \text{with} \quad \frac{KN}{r^2} \in \mathbb{Z}$$

By direct computation, we find the Witten index:

$$Z_{T^2\times S^1}[(SU(N)/\mathbb{Z}_r)_K] = \frac{1}{r^2} \sum_{d|r} \mathscr{J}_3^{N,K}(d) \binom{\frac{K}{d}-1}{\frac{N}{d}-1}$$

which is given in terms of Jordan's totient J_3 function as:

$$\mathscr{J}_3^{N,K}(d) \equiv \begin{cases} \frac{1}{7} J_3(d) & \text{for } N \text{ even, } \frac{N}{d} \text{ odd, } \frac{K}{d} \text{ even ,} \\ J_3(d) & \text{otherwise .} \end{cases}$$

This was known in the literature only when the $\frac{1}{7}$ subtlety can be ignored (see [Beauville, 1998; Oprea, 2010]). General result is (apparently) new. Non-trivial number-theoretic conjecture (fact, from physics): the index is an integer!

Chern-Simons theories with gauge algebra $\mathfrak{su}(N)$ – $\tilde{G} = SU(N)$

Explicit numbers for the Witten index of $PSU(N)_{\kappa N}$:

$\kappa \backslash N$	2	3	4	5	6	7	8	9	10
1	2	3	4	5	6	7	8	9	10
2	1	4	4	10	16	42	108	312	930
3	3	6	16	45	186	798	3860	19305	100235
4	2	9	32	160	942	6048	41144	290592	2119200
5	4	13	68	430	3328	27454	240448	2188095	20545320
6	3	18	116	955	9030	91770	982884	10942308	125656965
7	5	24	192	1860	20868	250446	3171084	41742027	566724020
8	4	31	288	3295	42628	591633	8645360	131347320	2058115980
9	6	39	420	5435	79794	1254589	20780280	357870942	6356282290
10	5	48	580	8480	139092	2446486	45294044	871916841	17310311600

Similar explicit results for higher-genus partition functions – *i.e.* Verlinde formula for any G:

$$Z_{\Sigma_g \times S^1}[(SU(N)/\mathbb{Z}_r)_K] = \frac{1}{r^{2g-1}} \sum_{d|r} J_{2g}(d) \sum_{\hat{u} \in \mathcal{S}_{\mathrm{BE}}^{\vartheta^{(\mathbb{Z}_r)}=1, \mathbb{Z}_d}} \mathcal{H}(\hat{u})^{g-1}$$

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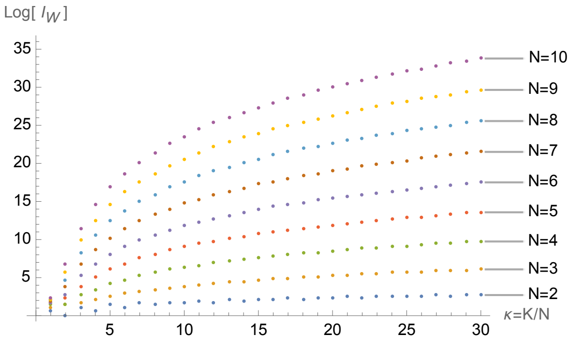

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A simple example: $SU(2)_K$ versus $SO(3)_K$

For $N = 2$, we have the $SU(2)_K$ theory with $\mathcal{W} = Ku^2$ and $K - 1$ Bethe vacua:

$$\hat{u}_l = \frac{l}{2K} \; , \qquad l = 1, \cdots, K-1$$

with $u \sum -u$ (Weyl symmetry). The Verlinde formula gives:

$$Z_{\Sigma_g \times S^1}[SU(2)_K] \equiv \langle 1 \rangle_{\Sigma_g} = \sum_{l=1}^{K-1} \left[\sqrt{\frac{2}{K}} \sin\left(\frac{\pi l}{K}\right) \right]^{2-2g} \; .$$

The action of $\Gamma^{(0)} = \mathbb{Z}_2$ on the Bethe vacua is:

$$\mathcal{U}(\mathcal{C}) |\hat{u}_l\rangle = |\hat{u}_{K-l}\rangle \; .$$

We can gauge the 3d one-form symmetry only if K is even, and we then find:

$$\begin{aligned} Z_{T^2 \times S^1}[SO(3)_K] &= \frac{1}{4} \sum_{n, n', n'' \in \mathbb{Z}_2} \left\langle \mathcal{U}(\tilde{\mathcal{C}})^n \mathcal{U}(\tilde{\mathcal{C}})^{n'} \Pi^{\frac{n''}{2}} \right\rangle_{T^2} \\ &= \frac{1}{4} \left(K - 1 + 3 + 1 + 3(-1)^{\frac{K-2}{2}} \right) = \begin{cases} \frac{K}{4} + \frac{3}{2} & \text{for } \frac{K}{2} \text{ odd} \; , \\ \frac{K}{4} & \text{for } \frac{K}{2} \text{ even} \; . \end{cases} \end{aligned}$$

This can be checked using anyon condensation in 3d TQFT as in [Hsin, Lam, Seiberg, 2018].

Future directions

The 3d A-twist on Seifert manifolds

For $G = \tilde{G}$, there is a well-developed formalism for 3d $\mathcal{N} = 2$ gauge theories on any Seifert manifold: [CC, Kim, Willett, 2018]

$$M_3 \cong [d; g; (q_1, p_1), \cdots, (q_n, p_n)]$$

For instance:

$$[0; g;] \cong \Sigma_g \times S^1 \;, \qquad [d; 0;] \cong S^3/\mathbb{Z}_d \;, \cdots$$

A more fun example is the Poincaré homology sphere:

$$S^3/\mathsf{BI} \cong [-1; 0; (2,1), (3,1), (5,1)] \qquad \pi_1(S^3/\mathsf{BI}) = \mathsf{BI} \quad \text{(binary icosehedral group).}$$

Using the Seifert fibration over a Riemann surface orbifold:

$$S^1 \longrightarrow M_3 \xrightarrow{\pi} \hat{\Sigma}_{g,n}$$

we can compute the partition function as the insertion of a Seifert fibering operator:

$$Z_{M_3}[\mathcal{T}] = \mathrm{Tr}_{\mathcal{H}_{S^1}} \left(\mathcal{H}^{g-1}\mathcal{G}_{M_3}\right) = \sum_{\hat{u} \in \mathcal{S}_{\mathrm{BE}}} \mathcal{H}(\hat{u})^{g-1}\mathcal{G}_{M_3}(\hat{u})$$

For $G = \tilde{G}/\Gamma$, we should now compute $\mathcal{G}_{M_3}$ in the $\Gamma^{(1)}_{3\mathrm{d}}$-gauged theory. In the A-model, we must wrap topological lines around orbifold points. [CC, Furrer, Keyes, Khlaif, *to appear*]

The 4d A-twist on elliptic quasi-bundles

We can similarly consider the 4d A-model – that is, the 2d A-model obtained from a 4d $\mathcal{N}=1$ gauge theory with $U(1)_R$ on:

$$T^2 \times \Sigma_g$$

The most general half-BPS geometries are complex manifolds called elliptic quasi-bundle:

$$M_4 \cong [d, d'; g; (q_i, p_i, p_i')] \ , \qquad\qquad T^2 \to M_4 \to \hat{\Sigma}_{g,n}$$

The partition functions on such manifolds can also be computed by fibering operator techniques. [CC, Keyes, arXiv:24xx.xxxxx]

The gauging of four-dimensional one-form symmetries become rather intricate. In the 2d language, we have:

$$\Gamma^{(1)}_{\rm 4d} \quad \longrightarrow \quad \Gamma^{(1)} \oplus \Gamma^{(0)} \oplus \Gamma^{(0)} \oplus \Gamma^{(-1)}$$

The gauging can be performed similarly to 3d. Many new questions arise due to intricate interplay with 0-form symmetries and their anomalies.

Summary and outlook

Summary:

- We generalised the 3d A-model formalism for 3d $\mathcal{N} = 2$ supersymmetric partition functions to the case of generic gauge group – that is, generic global structure.
- Equivalently, we studied in detail the insertion of topological defect operators for higher-form symmetries in these 3d $\mathcal{N} = 2$ gauge theories.
- We computed partition functions explicitly. Even for pure Chern–Simons theories without matter, this formalism subsumes and generalises many previous results.

Outlook:

- The 4d $\mathcal{N} = 1$ theories can be treated similarly. Lots of exciting question linked to *e.g.* higher-groups.
- How can one treat non-invertible symmetry operators in the A-model formalism?
- Similar questions can be asked about partition functions with 8 supercharges, *e.g.* for 5d SCFTs.
- $\{$generalised symmetry$\} \cap \{$SUSY localisation$\} \neq 0$ – lots more to do!