Extended phase space for sub^n-leading soft theorems

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Why symmetries ?

$$\text{asymptotic symmetries} \quad \xrightarrow[\text{id.}]{\text{Ward}} \quad \text{soft theorems}$$

- Important ingredient for flat space holography

Why symmetries ?

asymptotic symmetries $\xrightarrow[\text{id.}]{\text{Ward}}$ **subleading**/loop corrections for soft theorems

[Campligia,Ladha,Strominger,Lysov,Pasterski,Peraza,Donnelly,Freidel,Speranza,Geiller,Pranzetti,Ciambelli,Leigh, Pai,Oliveri,Speziale,Raclariu,Zwikel,Sahoo,Sen,Krishna,Pasterski,Donnay,Nguyen,Ruzziconi,Agrawal,Choi,Puhm, Bhatkar,Bianchi,He,Huang,Wen,Mitra,Conde,Mao,Wu,Bern,Davies,Di Vecchia,Nohle.....]

- Important ingredient for flat space holography

Strange symmetries

asymptotic symmetries $\xrightarrow[\text{id.}]{\text{Ward}}$ **subleading** soft theorems

$$\lim_{\omega\to 0} A_{n+1} = \underbrace{\left(\tfrac{1}{\omega}S^{(0)} + S^{(1)} + \ldots\right)A_n}_{\text{universal}} \quad + \quad \underbrace{\ldots}_{\text{non-universal}}$$

- Example: subleading order gauge theory extracted via

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)A_{n+1} = S^{(1)}A_n$$

was shown to arise as a Ward identity for an overleading gauge parameter [Campiglia,Laddha]

$$\Lambda(r,u,z,\bar{z}) = r\Lambda_1(u,z,\bar{z}) + \Lambda_0(u,z,\bar{z})$$

- Violates the fall-off of the fields

$$A_z(r,u,z,\bar{z}) = A_z^{(0)}(u,z,\bar{z}) + \ldots$$

Goals

$$\text{asymptotic symmetries} \xrightarrow[\text{id.}]{\text{Ward}} \textbf{subleading} \text{ soft theorems}$$

$$\lim_{\omega\to 0} A_{n+1} = \underbrace{\left(\tfrac{1}{\omega}S^{(0)} + S^{(1)} + \ldots\right)A_n}_{\text{universal}} \quad + \quad \underbrace{\ldots}_{\text{non-universal}}$$

- Goal 1: extend phase space such that symmetries act canonically (at all orders)
- Goal 2: consequences for soft theorems

Set-up

- Work in Yang-Mills

$$D^\mu \mathcal{F}_{\mu\nu} = \mathcal{J}_\nu \ (=0)$$

- Arbitrary gauge choice, coordinates $x = (\mathfrak{r}, \vec{\mathbf{y}})$, usually Bondi $(r, u, z, \bar{z})$.
- Very general expansion for

$$\mathcal{A}_\mu = \sum_{n,k} A_\mu^{(-n;k)}(\vec{\mathbf{y}}) \frac{\log^k \mathfrak{r}}{\mathfrak{r}^n} \ ,$$

with n, k such that $\lim_{r\to\infty} \mathcal{A}_\mu$ at most $\mathcal{O}(1)$.

- Phase space

$$\Gamma^0 = \left\{ \mathfrak{A}^0 | \text{with } \mathfrak{A}^0 \text{ constrained by e.o.m. and gauge choice } \right\}$$

Divergent gauge parameter

- Standard large gauge transformations give leading order soft theorems

$$\Lambda^{(0)} = \Lambda^{(0)}(\vec{\mathbf{y}})$$

- General divergent gauge parameter

$$\Lambda_+(x) = \sum_{n,k} \mathfrak{r}^n \log^k \mathfrak{r} \; \Lambda^{(n;k)}(\vec{\mathbf{y}}) \ ,$$

- May also be field dependent $\breve{\Lambda}_+ = \breve{\Lambda}_+(\mathcal{A}_\mu(x), \Lambda_+(x))$
- Violates gauge field fall-off
- Need new objects on which these symmetries act canonically.
- Helps to think of problem as a symmetry breaking...

Stueckelberg trick

- Stueckelberg trick: originally introduced to restore *broken local symmetry* in e.g. massive theories
- Promote gauge parameter of symmetry we want to restore to a field

$$\Lambda_+(x) \quad \rightarrow \quad \Psi(x),$$

with

$$\Psi(x) = \sum_{n,k} \mathfrak{r}^n \log^k \mathfrak{r}\ \Psi^{(n;k)}(\vec{\mathbf{y}}).$$

- the phase space is now

$$\Gamma^{\text{ext}}_{\infty} := \Gamma^0 \times \{\Psi(x)\}$$

- Ψ is Goldstone-type field

Consistency relation

- Ψ comes from the bulk
- Dressed gauge field

$$\tilde{\mathcal{A}}_\mu = e^{i\Psi}\mathcal{A}_\mu e^{-i\Psi} + ie^{i\Psi}\partial_\mu e^{-i\Psi}$$

- Consistency condition:

$$\delta_\Lambda \tilde{\mathcal{A}}_\mu = \tilde{D}_\mu \Lambda$$

  where

$$\Lambda = \Lambda^{(0)} + \Lambda_+$$

- The transformation of $\mathcal{A}_\mu$ is unchanged, and the overleading transformations only transform Ψ.

Stueckelberg field transformation

- The Stueckelberg (Goldstone) field transforms as

$$\delta_\Lambda \Psi = \mathcal{O}^{-1}_{-i\Psi}(\Lambda - e^{i\Psi}\Lambda^{(0)}e^{-i\Psi})$$

with

$$\mathcal{O}_X := \frac{1-e^{-ad_X}}{ad_X} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}(ad_X)^k$$

and its inverse

$$(\mathcal{O}_X)^{-1} = \left(\frac{1-e^{-ad_X}}{ad_X}\right)^{-1} = \sum_{m=0}^{\infty} \frac{B_m^+ ad_X^m}{m!}$$

where B_n^+ are the Bernoulli numbers

$$B_0^+ = 1, \quad B_1^+ = \frac{1}{2}, \quad B_2^+ = \frac{1}{6}, \quad \dots$$

Stueckelberg field transformation

- The Stueckelberg (Goldstone) field transforms as

$$\delta_\Lambda \Psi = \mathcal{O}^{-1}_{-i\Psi}(\Lambda - e^{i\Psi}\Lambda^{(0)}e^{-i\Psi})$$

- Perturbatively

$$\delta^{[m]}_\Lambda \Psi = \frac{B^+_m}{m!}(ad_{-i\Psi})^m \left[\Lambda + (-1 + 2\delta_{m,1})\Lambda^{(0)}\right]$$

- at 0th order in the field, it transforms via a shift

$$\delta^{[0]}_\Lambda \Psi = \Lambda - \Lambda^{(0)} = \Lambda_+$$

  i.e. Goldstone modes for the symmetry breaking in the bulk.

- recall that

$$B_{2k+1} = 0, \quad \text{for} \quad k > 0$$

  i.e. only even powers of the fields will contribute to the transformation

Charges in the bulk

- The *bulk* action,

$$S[\tilde{\mathcal{A}}_\mu] = \int_D \mathrm{tr}\left(\tilde{\mathcal{F}}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu}\right) dvol,$$

where $\tilde{\mathcal{F}}^{\mu\nu}$ is constructed from $\tilde{\mathcal{A}}_\mu$, and takes the form

$$\tilde{\mathcal{F}}_{\mu\nu} \equiv \partial_\mu \tilde{\mathcal{A}}_\nu - \partial_\nu \tilde{\mathcal{A}}_\mu - i[\tilde{\mathcal{A}}_\mu, \tilde{\mathcal{A}}_\nu] = e^{i\Psi}\mathcal{F}_{\mu\nu}e^{-i\Psi}$$

- same as without Ψ, but apply covariant phase space formalism on the tilded objects

$$\Omega^{\text{bulk}}[\delta_1, \delta_2] = -\int_\Sigma \mathrm{tr}\left(\delta_1\tilde{\mathcal{F}}^{\mu\nu} \wedge \delta_2\tilde{\mathcal{A}}_\nu\right) dS_\mu - (1 \leftrightarrow 2),$$

Charges at null infinity

- Covariant phase space formalism gives:

$$\tilde{Q}_\Lambda = \int_\Sigma \partial_\nu \, \mathrm{tr}(\Lambda \tilde{\mathcal{F}}^{\mu\nu}) dS_\mu .$$

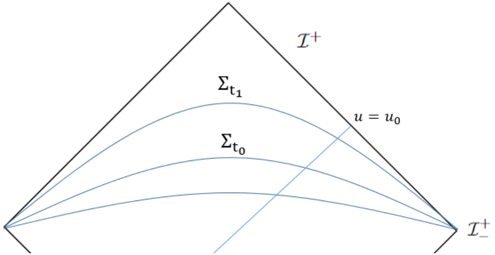


Figure: Cauchy Surfaces $\Sigma_{t_0}, \Sigma_{t_1}$, with $t_0 < t_1$, and constant u ray. Reach $\mathcal{I}^+$ by taking $r \to +\infty$ with u fixed.

Dressed charge algebra

- Charge on the celestial sphere

$$\tilde{Q}_\Lambda = \int_{S^2} \mathrm{tr}(\Lambda \tilde{\mathcal{F}}_{ru}) \sqrt{g_{S^2}}\, dz d\bar{z}.$$

- We can now define the charge density

$$\tilde{q}_\Lambda = \mathrm{tr}\left(\sqrt{g_{S^2}} \Lambda \tilde{\mathcal{F}}_{ru}\right)^{(0)}$$

In terms of the original field strength, the charge density is

$$\begin{aligned}\tilde{q}_\Lambda &= \mathrm{tr}\left(\sqrt{g_{S^2}} \Lambda e^{i\Psi} \mathcal{F}_{ru} e^{-i\Psi}\right)^{(0)} \\ &= \mathrm{tr}\left(\sqrt{g_{S^2}} e^{-i\Psi} \Lambda e^{i\Psi} \mathcal{F}_{ru}\right)^{(0)}\end{aligned}$$

- Charge algebra

$$\{\tilde{q}_{\Lambda_1}, \tilde{q}_{\Lambda_2}\} = \tilde{q}_{-i[\Lambda_1, \Lambda_2]}$$

Recursion relations to all orders

- Assume fall-off

$$\mathcal{A}_\mu(u,r,z,\bar{z}) = \sum_{n\in\mathbb{Z}} A_\mu^{(-n)}(u,z,\bar{z})\frac{1}{r^n},$$

- Further expand in u (dual of energy)

$$\mathcal{A}_\mu(u,r,z,\bar{z}) = \sum_{n,k\in\mathbb{Z}} A_\mu^{(-n,k)}(z,\bar{z})\frac{u^k}{r^n},$$

- The overleading gauge parameter and Stueckelberg field also have a simpler expansion in this case

$$\Lambda_+(u,r,z,\bar{z}) = \sum_{k=1}^{\infty} r^k \Lambda_+^{(k)}(u,z,\bar{z})$$

$$\Psi(u,r,z,\bar{z}) = \sum_{k=1}^{\infty} r^k \Psi^{(k)}(u,z,\bar{z})$$

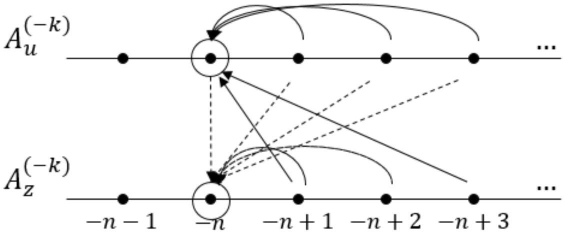
- Work in radial gauge

$$\mathcal{A}_r = 0, \quad \text{and} \quad A_u^{(0)} = 0,$$

Recursion relations to all orders

- Using e.o.m. and Bianchi idenities, we find

$$A_u^{(-n)} \quad \text{depends on} \quad \{A_z^{(-k)}, A_{\bar{z}}^{(-k)}\}_{k<n} \quad \text{and} \quad A_u^{(-n,0)}$$
$$A_z^{(-n)} \quad \text{depends on} \quad A_u^{(-n)}, \{A_z^{(-k)}, A_{\bar{z}}^{(-k)}, A_u^{(-k)}\}_{k<n} \quad \text{and} \quad A_z^{(-n,0)}.$$



Fall-offs in u

- the standard $u \to -\infty$ fall-offs for the radiative data $A_z^{(0)}$,

$$A_z^{(0)}(u, z, \bar{z}) = A_z^{(0,0)}(z, \bar{z}) + o(u^{-\infty}),$$

translates to

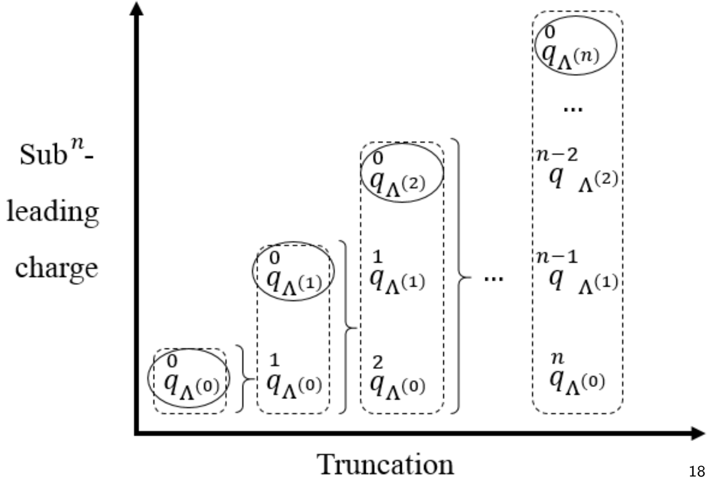
$o(u^{-\infty})$	$F_{uz}^{(0)}$	$F_{u\bar{z}}^{(0)}$				
$O(1)$	$F_{uz}^{(-1)}$	$F_{u\bar{z}}^{(-1)}$	$\mathbf{F}_{\mathbf{ru}}^{(-2)}$	$F_{z\bar{z}}^{(0)}$		
$O(u)$	$F_{uz}^{(-2)}$	$F_{u\bar{z}}^{(-2)}$	$\mathbf{F}_{\mathbf{ru}}^{(-3)}$	$F_{z\bar{z}}^{(-1)}$	$F_{rz}^{(-2)}$	$F_{r\bar{z}}^{(-2)}$
$O(u^2)$	$F_{uz}^{(-3)}$	$F_{u\bar{z}}^{(-3)}$	$\mathbf{F}_{\mathbf{ru}}^{(-4)}$	$F_{z\bar{z}}^{(-2)}$	$F_{rz}^{(-3)}$	$F_{r\bar{z}}^{(-3)}$
$\cdots$						
$O(u^n)$	$F_{uz}^{(-n-1)}$	$F_{u\bar{z}}^{(-n-1)}$	$\mathbf{F}_{\mathbf{ru}}^{(-\mathbf{n}-2)}$	$F_{z\bar{z}}^{(-n)}$	$F_{rz}^{(-n-1)}$	$F_{r\bar{z}}^{(-n-1)}$

Table: fall-offs in u

Charges revisited

- on the celestial sphere (i.e., $\mathcal{I}^+_-$), the charge is the limit $r \to +\infty$ and $u \to -\infty$ in the quantity

$$\tilde{Q}_\Lambda = \int_{S^2} \mathrm{tr}(\Lambda \tilde{\mathcal{F}}_{ru}) r^2 \gamma_{z\bar{z}} dz d\bar{z}.$$



Treuncated charge algebra and recursion relations

- Explicitly, *recursion for charges*

$$\overset{0}{q}_{\Lambda^{(n)}} = \mathrm{Tr}\left(\Lambda^{(n)} F_{ru}^{(-2-n)}\right)$$
$$\overset{n-j}{q}_{\Lambda^{(j)}} = \overset{0}{q}_{[e^{ad_{-i\Psi}}(\Lambda^{(j)})]^{\{n\}}},$$

- Truncated charge algebra : for $0 \leq j, k \leq n$,

$$\{\overset{n-k}{q}_{\Lambda_1^{(k)}}, \overset{n-j}{q}_{\Lambda_2^{(j)}}\} = \begin{cases} \overset{n-k-j}{q}_{-i[\Lambda^{(k)},\Lambda^{(j)}]} & \text{if } j+k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

With this result, we have a closed Poisson algebra, $\mathfrak{p}_n$ such that the action of the charges is canonical and we have the following chain

$$\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n.$$

- Truncation of

$$\{\tilde{q}_{\Lambda_1}, \tilde{q}_{\Lambda_2}\} = \tilde{q}_{-i[\Lambda_1,\Lambda_2]}$$

- match previous results for $n = 1$ YM and electromagnetism [Campiglia,Peraza]

Infinite algebras

- infinite dimensional YM version of the $w_{1+\infty}$ algebra

$$\left[S_m^{p,a}(\bar{z}), S_n^{q,b}(\bar{z})\right] = -i\ f^{ab}{}_c S_{m+n}^{p+q-1,c}(\bar{z})$$

- Following procedure from [Freidel,Pranzetti,Raclariu], construct

$$S_{m,n}^{1+\frac{s}{2},a} = \int dz d\bar{z} z^{m+\frac{s}{2}} \bar{z}^{n-\frac{s}{2}} r_s^a$$

where

$$r_s^a T^a \xleftarrow{\text{renormalised}} \mathcal{R}_s$$

and the charges $\mathcal{R}_s$ satisfy some recursion relations which are a subset of out recursion relations, corresponding to the self-dual sector.

Infinite algebras

- infinite dimensional YM version of the $w_{1+\infty}$ algebra

$$\left[S_m^{p,a}(\bar{z}), S_n^{q,b}(\bar{z})\right] = -i\ f^{ab}{}_c S_{m+n}^{p+q-1,c}(\bar{z})$$

- Following procedure from [Freidel,Pranzetti,Raclariu],

$$S_{m,n}^{1+\frac{s}{2},a} \quad \leftarrow \quad r_s^a$$

where we have restricted to the self-dual sector via

$$\begin{aligned} r_s^1 &= r_s^1(A_{\bar{z}}^{(0)}) \\ r_s^2 &= r_s^2(A_{\bar{z}}^{(0)}, A_z^{(0)}) \\ r_s^3 &= r_s^3(A_{\bar{z}}^{(0)}, A_z^{(0)}, A_z^{(0)}) \\ &\dots \end{aligned}$$

where A_z corresponds to the positive helicity, and $A_{\bar{z}}$ to the negative helicity modes.

Infinite algebras to boring algebras

- infinite dimensional YM version of the $w_{1+\infty}$ algebra

$$\left[S_m^{p,a}(\bar{z}), S_n^{q,b}(\bar{z})\right] = -i\ f^{ab}{}_c S_{m+n}^{p+q-1,c}(\bar{z})$$

- Recall that including all the charges we just get the gauge algebra

$$\{\tilde{q}_{\Lambda_1}, \tilde{q}_{\Lambda_2}\} = \tilde{q}_{-i[\Lambda_1,\Lambda_2]}$$

- Anything in between ?

∞ algebra ? gauge algebra

self-dual full

- Similar for gravity.

Back to soft limits

- Soft theorems

$$\lim_{\omega\to 0} A_{n+1} = \underbrace{(\tfrac{1}{\omega}S^{(0)} + S^{(1)} + \ldots)A_n}_{\text{universal}} \quad + \quad \underbrace{\ldots}_{\text{non-universal}}$$

- universal terms come from symmetries
- in some simple cases non-universal terms vanish
- Focus on subleading via

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)A_{n+1} = S^{(1)}A_n \quad + \quad \ldots$$

- take a more interesting scenario

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)A_{n+1} = S^{(1)}A_n \quad + \text{quasi-universal}$$

Quasi-universal terms from symmetries

- QED with higher derivative interactions, e.g. ϕF^2 [Elvang,Jones,Naculich]

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)A_{n+1} = S^{(1)}A_n \quad + \quad \tilde{S}^{(1)}[A_n]$$

where $S^{(1)}$ is the usual

$$S^{(1)} = -ie\sum_{k=1}^{n} Q_k \frac{p_s^{\mu}\varepsilon_s^{\nu}}{p_k \cdot p_s}\mathcal{J}_{k\mu\nu}$$

and $\tilde{S}^{(1)}$ is an *operator*

$$\tilde{S}_{+}^{(1)}A_n = \sum_{k=1}^{n}\frac{[sk]}{\langle sk\rangle}\mathcal{F}_k A_n, \quad \tilde{S}_{-}^{(1)} = (\tilde{S}_{+}^{(1)})^{\dagger}$$

where $\mathcal{F}_k$ is a particle changing operator.

Quasi-universal terms from symmetries

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)A_{n+1} = S^{(1)}A_n \quad + \quad \tilde{S}^{(1)}[A_n]$$

- [Laddha,Mitra] showed that both $S^{(1)}$ and $\tilde{S}^1$ follow as a Ward identity for an gauge symmetry which violates the fall-off of the fields

$$\Lambda(u,r,z,\bar{z}) = r\Lambda(z,\bar{z}) + \frac{u}{2}(D^2+2)\Lambda(z,zb)$$

and, schematically, the symplectic potential

$$\theta(\varphi,\delta\varphi) = \theta_{old}(\varphi,\delta\varphi) + \theta_{new}(\varphi,\delta\varphi)$$

- $\theta_{old}(\varphi,\delta\varphi)$ gives the universal soft factor
- $\theta_{new}(\varphi,\delta\varphi)$ is subleading relative to $\theta_{old}(\varphi,\delta\varphi)$ for a standard δ_Λ,
- it becomes of the same order when we allow $\delta_{r\Lambda}$ and gives the quasi-universal term

Quasi-universal terms in extended phase space

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)A_{n+1} = S^{(1)}A_n \quad + \quad \tilde{S}^{(1)}[A_n]$$

- Overleading part of gauge parameter acts on the Stueckelberg field

$$\Lambda(u,r,z,\bar{z}) = r\Lambda(z,\bar{z}) + \frac{u}{2}(D^2+2)\Lambda(z,zb)$$

- the symplectic potential is dressed with the Stueckelberg fields

$$\theta(\tilde{\varphi},\delta\tilde{\varphi}) = \theta_{old}(\tilde{\varphi},\delta\tilde{\varphi}) + \theta_{new}(\tilde{\varphi},\delta\tilde{\varphi})$$

- Universal and (mildly) non-universal terms arising from symmetries acting canonically on an extended phase space
- Different gauge choices can simplify things !

Future directions

- More structure from symmetries

$$\sum_{n,k} \mathfrak{r}^n \log^k \mathfrak{r}\ \Lambda^{(n;k)}(\vec{\mathbf{y}}) \xrightarrow{\text{Ward identity}} \text{universal}+(\text{mildly non-universal})\ \text{terms}$$

- Loop corrections
- Procedure is quite general, currently extending to gravity [Geiller,SN,Peraza,Pizzolo].

$$\tilde{g} = e^{\mathcal{L}_V} g.$$

where we have a Stueckelberg vector coming from the diffeo parameter

$$\xi^\mu \quad \rightarrow \quad V^\mu$$

Thank You !