ALGEBRAIC APPROACH TO PARAMETRIC RESURGENCE

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SAMUEL CREW
IMPERIAL

Based on work w. Inês Aniceto

① MOTIVATION

- This work is about linear singularly perturbed differential equations

  E.g. $\hbar^2 \psi''(x) + \psi(x) = 0$ $\qquad$ $\varepsilon y'(z) + y(z) = 1/z$

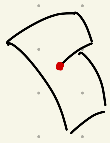
  → Broad application in hep-th and geometry

- Formal parametric transseries solutions

$$y(z) = y_0(z) + \varepsilon y_1(z) + \varepsilon^2 y_2(z) + \dots$$

- Parametric resurgence is hard e.g. new effects such as higher order Stokes' phenomenon

- We will study exactly soluble (algebraic curve) examples to gain insight!



② Parametric algebraic curves

Example

$\Sigma_z \subset \mathbb{C}_w \times \mathbb{C}_\varphi$ depending holomorphically on $z \in \mathbb{C}_z$

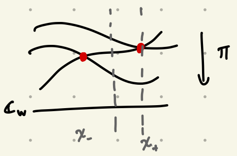
$\Sigma_z = \{ (w,\varphi) : P_z(w,\varphi) = 0 \}$

$P_z(w,\varphi) = (4w^2 - 2wz^2 - 4wz + 2w + 2z^3 - z)\varphi^3 + (2z^2 - 4w)\varphi + 1$

→ Could solve $\varphi(w,z) = \sqrt[3]{\dots}$ (w Borel variable, z parameter)

Branch / Ramification points

- $\pi : \Sigma_z \to \mathbb{C}_w$ defines a multi-sheeted Riemann surface structure



\# π^{-1} pre-images drops at branch points

- Branch pts. are detected by the discriminant $\Delta(w,z) := \prod_{i,j} (\varphi_i - \varphi_j)^2$

$$\Delta \propto (w - x_+(z))(w - x_-(z))$$

$$x_\pm(z) = -\frac{1}{3}(-1 + 3z \pm 2z^{3/2})$$

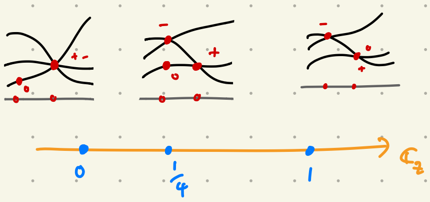
- Set of z-dependent branch pts $\Gamma_w(z) \subset \Sigma_z$

"Turning" Points

- As $z \in \mathbb{C}_z$ the structure varies. $\Gamma_w(z) = \{ \overset{x_0(z)}{0}, x_+(z), x_-(z) \}$

- Discriminant of discriminant tells us when $x_i(z)$ coalesce $\Delta_w(\Delta_\varphi(w,z))(z) = 0$

$z = 1 : \; x_0(z) = x_+(z)$
$z = 0 : \; x_+(z) = x_-(z)$ $\quad =: \Gamma_z \subset \mathbb{C}_z$
$z = 1/4 : \; x_0(z) = x_-(z)$



- $z = 1/4$ is a virtual turning point!

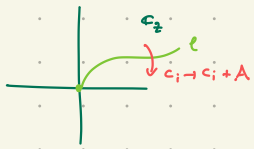
③ Transseries

Parametric transseries

- Formal algebraic asymptotic series + exponential corrections

$$
\begin{aligned}
y(z,\varepsilon) &= y_0(z) + y_1(z)\varepsilon + y_2(z)\varepsilon^2 + \dots \\
&\quad + c_1 e^{-x_1(z)/\varepsilon}\left(y_0^1(z) + y_1^1(z)\varepsilon + y_2^1(z)\varepsilon^2 + \dots\right) \\
&\quad + c_2 e^{-x_2(z)/\varepsilon}\left(y_0^2(z) + y_1^2(z)\varepsilon + y_2^2(z)\varepsilon^2 + \dots\right) \\
&\quad + \dots
\end{aligned}
$$

→ Together with Stokes phenomenon describe the asymptotics of sol^{n}s to singularly perturbed ODEs

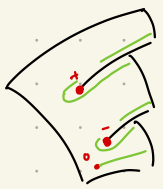


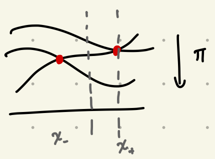
$G = \cup \ell \subset \mathbb{C}_z$

Transseries x Algebraic Curves

- We associate a parametric transseries to a parametric algebraic curve via the Borel transform.

- Consider the two branch point example w. marked point $\Sigma_z = \{ P_z(w,\varphi) = 0 \}$





- Associate Hankel contours $\mathcal{H}_+, \mathcal{H}_-, \mathcal{H}_0$ to each element $\Gamma_w(z) \subset \Sigma_z$

- The inverse Borel transform $y(z,\varepsilon) := \sum_x \int_{\mathcal{H}_x} dw\, e^{-w/\varepsilon} \varphi(w;z)$

  defines a formal transseries by asymptotic expansion:

$$
y(z,\varepsilon) := \sum_x e^{-x(z)/\varepsilon}\left[\varphi_0^x(z) + \varphi_1^x(z)\varepsilon + \varphi_2^x(z)\varepsilon^2 + \dots\right]
$$

  → $\varphi_i^x(z)$ are local (singular) germ expansion of the curve

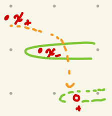
- Turning points / caustics occur at $\Gamma_z \subset \mathbb{C}_z$ where singularities coalesce

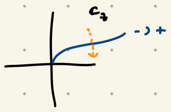
  → Asymptotic series breaks down – need new variables

  → Nothing special happens at virtual turning points

Stokes' Phenomenon

- The transseries jumps along Stokes lines $\ell \subset \mathbb{C}_z$

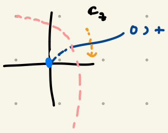




→ This occurs at ℓ: $\operatorname{Re} x_+(z) > \operatorname{Re} x_-(z)$ $\quad$ $\operatorname{Im} x_+(z) = \operatorname{Im} x_-(z)$

- The presence of virtual turning points modifies Stokes' lines

→ x_- can sneak over x_0 – hidden by x_+



Higher order Stokes' line

Moral: Algebraic parametric transseries shed light on HOSP

④ ODEs × Curves

Q: Do examples exist in the wild?

A: Algebraic curve $\Sigma_z = \{ (w, \varphi) \subset \mathbb{C}_w \times \mathbb{C}_\varphi \ : \ P_z(w, \varphi) = 0 \}$

B: ODE $\varepsilon^2 y''(z) + \varepsilon P(z) y'(z) + Q(z) y(z) = R(z)$

or Borel PDE $\partial_w^2 + P \partial_z \partial_w + Q \partial_z^2 = 0$

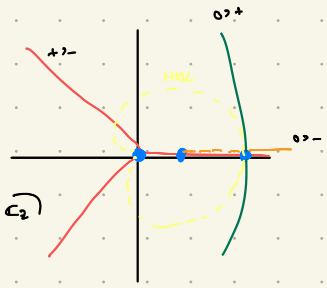
- We give an algorithm to find examples

Running Example

A: $P_z(w,\varphi) = (4w^2 - 2wz^2 - 4wz + 2w + 2z^3 - z)\varphi^2 + (2z^2 - 4w)\varphi + 1$

B: $\varepsilon^2 y''(z) + 2\varepsilon y'(z) + (1-z) y(z) = \varepsilon$

- One may then understand e.g. the HOSP purely in terms of the geometry of Σ_z



Outlook

We've developed a fun playground for parametric resurgence

1. Study two moduli curves Σ_{z_1, z_2} describing sd^{n}s to e.g. time dependent Schrödinger eq^{n}
2. Systematics of when P_B admits an algebraic sd^{n}
3. Nonlinear higher order Stokes' phenomenon

Thanks :)