# Phase shift and two-sided geodesics

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Understanding gravity and black holes from the holographic principle.

Start modestly: AdS spacetime and conformal field theories (CFT).

Holographic CFTs: minimal ingredients for a holographic description.

CFT correlation functions & Gravitational amplitudes

Consistency conditions (unitarity, causality, KMS condition, ...)

Crossing Equations & OPE:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_1(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\rangle$$

$$\mathcal{O}_1 \times \mathcal{O}_1 \to 1 + \mathcal{O}_3 + \cdots \to \mathcal{O}_2 \times \mathcal{O}_2, \ \text{direct-channel}$$

$$\mathcal{O}_1 \times \mathcal{O}_2 \to \mathcal{O}_1 \times \mathcal{O}_2, \qquad\qquad \text{cross-channel}$$

*Holographic CFTs:*

- Energy-momentum operator $T_{\mu\nu}$.
- $\langle T_{\mu\nu} T_{\rho\sigma} \rangle \propto c_T$. Focus on the large $c_T$ expansion.

$$\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \mathcal{O}_2 \rangle + \frac{1}{c_T}(\cdots)$$

- Another characteristic scale: $\Delta_{gap}$.

  Focus on $\Delta_{gap} = \infty$: the CFT contains only a finite number of primary single-trace operators with spin $j \leq 2$.

- “single-trace” primaries: $\mathcal{O}_1, \mathcal{O}_2, \cdots, J^\mu, \cdots, T^{\mu\nu}$.
- “double-trace” primaries:

$$M_2 : \mathcal{O}_1 \partial_{\mu_1 \ldots} \partial_{\mu_\ell} (\partial^2)^n \mathcal{O}_2, \quad \mathcal{O}_1 \partial_{\mu_1 \ldots} \partial_{\mu_\ell} (\partial^2)^n J^\mu, \cdots$$

- “multi-trace” primaries:

$$[\mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_2 \cdots J]_{\{a,\cdots,b\},\{m,\cdots,m\}} : \mathcal{O}_1 \partial_{\mu_1 \ldots} \partial_{\mu_a} \partial^{2n} \mathcal{O}_2 \cdots \partial_{\mu_1 \ldots} \partial_{\mu_b} \partial^{2m} J^\mu$$

$$\langle \mathcal{O}_1 \mathcal{O}_1 \rangle \sim 1 + \cdots, \qquad \langle [\mathcal{O}_2 \mathcal{O}_2][\mathcal{O}_2 \mathcal{O}_2] \rangle \sim 1 + \cdots$$

$$\langle \mathcal{O}_2 \mathcal{O}_2 [\mathcal{O}_2 \mathcal{O}_2]_{n,\ell} \rangle \sim 1 + \cdots, \qquad \langle \mathcal{O}_1 \mathcal{O}_1 [\mathcal{O}_2 \mathcal{O}_2]_{n,\ell} \rangle \sim \frac{1}{c_T} + \cdots,$$

$$\langle \mathcal{O}_1 \mathcal{O}_1 T \rangle \sim \frac{1}{\sqrt{c_T}} + \cdots$$

Results:

- Crossing equation and locality/unitarity of the holographic description.
- Unitarity (causality) imply that Einstein's theory of general relativity is the only consistent description.
- Correlation functions in certain kinematic regimes.
- Higher order CFT-data (OPE coefficients, anomalous dimensions etc).
- Thermal CFT structure.
- . . . . . . . . . . . .

[huge list of authors many of whom are present]

Multi-stress tensors: a class of operators present in generic CFTs.

$$T_{\mu\nu} \qquad t = d-2, s = 2$$

$$:T_{\mu_1\mu_2}\partial_{\nu_1}\partial_{\nu_2}\cdots\partial_{\nu_m}T_{\mu_3\mu_4}: \qquad t_{min} = 2d-4, s = 4+m$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$:T_{\mu_1\mu_2}T_{\mu_3\mu_4}\cdots\partial_{\nu_1}\partial_{\nu_2}\cdots\partial_{\nu_m}T_{\mu_{2k-1}\mu_{2k}}: \quad t_{min} = kd-2k, s = 2k+m$$

Multi-stress tensors play an important role in HHLL correlators:

$$\text{HHLL} \equiv \langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle$$

$\mathcal{O}_H$: characterised by $\Delta_H \sim \mathcal{O}(c_T)$ with $\frac{\Delta_H}{c_T} = \text{fixed}$.

$\mathcal{O}_L$: with $\Delta_L \sim \mathcal{O}(1)$.

When the state created by $\mathcal{O}_H$ is thermal:

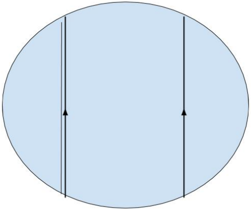
$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle \sim \langle \mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\rangle_T$$

Via the AdS/CFT dictionary the thermal two-point function can be obtained from the study of fluctuations around a black hole geometry.

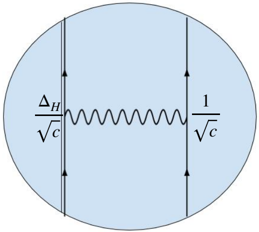
In the dual gravitational description (e.g. $d = 4$):

$$\mu \equiv \frac{r_H^2}{R_{AdS}^2} = \frac{M_{BH}\ell_p^3}{R_{AdS}^3} = (M_{BH}R_{AdS})\frac{\ell_p^3}{R_{AdS}^3} \sim \frac{\Delta_H}{c}$$

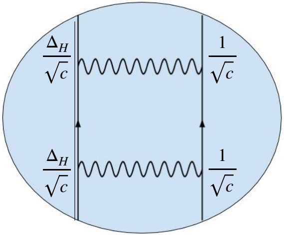
$\mathcal{O}(\mu^0)$

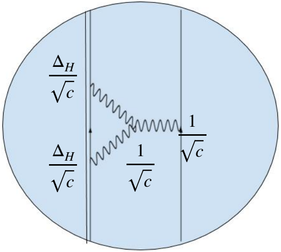


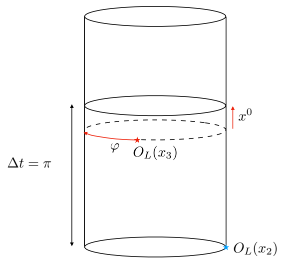
$\mathcal{O}(\mu)$




$\mathcal{O}(\mu^2)$






A physically relevant kinematic limit is the Regge/Eikonal limit.

$$z \to z\, e^{2\pi i}, \quad (z, \bar{z}) \to (1, 1) \quad \text{with} \quad \frac{1-z}{1-\bar{z}} = fixed,$$

where

$$z = e^{i\Delta t + \Delta\varphi}, \quad \bar{z} = e^{i(\Delta t - \Delta\varphi)}.$$

[Fitzpatrick, Huang, Karlsson, MK, Li, Ng, Parnachev, Sen,Tadić, ..]

The object of interest here is the bulk phase-shift.

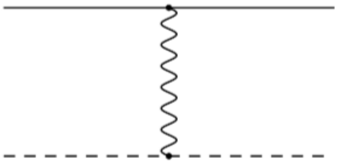
$$e^{i\delta(p)} \sim \int dx e^{ipx} \langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle$$

- What does the bulk phase shift compute in gravitaty?

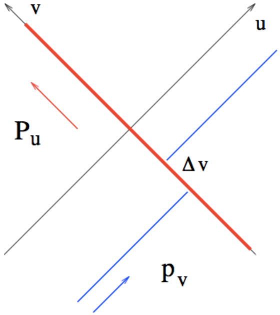
Consider first high energy (large s, finite t) two-to-two scattering in flat space. The amplitude is well described by the eikonal phase:

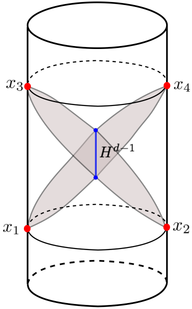
$$\mathcal{A} \sim e^{i\delta}(s,b)$$

with $b$ the impact parameter. $\mathcal{A}$ is the result of the summation of an infinite number of diagrams, which exponentiate the graviton exchange:



The eikonal phase can also be computed from the time-delay a particle experiences when traversing a shock-wave.




$$\underline{AdS_{d+1}/CFT_d}$$

$$e^{i\delta} \sim \int dx_3 dx_4 e^{ip_3 x_3} e^{ip_4 x_4} \langle \mathcal{O}_1(x_1)\mathcal{O}_1(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_3) \rangle$$

$$\delta = -p_v \cdot (\Delta v) \sim s\Pi_{d-1}(L)$$

So the bulk phase shift here is the eikonal phase induced when one particle is much heavier than the other.

CFT:

$$e^{i\delta(p)} \sim \int dx e^{ipx} \langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle$$

In holographic CFTs this regime is dominated in the direct channel by an infinite tower of spinning operators: the stress-tensor sector and double-traces.

$$\mathcal{O}_L \times \mathcal{O}_L \to 1 + \mu(T_{\mu\nu} + \cdots) + \cdots \to \mathcal{O}_H \times \mathcal{O}_H,$$

Their contribution can in principle be computed with Regge conformal theory order by order in $\mu$.

The Fourier transform is expected to eliminate the double-traces. Only the stress-tensor sector contributes.

The leading order phase shift is fixed by the stress-tensor contribution,

$$\delta^{(1)} \sim \sqrt{-p^2}\Pi_{d-1}(L)$$

where $\Pi_{d-1}(L)$ is the propagator in the transverse $H_{d-1}$ space of a particle with $m^2 = (d-1)^2$. The result nicely matches the AdS eikonal phase.

## AdS-black hole

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\varphi^2 + \sin^2\varphi d\Omega_{d-2}^2\right)$$
$$f(r) = 1 + \frac{r^2}{R^2} - \frac{\mu}{r^{d-2}},$$

A highly energetic particle exhibits a phase-shift $\delta$:

$$\delta \equiv -p \cdot (\Delta x) = p^t(\Delta t) - p^\varphi(\Delta\varphi)$$

with time-delay and deflection

$$\Delta t = 2\int_{r_0}^{\infty} \frac{dr}{f\sqrt{1-\frac{\alpha^2}{r^2}f}}, \quad \Delta\varphi = 2\alpha\int_{r_0}^{\infty} \frac{dr}{r^2\sqrt{1-\frac{\alpha^2}{r^2}f}}, \quad \alpha \equiv \frac{p^\varphi}{p^t}$$

Null geodesics are labelled by the impact parameter, $b$:

$$\frac{1}{b^2} = \frac{1}{\alpha^2} - \frac{1}{R^2}$$

For $\mu = 0$ it reduces to the standard pure AdS definition of impact parameter. For $R \gg \alpha$, $b$ ;eads to the flat space expression.

$$\delta = 2|p^t| \int_{r_0}^{\infty} \frac{dr}{f(r)} \sqrt{1 - \frac{\alpha^2}{r^2} f(r)}$$

To match the CFT result (alternative parametrisation):

$$e^{2L} \equiv \frac{p^t + p^\varphi}{p^t - p^\varphi}, \quad b = R\text{sinh}L$$

Integrals reduce to hypergeometrics with many variables, solvable in principle.

$$\delta = \sum_{k=1}^{\infty} \mu^k \delta^{(k)}, \quad \delta^{(k)} \sim \sqrt{-p^2}\,\Pi_{kd-2k+1}(L)$$

$\mathcal{O}(\mu^k)$ contribution as if produced by a spin-2 conserved operator with twist equal to the minimal twist of the corresponding multi-stress tensor operator at each order: $\Delta = kd - 2k + 2, s = 2$

The radius of convergence of the series for $L_c$ corresponds with the null geodesic approacing the circular null orbit.

For values of the impact parameter beyond $L_c$, the particle does not return to the boundary but rather falls into the black hole.

This is inelastic scattering. The phase shift develops an imaginary part related to the absorption cross section.

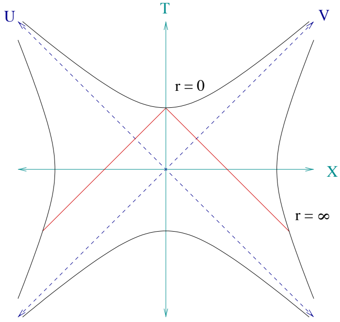
It is possible to compute the imaginary part by analytically continuing the exact result for the phase shift obtained.

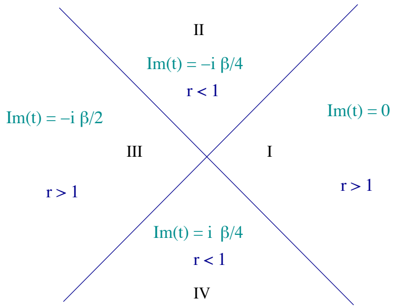
This was in fact computed and shown to reproduce the correct geometric absorption cross-section in the small impact parameter, equivalently, in the asymptotically flat space limit. [Sen, Parnachev]

An alternative approach: re-evaluate $\delta$ for the bouncing geodesic.

Recall what happens when $p^\varphi = 0$.

$$\Delta t = 2\int_0^\infty \frac{dr}{f(r)} = -i\frac{\beta}{2} + \text{real part}$$



The imaginary part is fixed and the particle "appears" on the other boundary.

Evaluated on the thermo-field double state

$$|\psi\rangle \equiv \frac{1}{\sqrt{\mathcal{Z}}} \sum_n e^{-\frac{1}{2}\beta E_n} |E_n\rangle_1 |E_n\rangle_2$$

the thermal correlator computed is really

$$\left\langle \psi | \mathcal{O}_L^{(1)}(0) \mathcal{O}_L^{(2)}(t) | \psi \right\rangle = \left\langle \mathcal{O}_L^{(1)}(0) \mathcal{O}_L^{(1)} \left( -t - i\frac{\beta}{2} \right) \right\rangle_T$$

[Fidkowski, Hubeny, Kleban, Shenker]

It is not difficult to incorporate angular momentum.

For values of $L < L_c$, the expression under the square root is always positive definite. A singularity appears in $\Delta t$ when crossing the horizon in the same way as for $p^\varphi = 0$.

$$\Delta t = 2 \int_0^\infty \frac{dr}{f\sqrt{1 - \frac{\alpha^2}{r^2} f}},$$

The imaginary part of the phase shift is then fixed:

$$Im(\delta) = -p^t \frac{\beta}{2}$$

*Are both analytic continuations meaningful?*

## $d = 2$

For $\mu < 1$: $AdS_3$ with conical deficit. Geodesics return to the boundary and the phase shift is:

$$\delta = \pi\sqrt{-p^2}e^{-L}\left(\frac{1}{\sqrt{1-\mu}} - 1\right)$$

CFT reproduces this result. The Virasoro vacuum block captures the stress-tensor sector of the correlator.

$$\langle \mathcal{O}_H(0)\mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\rangle \propto \frac{1}{\sin\left[\sqrt{1-\mu}(\pi + \frac{\Delta t \pm \Delta\varphi}{2})\right]}$$

The Fourier integral picks up the pole where the argument of the sine function vanishes reproducing the gravitational result.

For $\mu > 1$: all null geodesics fall into the black hole.

$$\Delta t = 2\int_0^\infty = -i\frac{\beta}{2}$$

CFT appears to reproduce the result from the Virasoro vacuum block.

Additional evidence that the phase shift is sensitive only to the stress-tensor sector.

The contribution of the stress-tensor sector of the correlator was recently studied [Čeplak, Liu, Parnachev, Valach].

$$G(\tau) = G_T(\tau) + G_{[\mathcal{O}_L\mathcal{O}_L]}(\tau), \quad \tau = t_E + it_L, \quad 0 \leq t_E \leq \beta$$
$$G_T(\tau) = \frac{1}{\tau^{2\Delta}} \sum_{k=0}^{\infty} \Lambda_k \left(\frac{\tau}{\beta}\right)^{dk}$$

The stress-tensor sector of the correlator behaves as the "naive" spacelike geodesic solution, which at $\tau \sim \tau_c$ approaches the null.

$$G_T(\tau \sim \tau_c) \sim_{\tau\to\tau_c} (\tau_c - \tau)^{-(2\Delta_L - 2)}, \quad \tau_c = \frac{\beta}{\sqrt{2}} e^{i\frac{\pi}{4} + i\frac{k\pi}{2}}$$

It appears as if the stress-tensor sector contains information about the black hole singularity.

The contribution from double-traces is expected to restore the periodicity and smoothness of the correlator.

For $\Delta_L \gg 1$, $G_T(\tau)$ is identified with a complex geodesic (order of limits issue).

The impact parameter representation of the correlator should confim the story.

Summary:

It is worth exploring the bulk phase shift in holographic CFTs and beyond.

- Stress-tensor sector.
- Singulaity.
- Inelastic scattering: is there more than one way?

*Thank you.*