Exploring thermal CFTs

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Motivation

* Quantum critical points: nonzero temperature in the lab.
* Study of Black Holes through AdS/CFT.
* Study CFTs on non-trivial manifolds.

Thermal QFTs

Thermal effects are captured by placing the QFT on a **circle**

$$S^1_\beta \times \mathbb{R}^{d-1} \qquad\qquad \beta = \frac{1}{T}$$

With periodic boundary conditions for the bosons

and anti-periodic for the fermions.

Thermal CFTs

In this talk <u>assume</u> we know the **zero temperature CFT data:**

$$\Delta_{\mathcal{O}} \ , \ f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$$

and are interested in computing **<u>new</u> finite temperature data:**

the non-zero ***thermal one-point functions***:

$$\langle \mathcal{O}(x) \rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}}$$

for neutral scalar operators.

And more generally for all traceless symmetric tensors:

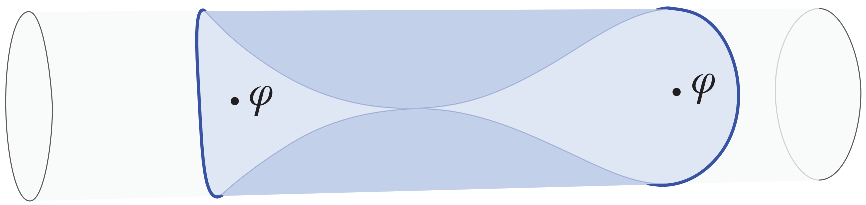
$$\langle \mathcal{O}^{\mu_1 \cdots \mu_J}(x) \rangle_\beta = \frac{b_{\mathcal{O}_J}}{\beta^{\Delta_{\mathcal{O}_J}}} (e^{\mu_1} \cdots e^{\mu_J} - traces)$$

Thermal CFTs

We can still use the OPE:

$$\mathcal{O}_1(x) \times \mathcal{O}_2(0) = \sum_{\mathcal{O}} f_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}} |x|^{\Delta_{\mathcal{O}}-\Delta_{\mathcal{O}_1}-\Delta_{\mathcal{O}_2}-J} x_{\mu_1}\ldots x_{\mu_J} \mathcal{O}^{\mu_1\ldots\mu_J}(0)$$

But now the radius of convergence is finite: $|x| < \beta$



$|x| = \sqrt{r^2 + \tau^2}$

[Iliesiu, Kologlu, Mahajan, Perlmutter, Simmons-Duffin 2018]

Thermal CFTs

The **two-point function** of identical scalars, using the OPE

$$\langle \phi(\tau, r)\phi(0)\rangle_\beta = \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}} \left(\sqrt{r^2+\tau^2}\right)^{\Delta_{\mathcal{O}}-2\Delta_\phi-J} x_{\mu_1}\ldots x_{\mu_J} \underline{\langle \mathcal{O}^{\mu_1\cdots\mu_J}\rangle}_\beta$$

and the definition of the Gegenbauer polynomials:

$$\langle \phi(\tau, r)\phi(0)\rangle_\beta = \sum_{\mathcal{O}} \frac{\underline{a_{\mathcal{O}}}}{\beta^{\Delta_{\mathcal{O}}}} \left(\sqrt{r^2+\tau^2}\right)^{\Delta_{\mathcal{O}}-2\Delta_\phi} C_J^{(\nu)}\left(\frac{\tau}{\sqrt{\tau^2+r^2}}\right)$$

$$\underline{a_{\mathcal{O}} = b_{\mathcal{O}} f_{\mathcal{O}\phi\phi}}\frac{J!}{2^J(\nu)_J} \qquad (\nu)_J = \frac{\Gamma(\nu+J)}{\Gamma(\nu)} \qquad \nu = \frac{d-2}{2}$$

New Finite Temperature data

Thermal CFTs

Periodicity of the two-point function is captured by the **KMS condition**:

[Kubo 1957]
[Martin,Schwinger 1959]

$$\langle \phi(\tau, r)\phi(0,0)\rangle_\beta = \langle \phi(\tau+\beta, r)\phi(0,0)\rangle_\beta$$

The OPE expression does not manifestly satisfy KMS, thus imposing it gives a nontrivial **"thermal crossing equation"**.

Variations of KMS:

$$\left\langle \phi\left(\beta/2+\tau\right)\phi(0)\right\rangle_\beta = \left\langle \phi\left(\beta/2-\tau\right)\phi(0)\right\rangle_\beta$$

[El-Showk, Papadodimas 2011]

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \langle \phi(-\tau)\phi(0)\rangle_\beta$$

[Iliesiu, Kologlu et al 2018]

Plan of the talk

* Derive and test thermal Sum Rules.
* Heavy operators: asymptotic OPE density.
* Setting up the numerical thermal bootstrap.
* Temporal line defects (Polyakov loops).

Sum Rules

Sum rules from KMS

✱ Expand both sides of the El-Showk - Papadodimas formula using the OPE

(KMS is outside of OPE regime)

$$\left\langle \phi\left(\beta/2+\tau\right)\phi(0)\right\rangle_{\beta} = \left\langle \phi\left(\beta/2-\tau\right)\phi(0)\right\rangle_{\beta}$$

✱ Then further expand the result in powers of τ and r.

✱ Use the definition of Gegenbauer polynomials and the binomial theorem.

$$\sum_{\mathcal{O}\in\phi\times\phi} \underbrace{b_{\mathcal{O}}}_{\text{New}} \underbrace{f_{\mathcal{O}\phi\phi} F_{\ell,n}(h,J)}_{\text{Known data}} = 0$$

Finite Temperature Zero Temperature

$$n \in \mathbb{N}\ , \ell \in 2\mathbb{N}+1$$

$$h = \Delta - J$$

$$F_{\ell,n}(h,J) = \frac{1}{2^{h+J}} \binom{\frac{h-2\Delta_\phi}{2}}{n} \binom{h+J-2\Delta_\phi-2n}{\ell} {}_3F_2\left[\left.\begin{matrix} \frac{1-J}{2}, -\frac{J}{2}, \frac{h}{2}-\Delta_\phi+1 \\ \frac{h}{2}-\Delta_\phi-n+1, -J-\nu+1 \end{matrix}\right| 1\right]$$

Sum rules from KMS

Similar to the **Gliozzi method** in standard zero temperature conformal bootstrap:

$$\frac{\partial^{2\ell+1}}{\partial\tau^{2\ell+1}}\frac{\partial^n}{\partial r^n}\langle\phi(\tau,r)\phi(0,0)\rangle_\beta\Big|_{\tau=\frac{\beta}{2},r=0}$$

The problem we are solving is **simpler**: we know the zero Temperature data.

We have an infinite set of linear equations for the combinations $a_{\mathcal{O}} \propto b_{\mathcal{O}} f_{\mathcal{O}\phi\phi}$

Difficulty: **Thermal one-point functions *not*** sign-definite.

Which is crucial for linear programming methods (standard numerical bootstrap).

Sum rules from KMS

The infinite set of linear equations for $a_{\mathcal{O}} \propto b_{\mathcal{O}} f_{\mathcal{O}\phi\phi}$

further simplify for r = 0 (zero spatial coordinates)

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta \neq 0} \frac{a_\Delta}{2^\Delta} \frac{\Gamma\left(\Delta - 2\Delta_\phi + 1\right)}{\Gamma\left(\Delta - 2\Delta_\phi - \ell + 1\right)}$$

$$\ell \in 2\mathbb{N} + 1$$

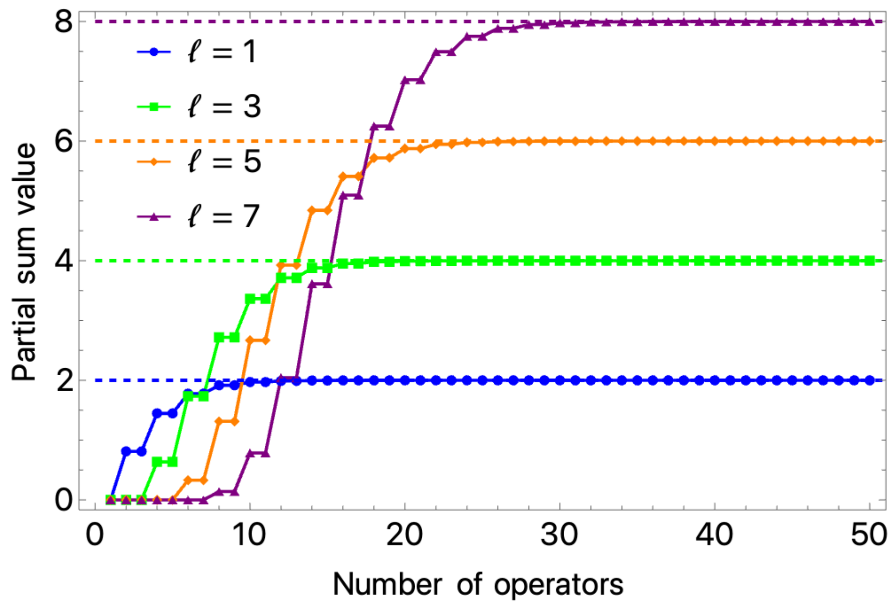
$a_\Delta = \sum_{\mathcal{O} \in \phi \times \phi} a_{\mathcal{O}} C_J^{(\nu)}(1)$ for fixed Δ

Operators of same Δ but different J cannot be distinguished because of r=0.

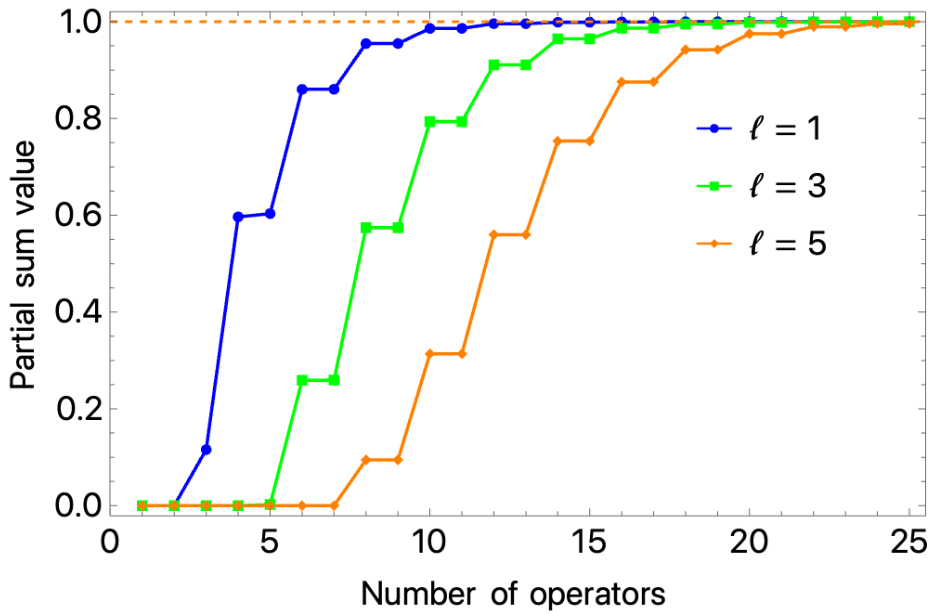
Generically this does not happen, only when there is extra symmetry like for free theory.

KMS sum rules: test & learn

4-dim free theory



$\mathrm{O}(N)$ **model at large N**



Dashed straight lines: the LHS of the sum rule (identity contribution). The RHS plot adding operators.

$$\frac{\Gamma\left(2\Delta_\phi + \ell\right)}{\Gamma\left(2\Delta_\phi\right)} = \sum_{\Delta \neq 0} \frac{a_\Delta}{2^\Delta} \frac{\Gamma\left(\Delta - 2\Delta_\phi + 1\right)}{\Gamma\left(\Delta - 2\Delta_\phi - \ell + 1\right)}$$

Observation: for **small** ℓ only **few light operators** contribute.

Bigger ℓ the more operators we need.

Heavy Operators

Asymptotic OPE density

We want to bound the OPE density for $\Delta \to \infty$.

Inspired by [Qiao, Rychkov 2017]

Consider the two-point function at r=0 (zero spatial coordinates):

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \tau^{-2\Delta_\phi} \sum_{\mathcal{O}\in\phi\times\phi} \frac{a_{\mathcal{O}}}{\beta^{\Delta_{\mathcal{O}}}} \tau^{\Delta_{\mathcal{O}}} = \tau^{-2\Delta_\phi} \int_0^\infty d\Delta\, \rho(\Delta)\, \frac{\tau^\Delta}{\beta^\Delta}$$

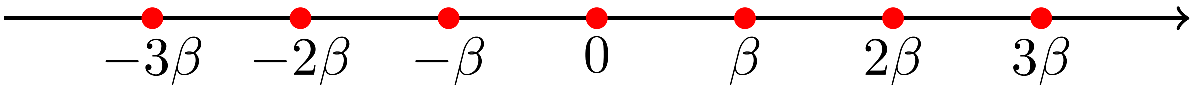
via introducing the spectral density $$\rho(\Delta) = \sum_{\Delta'} \delta(\Delta' - \Delta) a_\Delta$$

$$a_\Delta = \sum_{\mathcal{O}\in\phi\times\phi} a_{\mathcal{O}}\, C_J^{(\nu)}(1)$$ for fixed Δ.

Asymptotic OPE density

We will need:

The locations of the poles of the two-point function on the real axis:



due to periodicity.

$$\langle \phi(\tau)\phi(0)\rangle_\beta \overset{\tau\to k\beta}{\sim} \left(k\beta-\tau\right)^{-2\Delta_\phi}$$

Asymptotic OPE density

Make an **Ansatz** for the asymptotic behaviour of the density

$$\rho(\Delta) \overset{\Delta\to\infty}{\sim} A\Delta^{-\alpha}$$

Sensible due to Tauberian theory

Doing the integral

$$\langle\phi(\tau)\phi(0)\rangle_\beta \overset{\tau\to\beta}{\sim} A\,\Gamma(2-\alpha)\tau^{-2\Delta_\phi}\left(1-\frac{\tau}{\beta}\right)^{\alpha-2}$$

Comparing with the poles of the two-point function $\alpha = -2\Delta_\phi + 2\,, \quad A = \frac{1}{\Gamma(2\Delta_\phi)}$

$$\boxed{\rho(\Delta) \overset{\Delta\to\infty}{\sim} \frac{1}{\Gamma(2\Delta_\phi)}\Delta^{2\Delta_\phi-1}}$$

Asymptotic OPE density

Interpretation: the asymptotic density of OPE of Heavy operators.

$$\rho(\Delta) \overset{\Delta\to\infty}{\sim} \frac{1}{\Gamma(2\Delta_\phi)}\Delta^{2\Delta_\phi - 1}$$

Keep in mind, the physical spectrum is discreet $\rho(\Delta) \overset{\Delta\to\infty}{\sim} \sum_{\Delta'} \delta(\Delta' - \Delta) a_\Delta$

More correctly: **average density** of OPE **of Heavy operators**

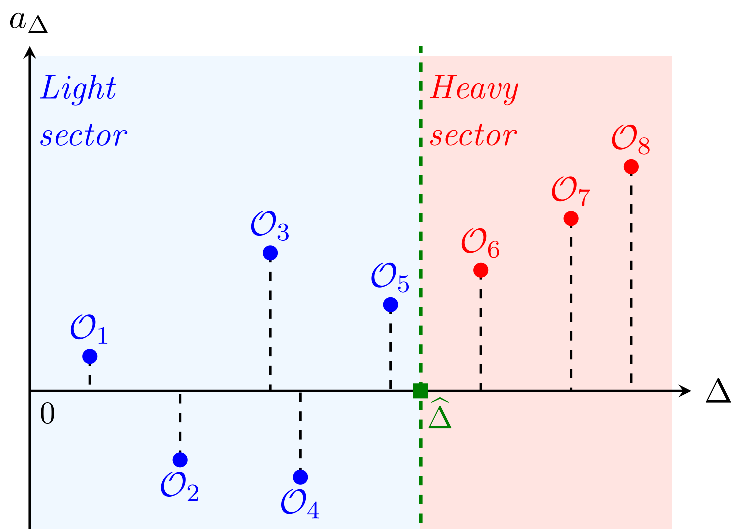
$$\int_0^\Delta \rho(\tilde{\Delta}) d\tilde{\Delta} \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi}}{\Gamma(2\Delta_\phi + 1)}\left(1 + \mathcal{O}\left(\frac{1}{\Delta}\right)\right)$$

This result is the correct formal math result (Tauberian theorems) with an error.

Asymptotic OPE density

This was a **heuristic derivation** (nonetheless captures the intuition).

In the paper we give a rigorous derivation using **Tauberian theorems** under the assumptions:



- ✱ Unitarity (reality of the OPE coefficients)
- ✱ **Boundedness** of the OPE from below (we proved)
- ✱ $\Delta_\phi > 1/2$ which is correct because of unitarity

$$a_\Delta \geq -c$$

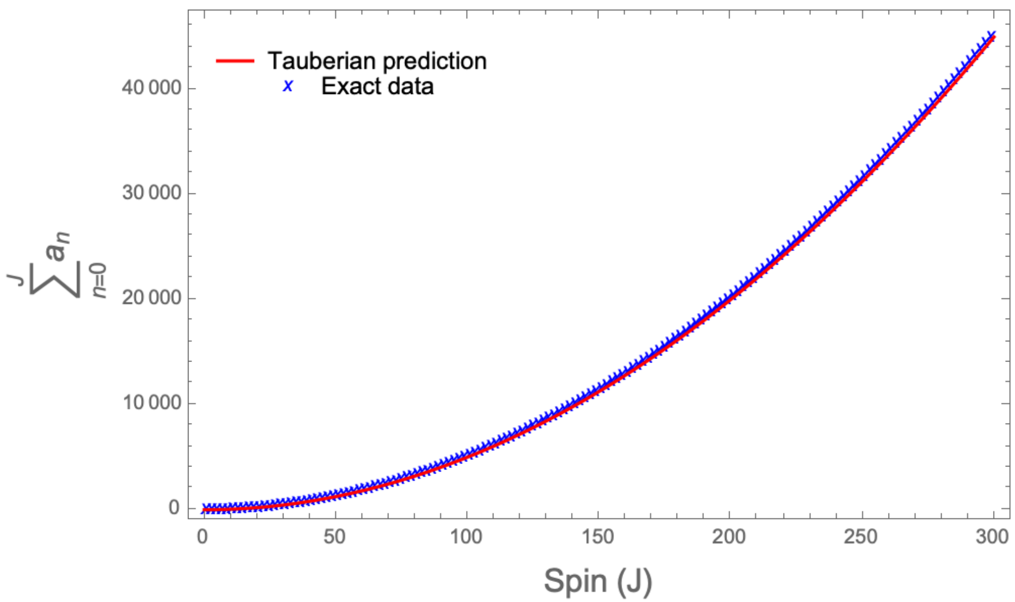
Free theory in 4d

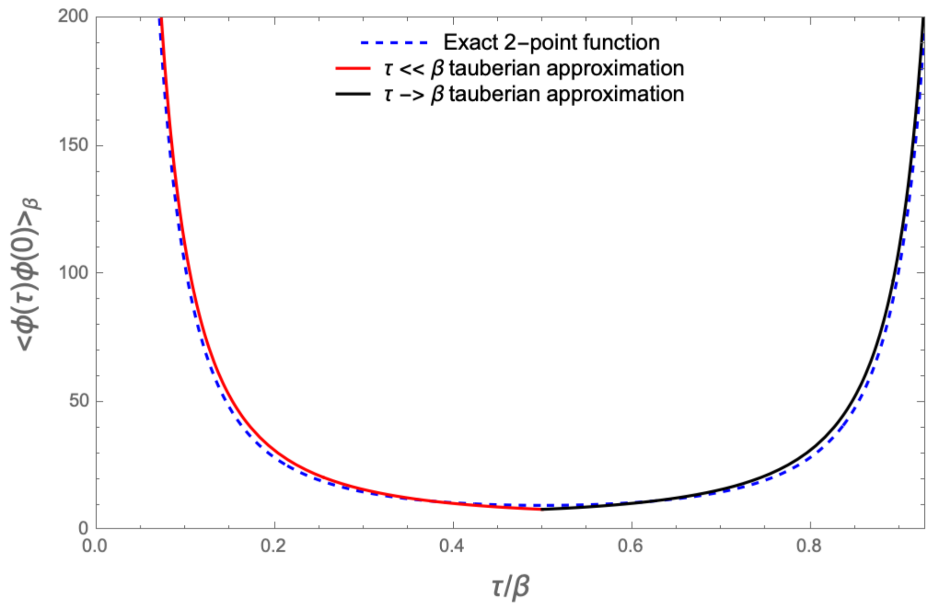
The exact two-point function:

$$\langle \phi(\tau)\phi(0)\rangle_\beta = \left(\frac{\pi}{\beta}\right)^2 \csc^2\left(\frac{\pi}{\beta}\tau\right)$$

Tauberian approximation:

$$\langle \phi(\tau)\phi(0)\rangle_\beta \simeq \begin{cases} \displaystyle\int_0^\infty d\Delta \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \frac{(\beta-\tau)^{\Delta - 2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \ll 1 \\ \displaystyle\int_0^\infty d\Delta \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)} \frac{\tau^{\Delta - 2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \sim 1 \end{cases}$$





3d O(N) model at large N

Lagrangian description :

Hubbard-Stratanovich field

$$\mathscr{L} = \frac{1}{2}(\partial\phi_i)^2 + \frac{1}{2}\sigma\phi_i^2 - \frac{\sigma^2}{4\lambda}$$

Wilson-Fisher expansion $\epsilon = 4 - d \ll 1$

weakly coupled in large N: $\lambda_* = \dfrac{8\pi^2}{N+8}\epsilon$

d=3 non-trivial IR fixed point.

The two-point function:

$$\langle\phi_i(\tau, r)\phi_j(0,0)\rangle_\beta = \delta_{ij}\sum_{m=-\infty}^{\infty}\int\frac{d^2k}{(2\pi)^2}\frac{e^{-i\vec{k}\cdot\vec{x}-i\omega_m\tau}}{\omega_n^2+\vec{k}^{\,2}+m_{th}^2} = \delta_{ij}\sum_{m=-\infty}^{\infty}\frac{e^{-m_{th}\sqrt{(\tau+m\beta)^2+r^2}}}{\sqrt{(\tau+m\beta)^2+r^2}}$$

$$\langle\sigma\rangle_\beta = m_{th}^2 = \frac{4}{\beta^2}\log^2\left(\frac{1+\sqrt{5}}{2}\right)$$

[Sachdev, Ye 1992]

[Iliesiu, Kologlu et all 2018]

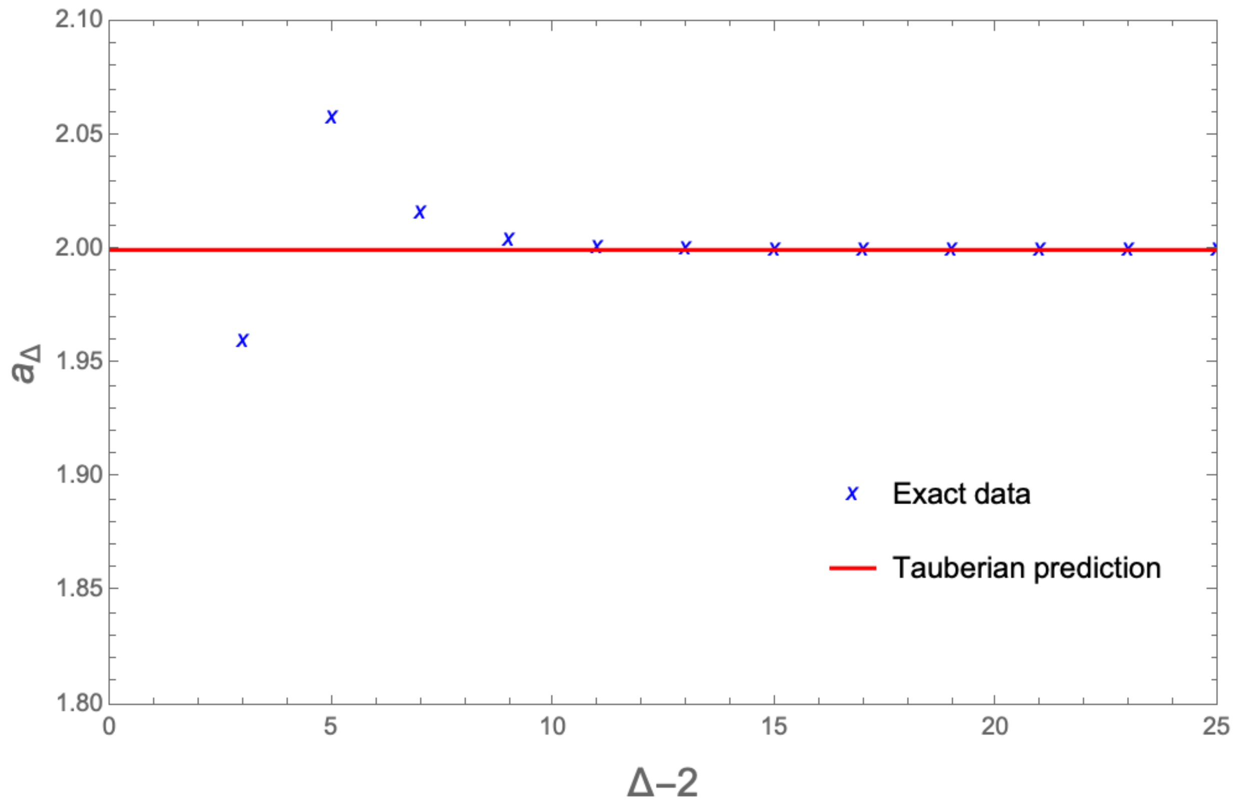
Planar 3d O(N) model

$$\Delta_{\phi_i} = \frac{1}{2} + \mathcal{O}\left(\frac{1}{N}\right)$$

$$\Delta_\sigma = 2 + \mathcal{O}\left(\frac{1}{N}\right)$$

$$a_\Delta \overset{\Delta\to\infty}{\sim} \frac{\Delta^{2\Delta_\phi - 1}}{\Gamma(2\Delta_\phi)}\delta\Delta$$

$$a_\Delta \overset{\Delta\to\infty}{\sim} 2$$



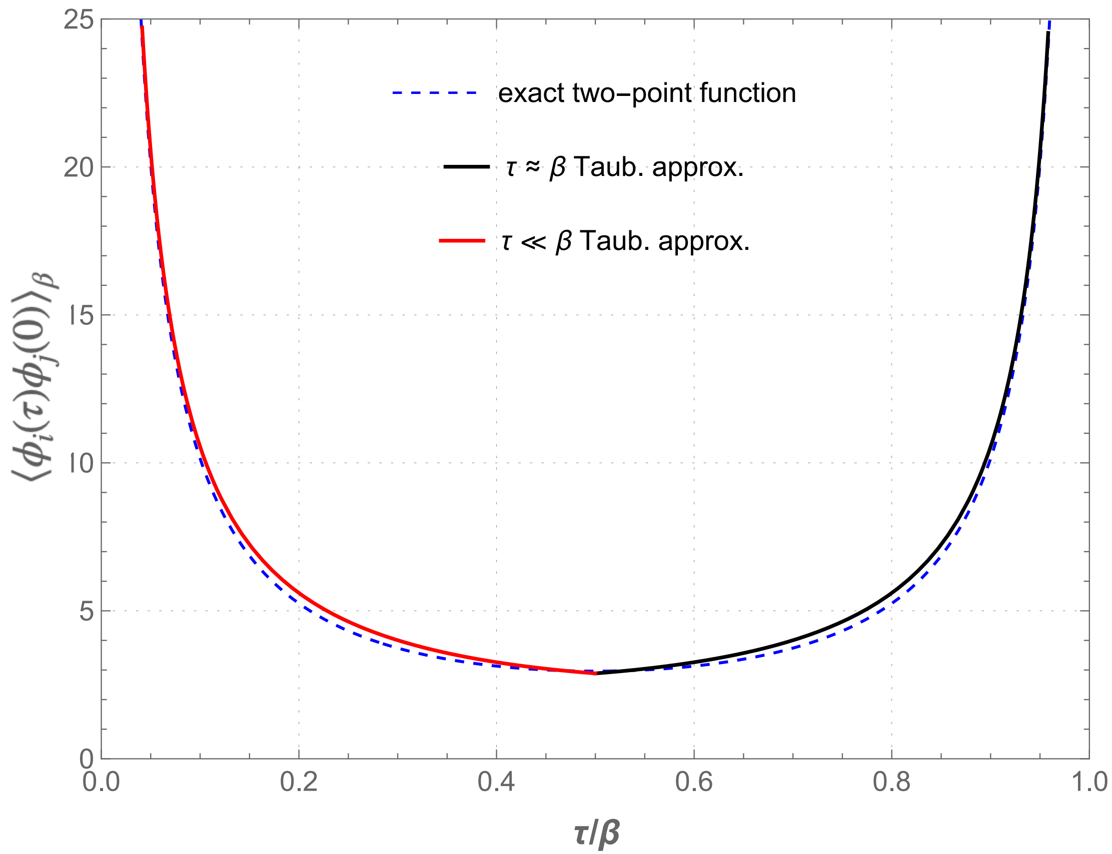
Closed form for r=0:

$$\langle \phi_i(\tau)\phi_j(0)\rangle_\beta = \delta_{ij}\left(\frac{e^{m_{th}(\tau-\beta)}}{\beta-\tau}\,{}_2F_1\left(\left\{\begin{matrix}1\ ,\frac{\beta-\tau}{\beta}\\ \frac{2\beta-\tau}{\beta}\end{matrix}\right\}\middle|\ e^{-m_{th}\beta}\right)+\frac{e^{-m_{th}\tau}}{\tau}\,{}_2F_1\left(\left\{\begin{matrix}1\ ,\frac{\tau}{\beta}\\ \frac{\beta+\tau}{\beta}\end{matrix}\right\}\middle|\ e^{-m_{th}\beta}\right)\right)$$

Planar 3d O(N) model

Tauberian approximation:

$$\langle \phi(\tau)\phi(0)\rangle_\beta \simeq \begin{cases} \displaystyle\int_0^\infty d\Delta \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)} \frac{(\beta-\tau)^{\Delta-2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \ll 1 \\ \displaystyle\int_0^\infty d\Delta \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)} \frac{\tau^{\Delta-2\Delta_\phi}}{\beta^\Delta} & \tau/\beta \sim 1 \end{cases}$$



The error is always less than 10%!

Exact:

$$\langle \phi_i(\tau)\phi_j(0)\rangle_\beta = \delta_{ij}\left(\frac{e^{m_{th}(\tau-\beta)}}{\beta-\tau}\,{}_2F_1\left(\left\{\begin{matrix}1\ ,\ \frac{\beta-\tau}{\beta}\\ \frac{2\beta-\tau}{\beta}\end{matrix}\right\}\middle|\, e^{-m_{th}\beta}\right)+\frac{e^{-m_{th}\tau}}{\tau}\,{}_2F_1\left(\left\{\begin{matrix}1\ ,\ \frac{\tau}{\beta}\\ \frac{\beta+\tau}{\beta}\end{matrix}\right\}\middle|\, e^{-m_{th}\beta}\right)\right)$$

Setting up the Numerics

Numerical method for sum rules

Our current approach (working well for all theories tested):

Inspired by [Gliozzi 2013] [Poland, Prilepina, Tadic' 2023] [W. Li 2023]

1. Input: zero Temperature spectrum and Output: a_Δ & c_i .

2. Truncate the sum + improved Tauberian asymptotic:

$$f(\ell) = \sum_{\Delta<\Delta_H} a_\Delta F(\Delta,\ell) + \sum_{\Delta>\Delta_H} a^T_\Delta F(\Delta,\ell)$$

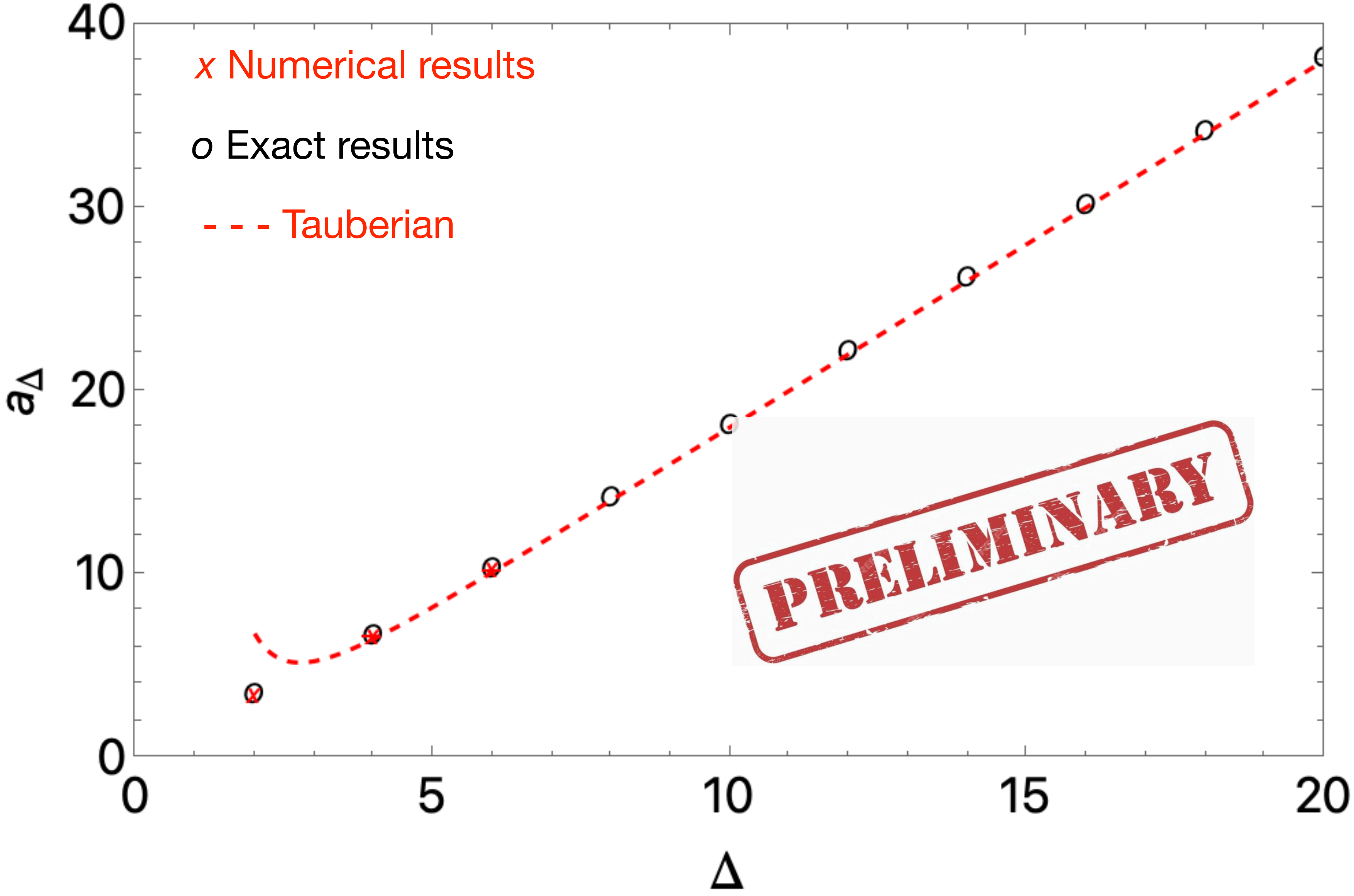
$$a^T_\Delta \sim \frac{\Delta^{2\Delta_\phi-1}}{\Gamma(2\Delta_\phi)}\delta\Delta\left(1+\frac{c}{\Delta}+\dots\right)$$

3. Numerically minimize with "random" coefficients the square of the sum rules.

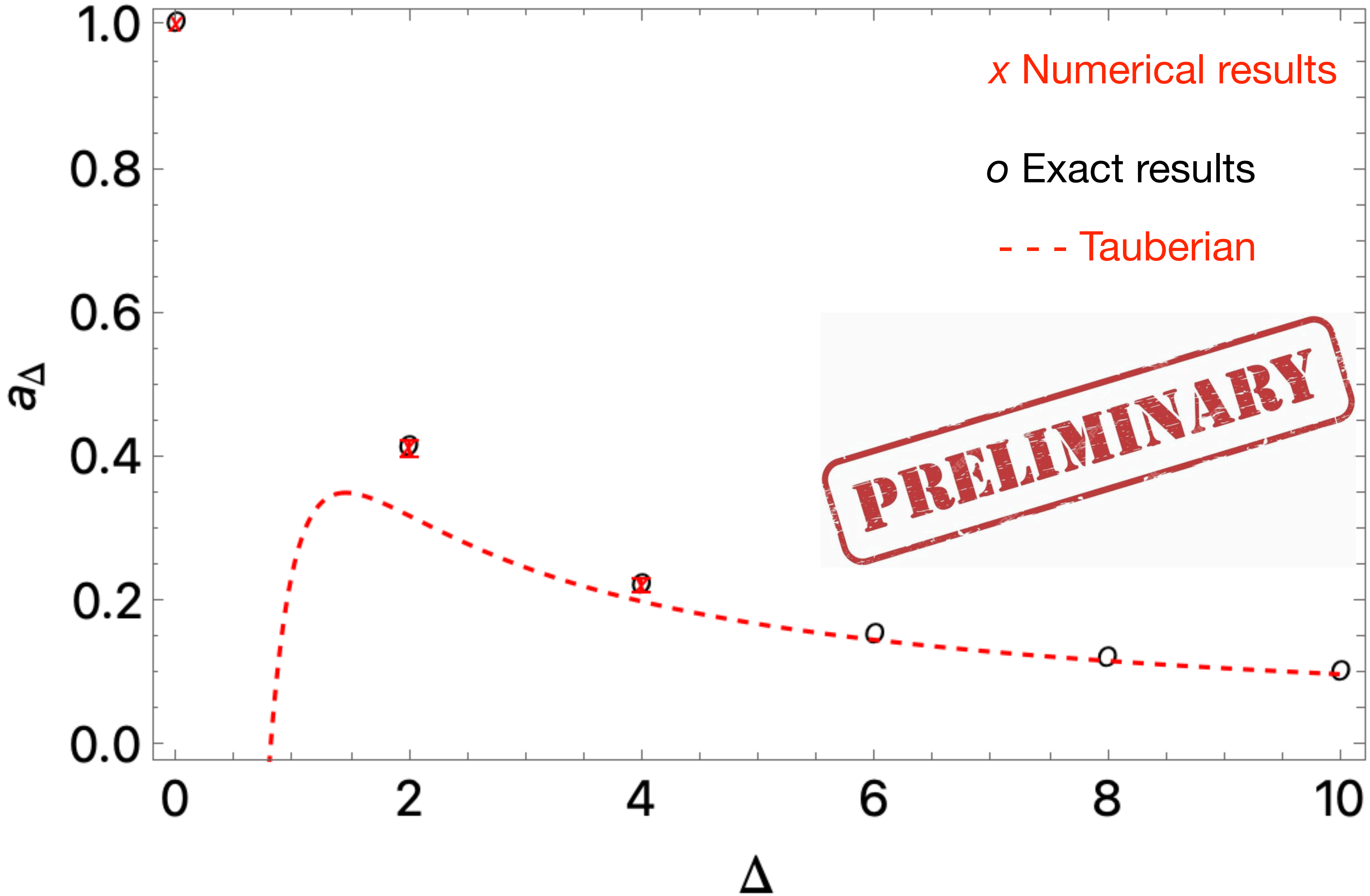
$$\min\left[\sum_{\ell\leq\ell_{max}} r_\ell\, f^2(\ell)\right]$$

Random coefficients

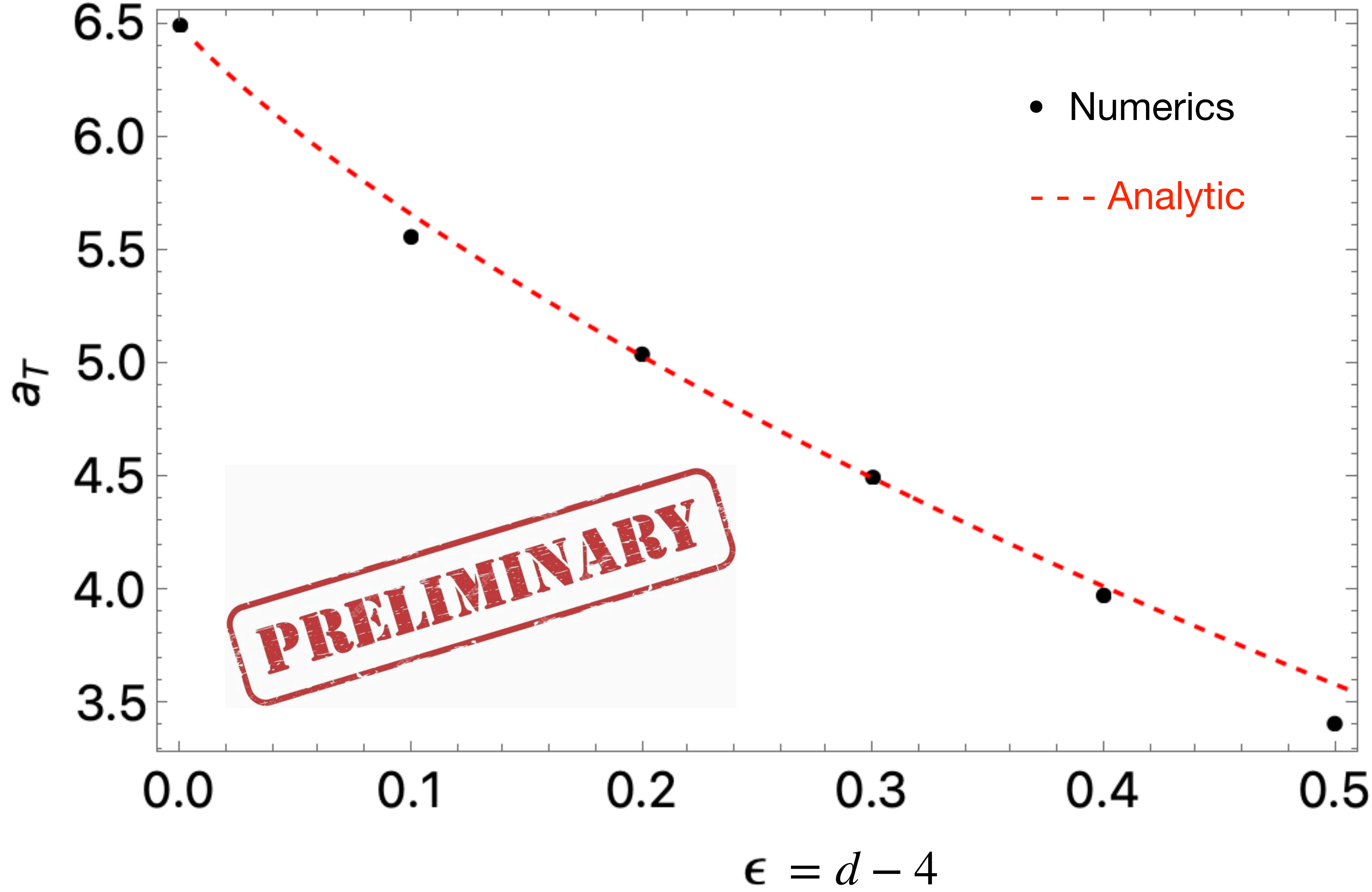
Free theory in 4d



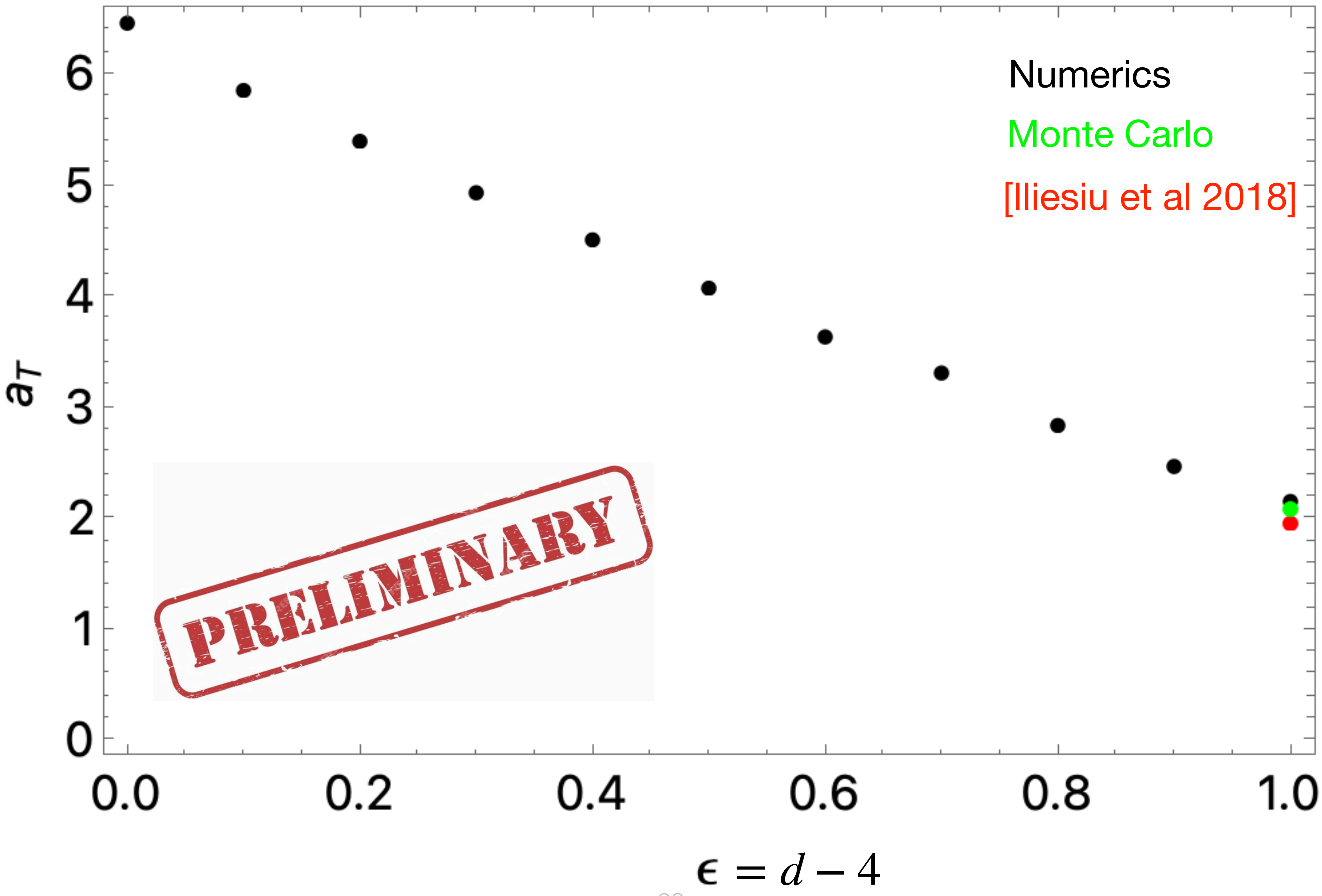
2D Ising $\langle \sigma\sigma \rangle_\beta$



Planar O(N) model



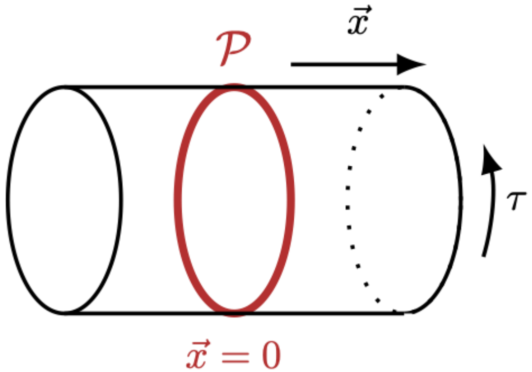
The Ising fixed points



Temporal line defects

Polyakov loops

Polyakov loops are temporal Wilson loops wrapping the thermal circle



They were introduced as a criterion for confinement. [Polyakov 1978]

* In the weak coupling using standard Feynman diagram techniques.
* In the strong coupling using holography (early days of AdS/CFT).

Can use the thermal bootstrap to compute them?

Temporal line defects

The line defects can be studied using “**thermal**” **1d defect CFT** methods.

OPE from bulk to defect
Zero temperature data

$$\mathcal{O}(\tau, \vec{x}) = \sum_{\hat{\mathcal{O}}^{i_1 \ldots i_s}} \mu_{\mathcal{O}\hat{\mathcal{O}}} |\vec{x}|^{\hat{\Delta}-\Delta-s} x_{i_1} \ldots x_{i_s} \hat{\mathcal{O}}^{i_1 \ldots i_s}(\tau)$$

“Thermal” 1d defect CFT, non-perturbative exact result:

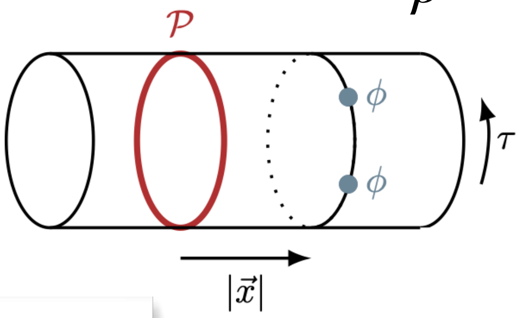
$$\langle \mathcal{O}(\tau, \vec{x}) \mathcal{P} \rangle_\beta = \sum_{\hat{\mathcal{O}}^{i_1 \ldots i_s}} \underline{b_{\hat{\mathcal{O}}}} \, \underline{\mu_{\mathcal{O}\hat{\mathcal{O}}}} \frac{|x|^{\hat{\Delta}-\Delta}}{\underline{\beta^{\hat{\Delta}}}}$$

New finite Temperature vevs of 1d defect CFT

Zero temperature data

Sum rules from KMS

$$\langle \phi(\beta/2-\tau,\vec{x})\phi(0,\vec{x})\mathscr{P}\rangle_\beta = \langle \phi(\beta/2+\tau,\vec{x})\phi(0,\vec{x})\mathscr{P}\rangle_\beta$$



The sum rules obtained are of the form:

$$\sum_{\Delta,\widehat{\Delta}} \widehat{b}_{\widehat{\mathcal{O}}}\, f_{\phi\phi\mathcal{O}}\lambda_{\mathcal{O}\widehat{\mathcal{O}}} \binom{\Delta-2\Delta_\phi}{n}\frac{z^{\widehat{\Delta}-\Delta}}{2^{\Delta}} = 0$$

New finite Temperature vevs of 1d defect CFT

Zero temperature data

Defect thermal sum rules are very similar to their bulk counterparts.

Where we currently are

* Derived and tested thermal Sum rules.
* Heavy operators: asymptotic OPE density.
* Setting up the numerical bootstrap.
* Temporal line defects (Polyakov loops).

Where are we going?

- More theories 3D Ising, $\mathcal{N}$=4 SYM, …
- Study the $S^1 \times S^{d-1}$ geometry.
- Black holes, hydro and CFT data.
- Numerical approach for temporal line defects.

Thank you!