RESURGENCE IN CFT IN 2D

Minimal models and beyond?

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2nd September 2024,
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Introduction

Resurgence of resurgence

- Resurgence has been very successful in Quantum Mechanics and weak coupling expansions in Quantum Field Theory.
- A developing frontier for application of resurgence is CFTs. There have been developments in $\mathcal{N} = 4$ SYM and SCFTs [Dorigoni et al., Perlmutter et al.], 3d sigma models [Reffert et al.], and two dimensional CFTs.
- We want to focus on the last case, CFT$_2$ in the large central charge c expansion. We expand the direction of [Benjamin Collier Maloney Merulyia '23], which is related to the work of [Fiztpatrick Kaplan et al. '14 – '16]. Last week a new paper of [Benjamin et al.] came out which is also related.

It was quantum gravity all along

- A big motivation is AdS/CFT. The large charge expansion of the CFT is related to the weak coupling expansion (small G_N) in gravity. So by studying the more accessible CFT side we have a model for resurgence in the much harder graviton expansions in quantum gravity.
- We specialize to two dimensions because of many powerful exact techniques in CFT_2. Minimal models provide very simple cases where we can do the analysis thoroughly. The history of resurgence suggests that we should start from the simpler solvable models.
- Note that even though there are no gravitons in AdS_3, there is a small G_N perturbation theory from offshell virtual gravitons.

Resurgence in an instant

Many if not most series in QFT are asymptotic, i.e. divergent (Dyson 1953). Typically they are of the form:

$$F_N(g) = \sum_{k=1}^{N} a_k g^k, \qquad a_k \sim A^{-k} k! \quad k \gg 1. \tag{2.1}$$

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$$\varphi(z) \approx \sum_{k \geq 0} c_k z^k \to \widehat{\varphi}(\zeta) = \sum_{k \geq 0} \frac{c_k}{k!} \zeta^k \tag{2.2}$$

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If φ is *Borel summable*, we recover a well defined function $\varphi(z)$ from the Borel sum

$$s(\varphi)(g) = \int_0^\infty \mathrm{e}^{-\zeta} \widehat{\varphi}(g\zeta)\, d\zeta. \tag{2.3}$$

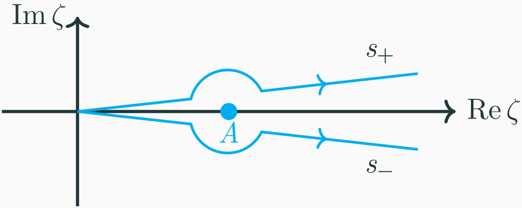
But in most physical theories this is not enough.

If we Borel transform the example from before with $A > 0$

$$F_p(g) \sim \sum_{k\geq 0}^{\infty} (A^{-k} k!) g^k \Rightarrow \widehat{F}(\zeta) = \frac{1}{1 - \zeta/A} \tag{2.4}$$

There's a pole on $\mathbb{R}^+$! We can deform the contour to go slightly above or below the real axis. But an ambiguity remains

$$s_+(F)(g) - s_-(F)(g) = 2\pi \mathrm{i} A g^{-1} \mathrm{e}^{-A/g} \tag{2.5}$$



Or how I learned to stop worrying and love divergent series

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Incorporating these effects introduces **trans-series**

$$\Phi(z) = \sum_{k\geq 0} c_k g^k + \sum_i \mathcal{C}_i^{\pm} \mathrm{e}^{-A_i/g} g^{b_i} \sum_{k\geq 0} c_k^{(i)} g^k + \cdots \tag{2.6}$$

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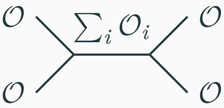
Resurgence helps both **make sense** of what we know and **explore** what we don't know.

2d CFT at large charge

Fantastic Four point functions



The four-point function in a CFT can be expanded as an OPE,

$$\langle \mathcal{O}(0)\mathcal{O}(z)\mathcal{O}(1)\mathcal{O}(\infty)\rangle = |\mathcal{F}_{\mathbb{I}}(c,z)|^2 + \sum_i C_{\mathcal{O}\mathcal{O}i}\,|\mathcal{F}_i(c,z)|^2 \quad (3.7)$$

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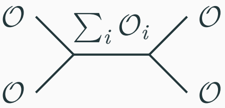

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And it is also known that (in certain setups), the blocks $\mathcal{F}_i$ admit an expansion as an asymptotic series in $1/c$

$$\mathcal{F}_i(c,z) \sim \mathrm{e}^{cS_i(z)} \sum_{n\geq 0} \frac{f_n(z)}{c^n} \quad (3.8)$$

So the 4pt function has a trans-series structure. Can blocks "discover each other" through asymptotic behaviour?

Minimal models are constructed with a finite number of (degenerate) operators constrained by Virasoro symmetry. For the simplest case we take the operator $\phi_{2,1}$, whose OPE is constrained to be $\phi_{2,1} \times \phi_{2,1} = \phi_{1,1} + \phi_{3,1}$ (where $\phi_{1,1} = \mathbb{I}$)

$$\langle \phi_{2,1}\phi_{2,1}\phi_{2,1}\phi_{2,1} \rangle = |\mathcal{F}_{1,1}(c,z)|^2 + g(c)\,|\mathcal{F}_{3,1}(c,z)|^2 \tag{3.9}$$

The identity block is known exactly

$$\mathcal{F}_{1,1}(z,b^2) = z^{1+\frac{3b^2}{2}}(1-z)^{-\frac{b^2}{2}}\,{}_2F_1\left(-b^2, 1+b^2, 2+2b^2; z\right), \tag{3.10}$$

where $c = 13 + 6(b^2 + b^{-2})$. From now on, we change $b^2 = 1/\epsilon - 3/2$, keep in mind $\epsilon \sim \mathcal{O}(1/c)$.

Hypergeometric functions, deconstructed

While saddle point techniques are available, they are cumbersome. Instead, by using the BPZ differential equations one can find explicitly the asymptotic series at finite z,

$$\mathcal{F}_{1,1}(z,b^2) \sim z^{-\frac{5}{4}+\frac{3}{2\epsilon}}(1-z)^{\frac{3}{4}-\frac{1}{2\epsilon}}\mathrm{e}^{\left(1-\frac{1}{\epsilon}\right)S_0(z)}A_0(z)\sum_{n\geq 0} f_n(r(z))\epsilon^n \tag{3.11}$$

where

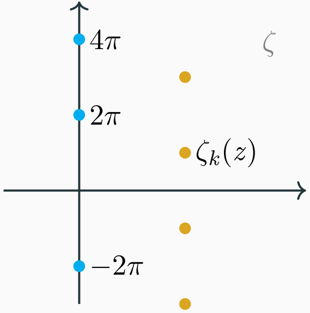
$$S_0(z) = \frac{1}{2}\left(\log\left(\frac{1-r(z)}{r(z)}\right) + \log\left(\frac{z^2}{(z-1)^2}\right) + \log\left(\frac{27}{16}\right)\right),$$
$$A_0(z) = (1-(1-z)z)^{-\frac{1}{4}}, \quad r(z) = \frac{1}{4}\left(\frac{(z-2)(z+1)(1-2z)}{((z-1)z+1)^{3/2}}+2\right). \tag{3.12}$$

And the Borel transform of the f_n series is

$$\widehat{\varphi}(r(z),\zeta) = \frac{5r\zeta}{36}\,{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;r(z)(1-e^{-\zeta})\right). \tag{3.13}$$

Two towers of Borel singularities

At any value of z, in the Borel plane dual to ϵ there are two families of singularities,



$$\zeta_k^i = (2k)\pi\mathrm{i}, \quad k \in \mathbb{Z}\backslash\{0\}$$

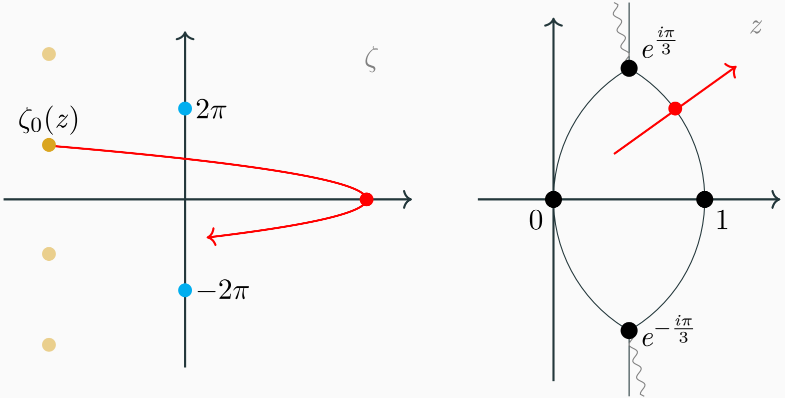
$$\zeta_k = \log\left(\frac{r(z)}{1-r(z)}\right) + (2k-1)\pi\mathrm{i},$$

$$k \in \mathbb{Z}$$

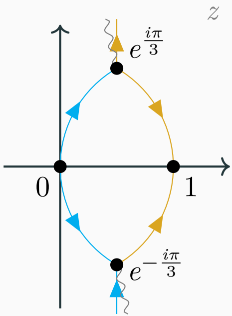
The Stokes jump at ζ_k give the same series with $z \to 1 - z$.

From $\mathcal{B}$ to z

The map $r(z)$ leads to non-trivial lines in the z-plane. These lines happen when the a singularity crosses the positive real line. Similar to WKB (see Aoki et al.).



These jumps are Airy-like when appropriately normalized.



$$\begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_+ = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_-$$

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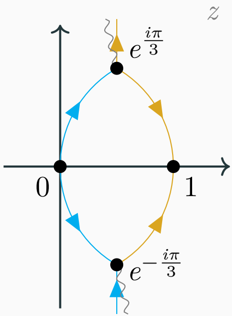

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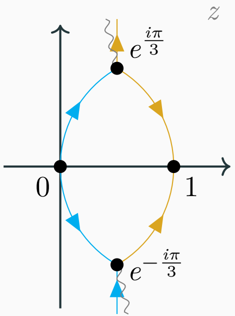

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Love is in the Airy

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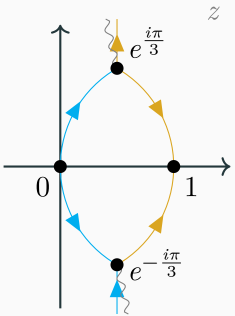

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Resurgence bootstrap?

If we write the four point function in terms of the asymptotic series

$$\langle \phi_{2,1}\phi_{2,1}\phi_{2,1}\phi_{2,1} \rangle = f(\epsilon) \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}^T \cdot \mathcal{M} \cdot \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix} . \quad (3.14)$$

The matrix $\mathcal{M}$ can be fixed by demanding invariance under Stokes jump. This is a stronger requirement than single valuedness of the four-point function.

There are many ways of fixing this four-point function (e.g. crossing) but this suggests that we can constraint observables from resurgence.

Conclusion

Future directions

- We still have more to do! More complicated minimal model (e.g. $\phi_{3,1}$) where not all is analytically available, unitary non-minimal models (numerically through Zamolodchikov recursion relations, an analysis already initiated in Benjamin et al. for $z \to 0$).
- A more specific holographic interpretation of this relation (particularly in unitary non-minimal models) could give insights into resurgence in quantum gravity. In Benjamin et al., they identify a Borel singularity which they associate to unphysical excess angle geometry. Could there be more? Can black holes be seen?
- Are some of these insights valuable for higher dimensions?

Fin

Thank you!