Replica wormholes and generalised Rényi entropies

Eurostrings meets FPUK, Southampton 2024

S. Prem Kumar (Swansea U.)

September 5, 2024

Based on: arXiv:2406.16339 with Tim Hollowood & Luke Piper

Black hole information problem

- How does Hawking radiation **R** from evaporating black hole (**BH**) encode information of quantum state of matter collapsing to form the back hole ?

- Hawking's semiclassical calculation yields thermal radiation whose entropy grows unbounded. (Hawking '76)

- **BH** and its radiation **R** are entangled: we need to consider the **von Neumann** entropy $S_{\rm EE}(\mathbf{R})$:

$$S_{\rm EE}(\mathbf{R}) = -\mathrm{Tr}_{\mathbf{R}}\left(\rho_r \log \rho_r\right)$$

ρ_r: reduced density matrix for **R**

Page curve

- $S_{\rm EE}(\mathbf{A}) \leq \log \dim \mathscr{H}_{\mathbf{A}}$ i.e. $S_{\rm EE}$ saturated by thermal entropy for large subsystems.

- In the thermodynamic limit,

$$S_{\rm EE}(\mathbf{A}) = \min\left(\log \dim \mathscr{H}_{\mathbf{A}},\ \log \dim \mathscr{H}_{\mathbf{B}}\right)$$

(Page's theorem)

- **Page curve** for BHs:

$$S_{\rm EE}(\mathbf{R}) = \min\left(S_{\rm BH}(t), S_{\rm rad}(t)\right)$$

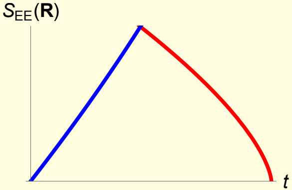
$S_{\rm rad}(t) =$ thermal entropy of all Hawking radiation collected up to time t.

Page curve for BHs in adiabatic limit

- **Schwarzschild:** $\dot{M} = -\# T(t)^2$; $\delta S_{\rm BH}(t) = \delta M/T$ (first law) and $T(t) = \frac{1}{8\pi G_N M}$,

$$S_{\rm BH} = \# T_0 \left(1 - \frac{t}{t_0}\right)^{\frac{2}{3}}, \qquad S_{\rm rad} = 2\left(S_{\rm BH}(0) - S_{\rm BH}(t)\right)$$

- $S_{\rm EE}(\mathbf{R})$:



- Page time: $S_{\rm BH}(t) = \frac{2}{3} S_{\rm BH}(0)$

Generalised entropy

- Generalised, "fine grained" entanglement entropy

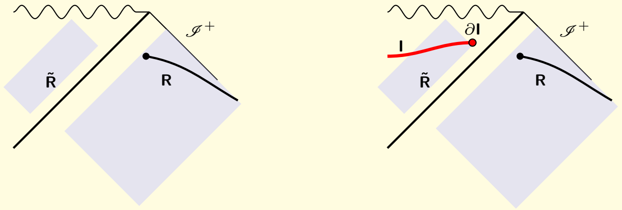
  (Almheiri-Mahajan-Maldacena-Zhao '19)

- Precursor: Holographic entanglement entropy & Quantum Extremal Surfaces (QES) Faulkner-Lewkowycz-Maldacena (2013); Engelhardt-Wall (2014)

Generalised entropy and island mechanism

(Almheiri-Hartman-Maldacena-Shaghoulian-Tajdini (2019); Penington-Shenker-Stanford-Yang (2019))

$$S^{\rm EE}_{\rm gen}(\mathbf{R}) = \min_{\mathbf{I}} {\rm ext}_{\partial\mathbf{I}} \left\{ \sum_{\partial\mathbf{I}} \frac{\mathbf{A}(\partial\mathbf{I})}{4G_N} + S^{\rm EE}_{\rm QFT}(\mathbf{R}\cup\mathbf{I}) \right\}$$



- **I**: "Island" ∂**I**: "Quantum Extremal Surface" (QES)
- Early times, No-island: $S_{\rm gen}(\mathbf{R}) = S_{\rm QFT}(\mathbf{R})$
- Late times, Island saddle: **I** gathers purifiers $\tilde{\mathbf{R}}$ of **R** $\rightarrow$ Page curve

Generalised Rényi entropies ?

- $$S_{\rm EE}(\mathbf{R}) = \lim_{n\to 1} \tfrac{1}{1-n} \log {\rm Tr}\rho_r^n$$

- Can we compute Rényi entropies for general (integer) $n \neq 1$

$$S_n(\mathbf{R}) = \frac{1}{1-n} \log {\rm Tr}\rho_r^n$$

- Computing ${\rm Tr}\rho_r^n$ yields the entanglement spectrum.

- Is there a "generalised Rényi entropy" for $n \neq 1$?

JT gravity setup

- Jackiw-Teitelboim (JT) gravity: 2d dilaton-gravity Teitelboim '83, Jackiw '85
- ◇ Effective framework: s-wave truncation of $d > 2$ gravity.

  ◇ Near horizon description of higher d near extremal BHs.
-

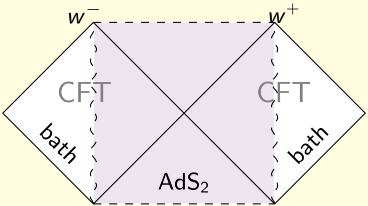
$$I_{\rm JT} = \frac{1}{16\pi G_N}\int d^2x\sqrt{-g}\,\Phi(R+2) + \frac{1}{8\pi G_N}\int_{bdry} d\tau\sqrt{-g}\,\Phi K$$

- ◇ Locally AdS_2: $R = -2$.

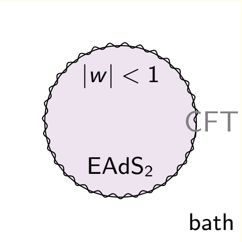
  ◇ All dynamics $\leftrightarrow$ Boundary term

  ◇ Dilaton Φ $\leftrightarrow$ Area

Black hole coupled to radiation (CFT) bath



Lorentzian



Euclidean

- BH coupled to flat nongravitating bath

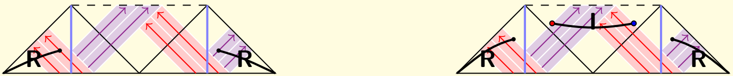
$$ds^2 = \frac{4 dw d\bar{w}}{(1-|w|^2)^2}\,, \qquad \Phi = \Phi_0 + \frac{2\pi\phi_r}{\beta}\frac{1+|w|^2}{1-|w|^2}\,.$$

- BH entropy:

$$S_{\rm BH} = \left.\frac{\Phi}{4G_N}\right|_{w=0} = S_0 + \frac{\pi}{2G_N\beta}\phi_r$$

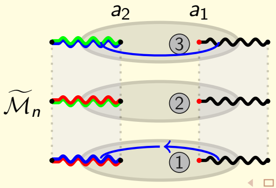
Replica problem

- Consider e.g. Rényi entropies for union of two intervals. Late times, we expect two QES:



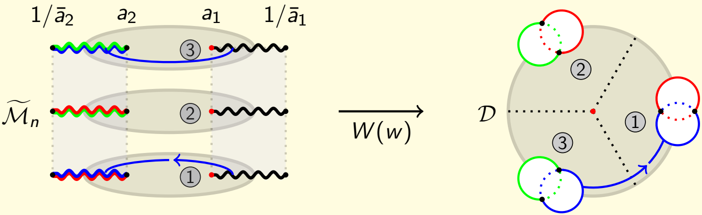
- In general: we want a (Euclidean) replica calculation with N QES's/punctures (determined dynamically)

- Replica of AdS_2 geometry $\widetilde{\mathcal{M}}_n$: n-fold cover of base $\mathcal{M}_n$ with N punctures: $w = \{a_j\}_{j=1,\ldots N}$.

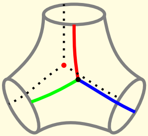


Uniformisation

- Replica geometry from Fuchsian uniformisation: (Hadasz-Jaskolski '06; Hulík-Procházka-Raeymaekers '16; Faulkner '13)



Replica wormhole



Replica problem

- $-\log Z_n = I_{\rm JT}^{(n)} - \log Z_n^{\rm CFT}$ (τ =bath time):

$$I_{\rm JT}^{(n)}[a_j; w(\tau)] \quad = \quad -NS_0 - \frac{n\phi_r}{8\pi G_N}\int_{|w|=1} d\tau \left[\{w,\tau\} + \mathcal{T}_L\, \dot{w}^2\right]$$

- Liouville stress tensor $\mathcal{T}_L = \frac{1}{2}\{W, w\}$,

$$ds^2\Big|_{\mathcal{M}_n} = e^{2\rho(w,\bar{w})}|dw|^2\,, \qquad \mathcal{T}_L = -(\partial_w\rho)^2 + \partial_w^2\rho$$

- Near twist/antitwist branch points,

$$W(w\to a_i) \sim (w-a_i)^{\pm 1/n} + \ldots$$

- Liouville eqn: $-4\partial_w\partial_{\bar{w}}\rho + e^{2\rho} = 2\pi\left(1-\frac{1}{n}\right)\sum_{j=1}^N \delta^2(w-a_j)$

Equations of motion and a puzzle

- JT gravity Einstein eqn:

$$e^{2\rho}\partial_w\left(e^{-2\rho}\partial_w\Phi\right) = 8\pi G_N \langle T_{ww}\rangle_n$$

- Near $w = a_j$, $\rho \approx -\frac{1}{2}\left(1 - \frac{1}{n}\right)\log(w - a_j)$

- CFT Ward identity for $\langle T_{ww}\rangle_n$ and residue at $w = a_j$ yields

$$\boxed{\frac{1}{4G_N}\,\partial_w\Phi\big|_{w=a_j} + \partial_{a_j} S^{(n)}_{\rm CFT} = 0}$$

- Superficially resembling, but **not** a QES condition.

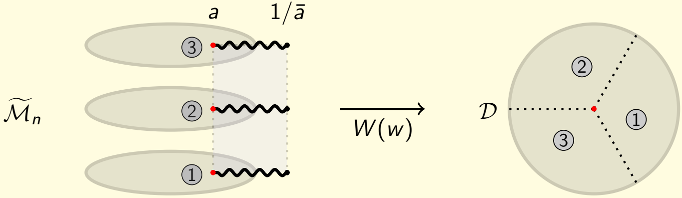
- How is $\partial_w\Phi(a_j)$ related to $\partial_{a_j}\Phi(a_j)$?

One QES problem

- The uniformisation map is immediate:

$$W(w) = \left(\frac{w - a}{1 - \bar{a}w}\right)^{1/n}$$

- Base $\mathcal{M}_n$: $|w| < 1$, and cover $\mathcal{D}$: $|W| < 1$ unit disc.



One QES problem

- Integral representation for dilaton on $\mathcal{D}$ (from EoM + energy balance at boundary):

$$\Phi(W,\bar{W}) = -\frac{\phi_r(1-|W|^2)^2}{\beta}\oint_{\partial\mathcal{D}} d\tilde{W}\,\frac{\dot{W}(\tilde{W})}{(\tilde{W}-W)^2(1-\bar{W}\tilde{W})^2}$$

- Evaluate $\Phi|_{w=a}$ and $\partial_w\Phi|_{w=a}$: $\boxed{\partial_w\Phi(w)\big|_{w=a} = \frac{n+1}{2}\partial_a\Phi(a)}$

Generalised modular entropy

1. Einstein eq & CFT Ward identity imply a QES condition:

$$\partial_a \left(\frac{\Phi(a)}{4G_N} + \frac{2}{n+1} S_n^{\rm CFT}(a) \right) = \partial_a \tilde{S}_n^{\rm gen} = 0\,.$$

2. Independently, off-shell action

$$-\log Z_n \;=\; \overbrace{\frac{(n^2-1)\Phi(a,\bar a)}{8G_N} + \ldots}^{I^n_{\rm JT}} - \log Z_n^{\rm CFT}[a]$$

3. Extremising yields exactly 1
4. Generalised entropy for general n is a modular entropy:

$$\boxed{\tilde{S}_n^{\rm gen}(a) = \frac{\Phi(a)}{4G_N} + \tilde{S}_n^{\rm CFT}} \qquad \tilde{S}_n^{\rm CFT} = \frac{\partial}{\partial n^{-1}}\left(n^{-1}\ln Z_n^{\rm CFT}\right)$$

(consistent with Dong '16)

Multiple QES

- For general n and multiple QES, mutual backreaction of branch points nontrivial (unlike $n \to 1$ limit)
- Nontrivial uniformisation problem. Map to fundamental domain $\mathcal{D}$: $W(w) = \psi_1/\psi_2$

$$\left(\partial_w^2 + \mathcal{T}_L(w)\right)\Psi = 0\,, \qquad \Psi = (\psi_1, \psi_2)$$

- Liouville stress tensor,

$$\mathcal{T}_L = \sum_{i=1}^{N}\left(\frac{1-\frac{1}{n}^2}{4(w-a_i)^2} + \frac{1-\frac{1}{n}^2}{4(w-1/\bar{a}_i)^2} + \frac{c_i^+}{(w-a_i)} + \frac{c_i^-}{(w-1/\bar{a}_i)}\right)$$

- Replica partition function completely determined by the accessory parameters $\{c_i^\pm\}$ and boundary map $\dot{w}(w)$

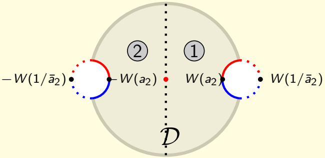
Multiple QES

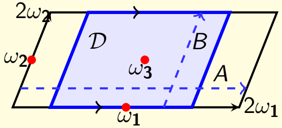
- Accessory parameters fixed by requirement that monodromies around singular points are in $SU(1,1)$, leaving e^{ρ} invariant
- Gravity action depends on $\{a_j\}$ and nontrivially on *a priori* unknown boundary map $w(\tau)$
- High temperature simplifying limit: $w \approx e^{i\tau}$, and

$$I_{\rm JT}^{n} \quad \to \quad \frac{n\pi\phi_r}{\beta G_N} \sum_{j=1}^{N} c_j^{+} a_j$$

Case $n = 2,\ N = 2$

- For $n = 2$ and two branch points, the problem is elliptic, and $\mathcal{D}$ is a strip on the torus





- Complex structure $\tau = i\frac{K(1-x)}{K(x)}$

- Cross-ratio $x = \frac{(a_1 - \frac{1}{\bar{a}_1})(a_2 - \frac{1}{\bar{a}_2})}{(a_1 - \frac{1}{\bar{a}_2})(a_2 - \frac{1}{\bar{a}_1})}$. Small x: $\tau \approx -i \ln x$

- Double trumpet metric: $ds^2 \sim dz d\bar{z} / \sin^2(\mathrm{Re}\, \alpha z)$

Case $n = 2$, $N = 2$

- The high temperature action for $n = 2$ and 2 branch points:

$$I_{\rm JT}^{(2)} = \frac{3}{2} \cdot \frac{1}{4G_N} \left(\Phi_1(a_1) + \Phi_1(a_2)\right)\left(1 - \mathcal{K}(x)\right)$$

$$\mathcal{K}(x) = \tfrac{1}{12}\left(-\tfrac{\pi^2}{K(x)^2} + 4 - 2x\right) = \tfrac{x^2}{32} + \tfrac{x^3}{64} + \ldots$$

- In Lorentzian signature $x \ll 1 \leftrightarrow$ late times, when there is a "factorisation" of QES positions.
- $x \ll 1$ limit: cut-out arcs become small and $\mathcal{D} \to$ unit disk

Multiple QES

- For general n, accessory parameters related to semiclassical conformal block in Liouville theory

$$c_i^+ = \lim_{c_L\to\infty} -\frac{6}{c_L}\,\partial_{a_i}\log\mathcal{F}(\{a_j\};\Delta_n c_L)$$

Determined as an expansion in small x limit

- $$I_{\rm JT} = \frac{(n^2-1)}{8G_N}\left(\sum_j \Phi(a_j,\bar{a}_j) + \delta\Phi_n\right) + \ldots$$

$\delta\Phi_n$: contributions from cut-out arcs of $\mathcal{D}$, suppressed in late time limits.

- $\tilde{S}_n^{\rm gen} = \sum_j \frac{\Phi(a_j)}{4G_N} + \tilde{S}_n^{\rm CFT}(a_j) + (small)$ late time limit.

Summary and further questions

- Summary:
  - Generalised Rényi entropies computable for finite n.
  - n-dependent Page times $n > 1$:

$$t_0 = \frac{6\beta n S_0}{\pi c(n+1)} + \frac{\beta}{2\pi}\left(\frac{1}{\kappa} - \frac{\kappa(19n^2-7)}{2(4n^2-1)} + \cdots\right) \qquad \kappa = \frac{\beta c G_N}{6\pi\phi_r} \ll 1$$

- Open questions:
  - Real time Rényi entropy evolution for single sided evaporating BH
  - Computing corrections to $\tilde{S}_n^{\mathrm{gen}}$ away from factorisation limit. Compare with holographic arguments (Dong '16).