# Spread Complexity and Bulk Momentum

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## Outline:

- Introduction/Motivation
- Spread complexity
- Relation to the bulk momentum
- Discussion and Open Questions

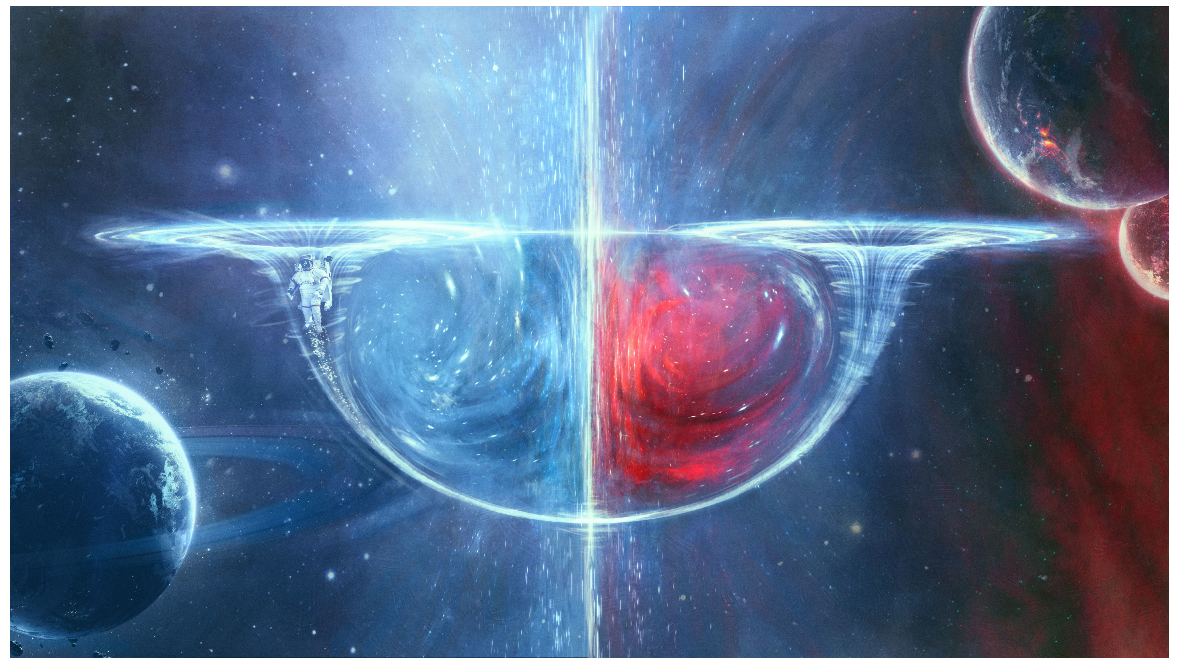
Based on:

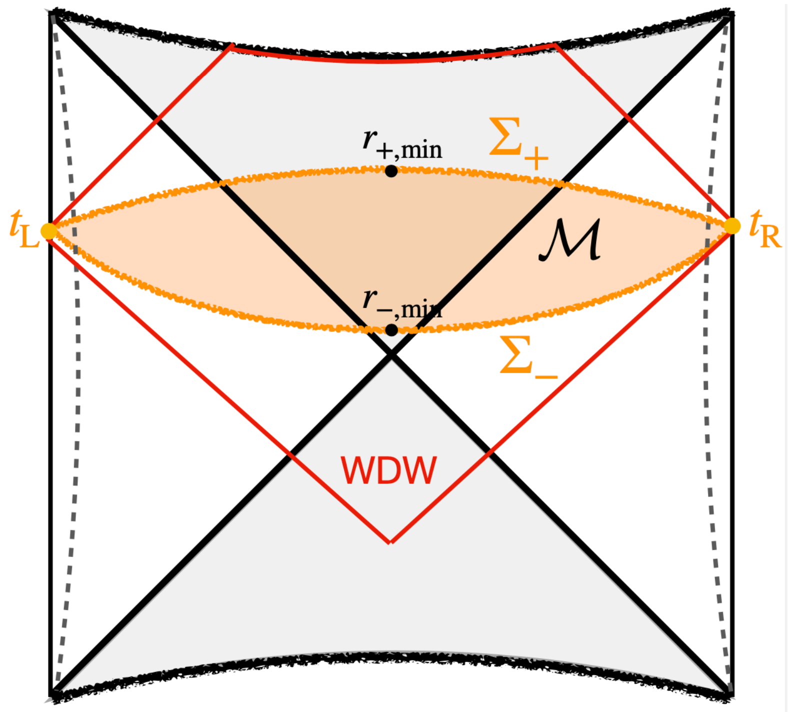
To appear soon with B. Chen (UW->Beijing), R. McDonald, J. Simon, B. Strittmatter (U. of Edinburgh)

Previous works with V. Balasubramanian, J. Magan, S. Liu, D. Patramanis, Q.Wu,

## Complexity = Volume, WdW Action, (Almost) Anything …

$$\text{“Complexity”}(\,|\psi(t)\rangle = e^{-iHt}\,|\psi_\beta\rangle\,)$$






[Susskind’14,Stanford,Susskind’14,Brown et al.’15,Belin et al.’21]

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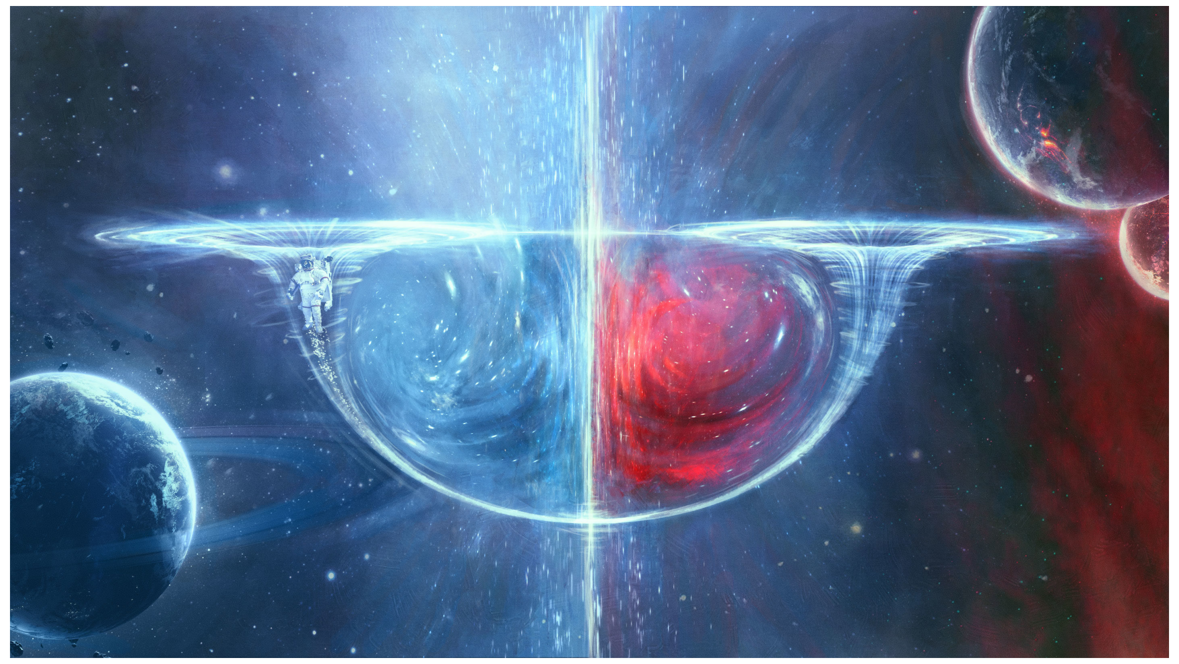
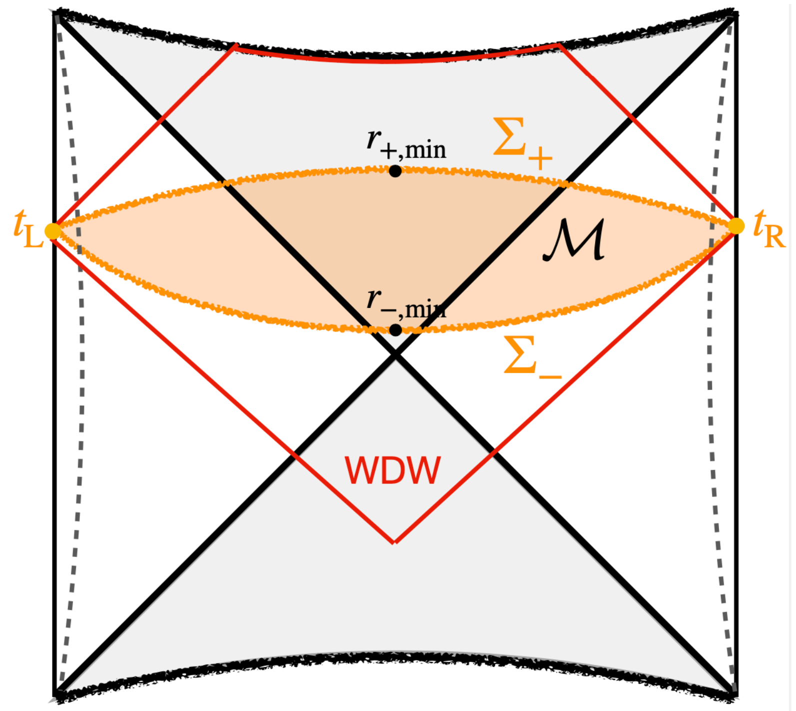


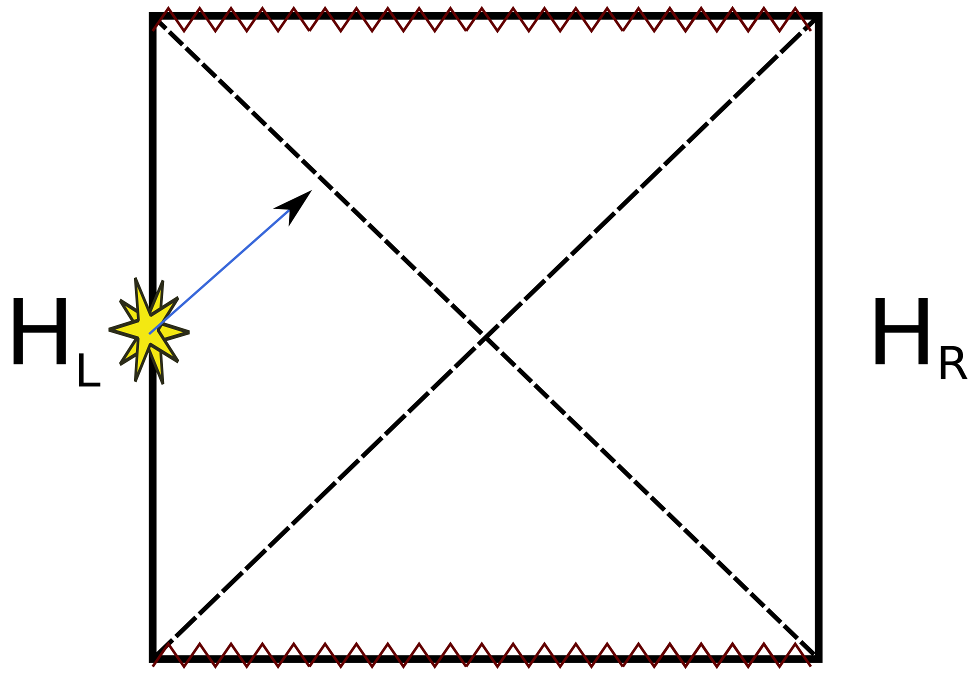



[Susskind'14,Stanford,Susskind'14,Brown et al.'15,Belin et al.'21]

## Complexity ~ Momentum

[Susskind'19,Lin,Maldacena,Zhao'19,Susskind,Zhao'20]
[Barbon,Martin-Garcia,Sasieta'19…]

$$\text{"Complexity"}(\mathcal{O}(t) = e^{iHt}\mathcal{O}(0)e^{-iHt})\text{~size(t)}$$




$$P_r = \frac{\partial \mathcal{L}}{\partial r'(t)}$$

$$\partial_t C(t) \sim P_r(t)$$

“Complexity”?

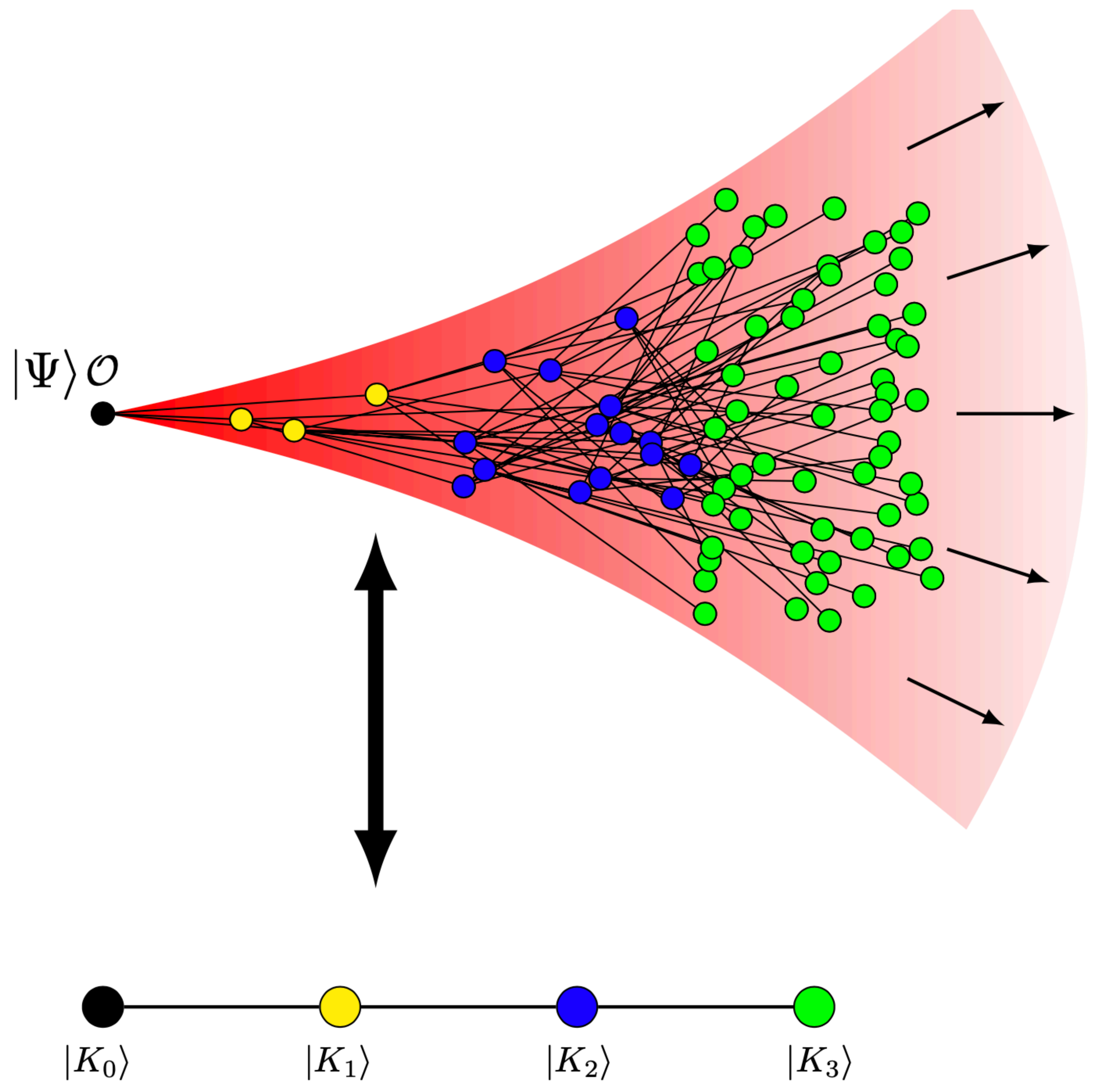
## Spread/Krylov Complexity

Map the unitary evolution into a 1d chain
**Complexity** = distance from the origin

$$|\Psi(t)\rangle = e^{-iHt}\,|\Psi_0\rangle = \sum_n \phi_n(t)\,|K_n\rangle$$

Krylov basis coefficients => probability distribution

$$\sum_n |\phi_n(t)|^2 \equiv \sum_n p_n(t) = 1$$




…. [Parker et al. ’19] …. [Balasubramanian,PC,Magan,Wu ’22]….

Spread complexity

$$\mathcal{C}_K(t) = \langle n(t)\rangle = \sum_n n\,p_n(t)$$

## Spread/Krylov Complexity

[Parker et al. '19, Balasubramanian,PC,Magan,Wu '22]

$$|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle = \sum_{k=0}^{\infty}\frac{(-it)^k}{k!}|\Psi_k\rangle \equiv \sum_{n=0}\phi_n(t)|K_n\rangle$$

Krylov subspace

$$|\Psi_n\rangle \equiv \{|\Psi_0\rangle, H|\Psi_0\rangle, ..., H^n|\Psi_0\rangle, ...\}$$

construct a basis $|K_n\rangle$ recursively (Lanczos algorithm, GS):

$$|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle, \qquad |K_n\rangle = b_n^{-1}|A_n\rangle$$

Lanczos coefficients

$$a_n = \langle K_n|H|K_n\rangle, \qquad b_n = \langle A_n|A_n\rangle^{1/2}$$

Evolution (Schrödinger) equation

$$i\partial_t\phi_n(t) = a_n\phi_n(t) + b_n\phi_{n-1}(t) + b_{n+1}\phi_{n+1}(t)$$

Moments of the return amplitude

$$S(t) \equiv \langle\Psi(t)|\Psi(0)\rangle = \langle\Psi_0|e^{iHt}|\Psi_0\rangle = \phi_0^*(t)$$

## Spread/Krylov Complexity

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Moments of the return amplitude → $a_n, b_n$ → $p_n(t)$ → $\mathcal{C}_K(t) = \sum_n n p_n(t)$

$$S(t) \equiv \langle\Psi(t)|\Psi(0)\rangle = \langle\Psi_0|e^{iHt}|\Psi_0\rangle = \phi_0^*(t)$$

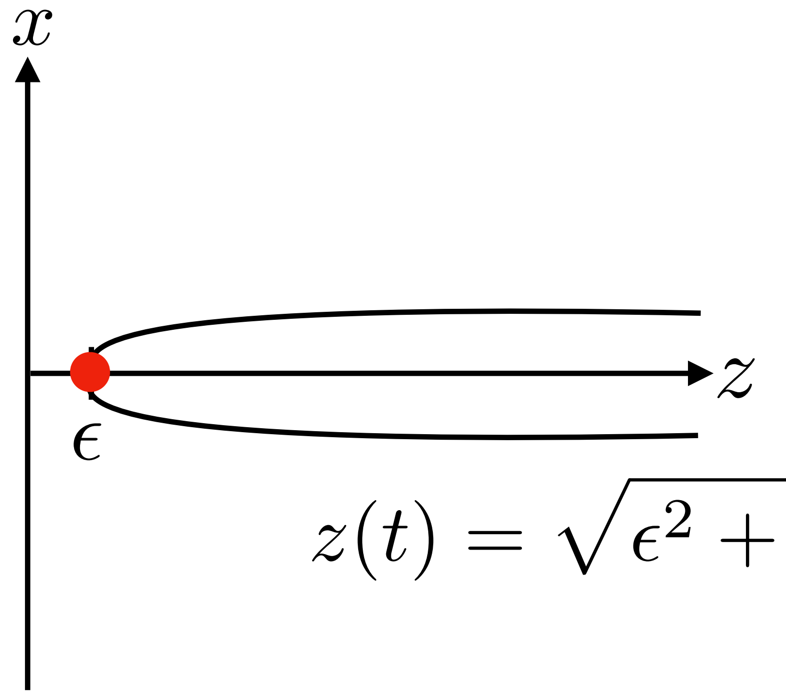
[Length basis in JT, Lin '23, Successful in DSSYK/JT Rabinovici et al. '23]

Now that we have developed a precise definition of “Complexity” in CFTs,
can we finally make progress on holographic complexity?

# Setup: Local Operator Quench in AdS/CFT

$$ds^2 = R^2 \frac{-dt^2 + dx^2 + dz^2}{z^2}$$

$$|\psi(t)\rangle = \mathcal{N} e^{-iHt} e^{-\epsilon H} \mathcal{O}(x) |0\rangle$$




$$S = -mR \int dt \frac{\sqrt{1 - z'(t)^2}}{z(t)}$$

$$E = \int dx \langle T_{00} \rangle = \frac{\Delta}{\epsilon} \qquad \Delta = h + \bar{h}$$

$$z(t) = \sqrt{\epsilon^2 + t^2}$$

$$mR = \Delta$$

[Nozaki,Numasawa,Takayanagi'13,PC,Nozaki,Takayanagi'14]

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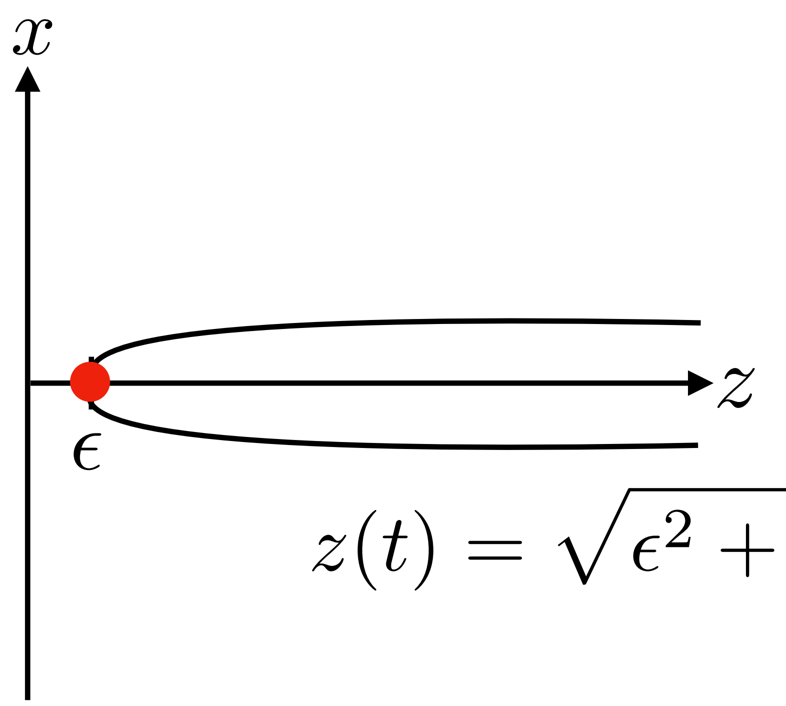


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[Nozaki,Numasawa,Takayanagi'13,PC,Nozaki,Takayanagi'14]

$$|\psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2} E_n} |E_n\rangle_L \otimes |E_n\rangle_R$$

$$|\psi(t)\rangle = \mathcal{N} e^{-iH_L t} e^{-\epsilon H_L} \mathcal{O}_L(x) |\psi_\beta\rangle$$

[PC,Simon,Stikonas,Takayanagi,Watanabe'14'15]

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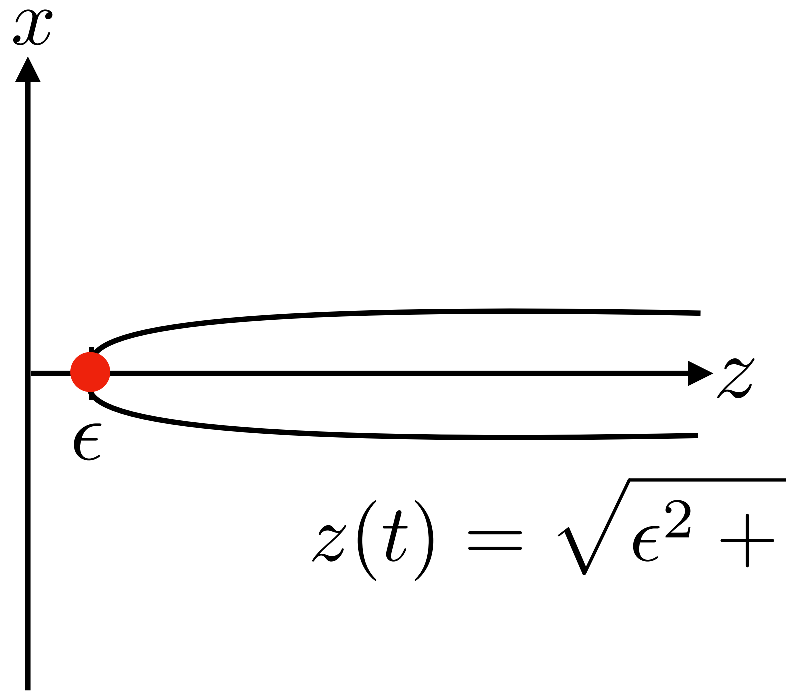


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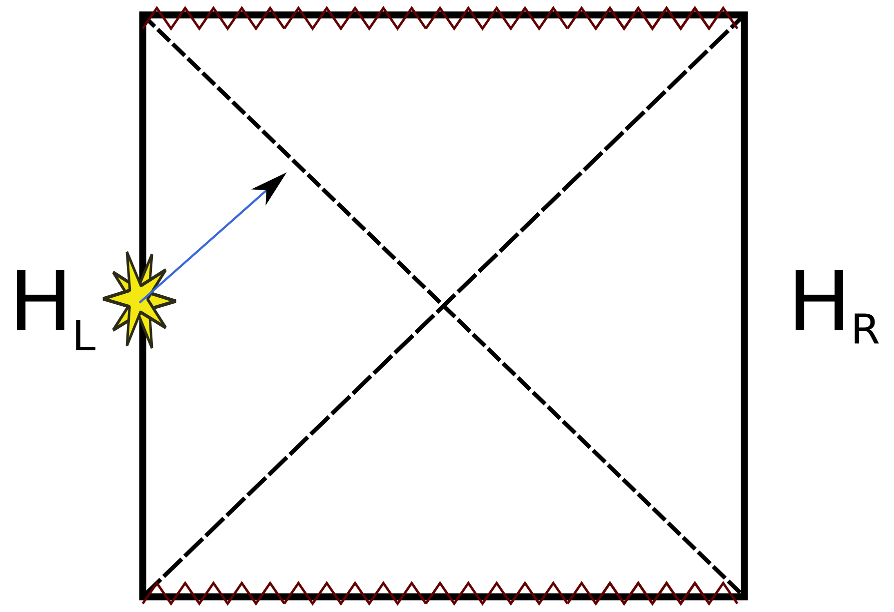


$$ds^2 = \frac{R^2}{z^2}\left(-(1 - Mz^2)dt^2 + \frac{dz^2}{1 - Mz^2} + dx^2\right)$$

$$z(t) = \frac{\beta}{2\pi}\sqrt{1 - \left(1 - \frac{4\pi^2\epsilon^2}{\beta^2}\right)\cosh^{-2}\left(\frac{2\pi t}{\beta}\right)}$$

$$z(t \gtrsim \frac{\beta}{2\pi}) = \frac{\beta}{2\pi} \qquad \sqrt{M} = 2\pi/\beta$$

[PC,Simon,Stikonas,Takayanagi,Watanabe'14'15]

[PC,Magan,Patramanis,Tonni ’23]

## Spread Complexity of local operators

$$S(t) = \frac{\langle \mathcal{O}(x + i(\epsilon + it))\mathcal{O}(x - i\epsilon)\rangle}{\langle \mathcal{O}(x + i\epsilon)\mathcal{O}(x - i\epsilon)\rangle} \longrightarrow a_n, b_n \longrightarrow p_n(t)$$

## Rate of growth: CFT predictions

[PC,Chen,McDonald,Simon,Strittmatter…]

$$\partial_t C_K(t) = \frac{1}{\epsilon}\frac{\Delta}{\epsilon} t \qquad \partial_t C_K(t) = \frac{1}{\epsilon}\frac{\Delta}{\epsilon}\frac{L}{2\pi}\sin\left(\frac{2\pi t}{L}\right) \qquad \partial_t C_K(t) = \frac{1}{\epsilon}\frac{\Delta}{\epsilon}\frac{\beta}{2\pi}\sinh\left(\frac{2\pi t}{\beta}\right)$$

Can we reproduce this from the gravity computation? Basis dependent quantity?

# Radial momentum in AdS

[PC,Chen,McDonald,Simon,Strittmatter…]

$$S = -m \int ds \qquad t,\ r(t),\ x = x_0 \qquad P_r = \frac{\partial \mathcal{L}}{\partial r'(t)}$$

Poincare:

$$S = -mR \int dt \frac{\sqrt{1 - z'(t)^2}}{z(t)} \qquad P_z = \frac{mR\, z'(t)}{z(t)\sqrt{1 - z'(t)^2}} \to \frac{mR\, t}{\epsilon\sqrt{t^2 + \epsilon^2}}$$

BTZ:

$$S = -mR \int \frac{dt}{z(t)} \sqrt{1 - Mz(t)^2 - \frac{z'(t)^2}{1 - Mz(t)^2}} \qquad P_z \to \frac{mR}{\epsilon} \cosh^2\left(\frac{2\pi t}{\beta}\right)$$

[PC,Chen,McDonald,Simon,Strittmatter…]

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## Proper distance momentum

[PC,Chen,McDonald,Simon,Strittmatter…]

$$ds^2 = d\rho^2 + \frac{4\pi^2}{\beta^2}\left(-\sinh^2(\rho)dt^2 + \cosh^2(\rho)dx^2\right)$$

$$ds^2 = d\rho^2 + \frac{4\pi^2}{L^2}\left(-\cosh^2(\rho)dt^2 + \sinh^2(\rho)dx^2\right)$$

$$ds^2 = d\rho^2 + e^{2\rho}\left(-dt^2 + dx^2\right)$$

Proper distance

$$\cosh(D_{if}) = -X_i(\rho_i)\cdot X_f(\rho_f)|_{t,x} = \cosh(\rho_i - \rho_f)$$

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$$P_\rho = \frac{mR}{\epsilon}\frac{\beta}{2\pi}\sinh\left(\frac{2\pi t}{\beta}\right)$$

$$P_\rho = \frac{mR}{\epsilon}\frac{L}{2\pi}\sin\left(\frac{2\pi t}{L}\right)$$

$$P_\rho = \frac{mR}{\epsilon}t$$

[PC,Chen,McDonald,Simon,Strittmatter…]

## Proper distance momentum

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$$P_\rho = \frac{mR}{\epsilon}t$$

$$\partial_t C_K(t) = \frac{1}{\epsilon}\frac{\Delta}{\epsilon}t \qquad \partial_t C_K(t) = \frac{1}{\epsilon}\frac{\Delta}{\epsilon}\frac{L}{2\pi}\sin\left(\frac{2\pi t}{L}\right) \qquad \partial_t C_K(t) = \frac{1}{\epsilon}\frac{\Delta}{\epsilon}\frac{\beta}{2\pi}\sinh\left(\frac{2\pi t}{\beta}\right)$$

$$\boxed{\partial_t \mathcal{C}_K(t) = \frac{1}{\epsilon}P_\rho}$$

## Conclusions/Discussion

- Rate of growth of the Spread Complexity in CFT is the “proper” bulk momentum!
- Krylov basis in the boundary and proper distance (n vs rho, wormhole length)?
- Lots of generalisations, back-reaction, more complicated states….
- Isomorphism between the Boundary and the Bulk Hilbert spaces?
- Basis dependent tools in AdS/CFT? Gravitational dressing [Lin,Maldacena,Zhao’19]?

**Thank you!**